CIVIL ENGINEERING

For

UPSC Engineering Services Examination, GATE,

State Engineering Service Examination & Public Sector Examination.

(BHEL, NTPC, NHPC, DRDO, SAIL, HAL, BSNL, BPCL, NPCL, etc.)

STRENGTH OF MATERIALS

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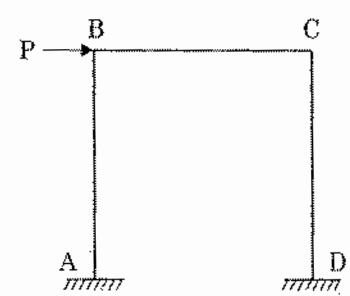
Strength of Materials

INTRODUCTION

- Our concern in Strength of Materials topic is to study the behaviour of a material when it is subjected to forces and moments.
- Each member of a structure is made up of certain materials, which could be a rigid material or a deformable material.

Rigid and Deformable Material

A **rigid material** is one which does not undergo any change in its geometry, size or shape. On the other hand, a deformable material is the one in which change in size, shape or both will occur when it is subjected to a force/moment. The geometrical changes produced are called **deformations** and hence the name deformable material. All materials are actually deformable and the idea of rigid material is only a conceptual idealization. A rigid material term has been used just for the simplification in the analysis.



For example when we perform the analysis of a frame as shown here, we assume that length of member BC will not change. Which means that member BC has been assumed to be axially rigid. Thus joint B and C will move rightwards by equal amount and the analysis will become simplified.

STRESSES AND STRAINS

- When we apply forces on solids, deformations are produced if the solid is restrained from motion either fully or partially.
- If the solid is not restrained, it may undergo displacements without change in shape or size and these displacements are termed as rigid body displacements. If the solid is restrained by some other force, known as **reaction**, which keeps the solid in equilibrium, the force will be transmitted through the medium of the solid to the restraining support.
- Stresses (defined as force/area) are generated as a resistance to the applied external forces or as a result of restrainted deformations.
- In the analysis and design of structures we are required to find out stresses and deformations/deflection (which is related to strain).
- Stresses are broadly classified as:
  (a) Normal Stress
  (b) Shear Stress

NORMAL STRESS

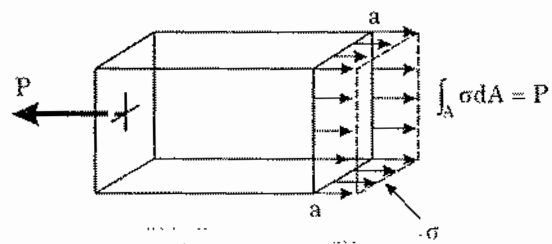
Normal stresses can be:

(a) Axial stress

(b) Bearing stress

(c) Bending stress

Axial stress: It is the load directed along the axis of the member (i.e. Normal to the section).

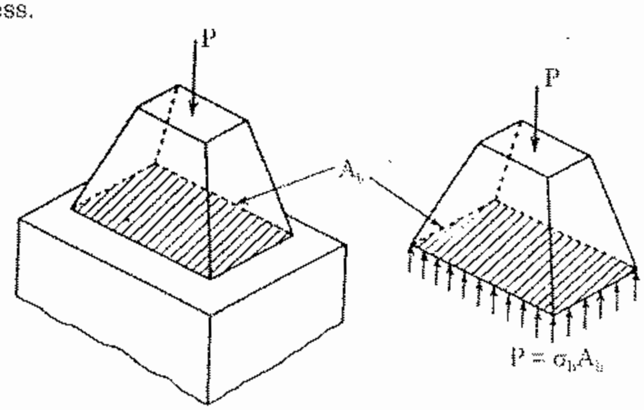


$$\text{Normal stress at a point} = \frac{dP}{dA} = \sigma$$

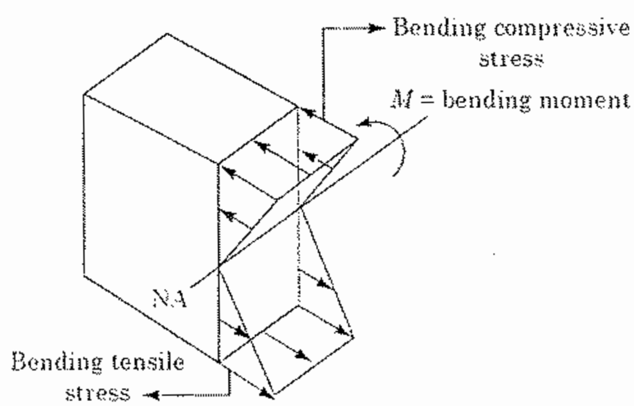
$$P = \int \sigma dA$$

Note: *Normal stress could be tensile or compressive. [When stress is constant or uniform over a section, the stress is called simple stress].*

Bearing stress: Compressive stress arising when one body is supported by another is called *bearing stress*. It is a type of normal stress.

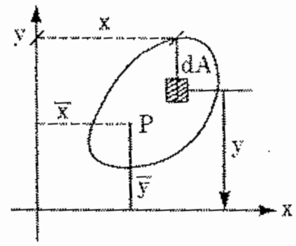


Bending stress: Bending tension and compression produces normal stress.



Sign convention → Tensile stress (+) ve; Compressive stress (–) ve

Note: *Line of action of the axial force for uniform stress distribution passes through the centroid of the section.*



$$\int \sigma dA y = \text{Moment about x-axis due to uniform stress}$$

$$\int \sigma dA.x = \text{Moment about y-axis}$$

$$\sigma = \text{constant (uniform stress)}$$

$$P.\bar{y} = \text{Moment about x-axis due to load P}$$

$$P.\bar{x} = \text{Moment about y-axis due to load P}$$

$$\Rightarrow \quad P.\bar{y} = \int \sigma dA y = \int \frac{P}{A} dA.y \quad \Rightarrow \quad \bar{y} = \frac{\int y dA}{A} \qquad \text{...(1)}$$

$$P.\bar{x} = \int \sigma dA x = \int \frac{P}{A} dA.x \quad \Rightarrow \quad \bar{x} = \frac{\int x dA}{A} \qquad \text{...(2)}$$

Equations (1) and (2) are defining the coordinate of centroid of an area.

SHEARING STRESS

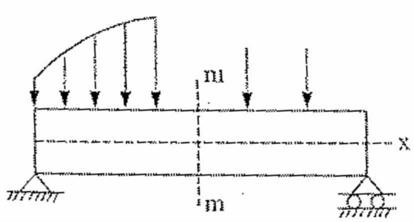
It is the stress acting in the plane of a section. The shearing stress could be

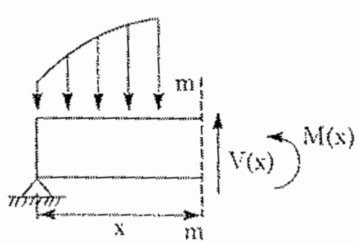
(a) Direct shear stress

(b) Indirect shear stress

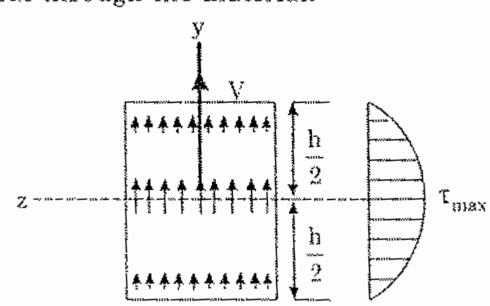
Direct Shear Stress

Shear stress is created due to direct action of forces in trying to cut through the material.





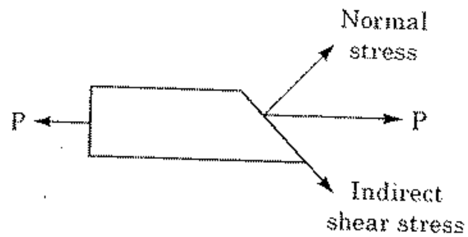
V(x) is the direct shear stress



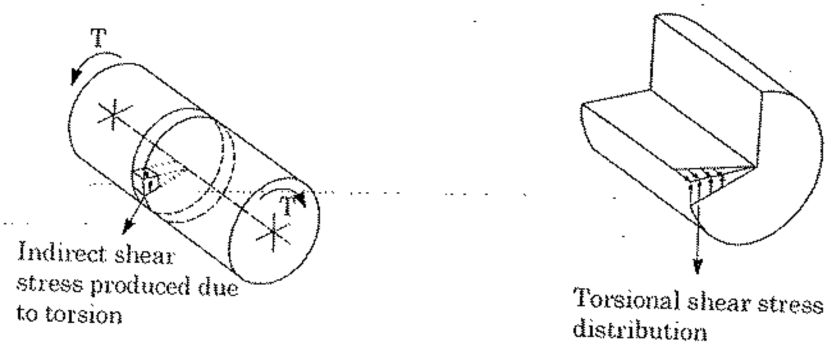
Direct shear stress distribution is parabolic.

Indirect Shear Stress

Indirect shear stress arises due to (a) tension or compression (b) torsion. The figure below shows that shear stress arises in an indirect manner when members are subjected to tension/compression



Torsional shear stress is zero at centre and max at extreme point.



EQUALITY OF SHEAR STRESS ON PERPENDICULAR PLANES

1. Shear stress on *opposite faces* of an element are equal in magnitude and opposite in direction
2. Shear stress on adjacent and perpendicular faces of an element are equal in magnitude and have directions such that both stresses point towards or both point away from the line of intersection of the faces. These shear stresses are called complimentry shear stresses.

To prove the above statements we consider a case of pure shear i.e. when only shear stress is acting and no normal stress is acting.

If we cut out an element as shown in the figure (a) below and consider its equilibrium, then as the stress element shown is very small, shear stress over the face will be uniform, hence

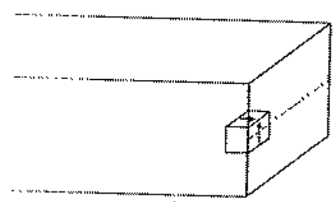
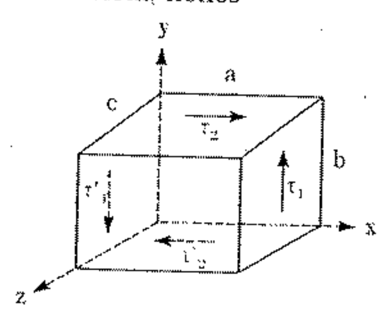


Fig. a



$$\tau_1 \times b \times c = \tau_1' \times b \times c \quad \text{(Force equilibrium)}$$

$$\Rightarrow \quad \tau_1 = \tau_1'$$

$$\tau_2 \times a \times c = \tau_2' \times a \times c \quad \text{(Force equilibrium)}$$

$$\Rightarrow \quad \tau_2 = \tau'_2$$

(Shear stress on opposite faces are equal and opposite)

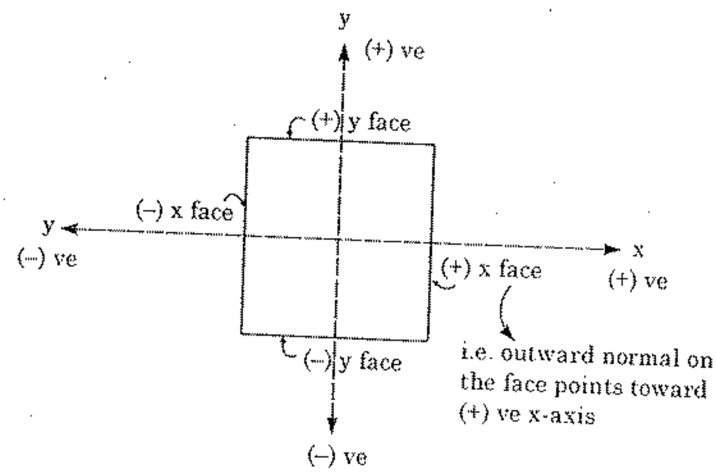
Similarly, Anti clockwise couple = clockwise couple (Moment equilibrium)

$$\Rightarrow \quad (\tau_1 \times b \times c)\, a = (\tau_2 \times a \times c)\, b$$

$$\Rightarrow \quad \tau_1 = \tau_2$$

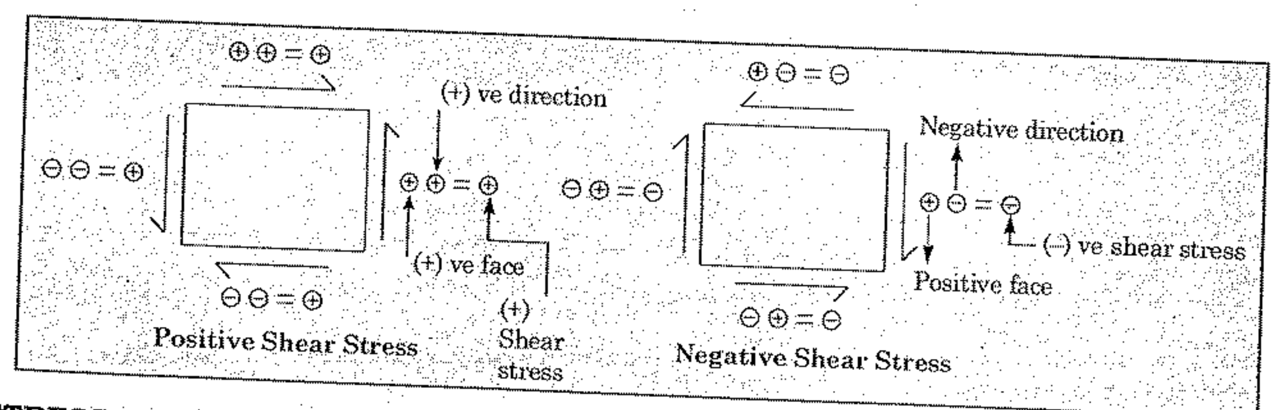
> **Shear stress on adjacenet ⊥r faces are equal in magnitude and are directed such that both points towards or both points away from the line of intersection of planes.**

SIGN CONVENTION FOR SHEAR STRESS



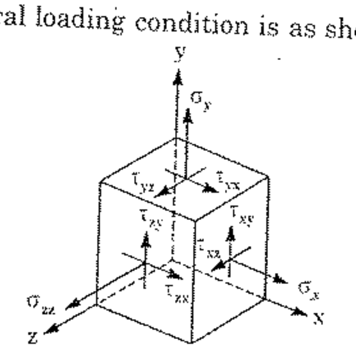
- A shear stress acting on positive face is
  - (+) ve if it acts in (+) ve coordinate direction
  - (–) ve if it acts in (–) ve coordinate direction
- A shear stress acting on negative face is
  - (+) ve if it acts in (–) ve coordinate direction
  - (–) ve if it acts in (+) ve coordinate direction

Thus



STRESS UNDER GENERAL LOADING CONDITIONS

Stress at any point under most general loading condition is as shown below in the figure.



Stress at any point (stress

τ_{xz} = shear stress on x-face in z-direction.

σ_{xx} = normal stress on plane normal to x face in x-direction.

Stress is not a vector because its resultant cannot be obtained by parallelogram law of vector addtion. It is a mathematical quantity called tensor.

Stress tensor is represented as:

$$\sigma \text{ (stress tensor)} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$$

Stress is a 2nd order tensor.

Note: *Magnitude has only one dimension hence it is a 3^0 = zero-order tensor.*

Direction has three dimension (x-direction, y-direction and z-direction) hece it is a 3^1 = 1st order tensor. Stress has 9-dimensions (3^2 = 2nd order tensor).

- At any point we have 9-stress component

  3—Normal stress component ($\sigma_{xx}, \sigma_{yy}, \sigma_{zz}$)

  6—Shear stress component ($\tau_{xy}, \tau_{xz}, \tau_{yx}, \tau_{yz}, \tau_{zx}, \tau_{zy}$]

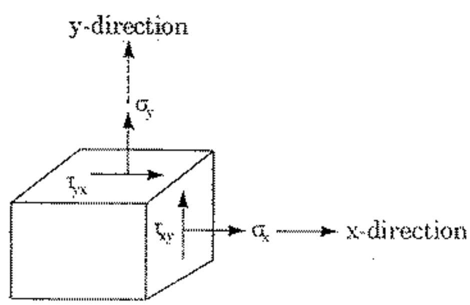
  But we know from complimantry shear stress concept that

$$\tau_{xz} = \tau_{zx}$$
$$\tau_{xy} = \tau_{yx}$$
$$\tau_{yz} = \tau_{zy}$$

Hence only 6-stress components are require to define the condition of stress at a given point. They are

$$\boxed{\sigma_{xx}, \sigma_{yy}, \sigma_{zz} \quad \tau_{xy}, \tau_{yz}, \tau_{zx}}$$

In 2D conditions:



There are 4-stress elements at a point. They are σ_x, σ_y τ_{xy}, τ_{yx}. But $\tau_{xy} = \tau_{yx}$. Thus only three stresses components are required to define condition of stress at a point. They are

$$\sigma_{yy}, \sigma_{xx}, \tau_{xy}$$

Note: *In 3D-condition, 9 stress elements exist → 6 stress components are require to define condition of stress at a point.*

In 2D-condition, 4 stress elements exist → 3 stress components are require to define conditions of stress at a point.

DESIGN OF MEMBERS

- For a member to be safe, stresses in the member generated due to external effects must be less than the allowable stress.

$$\text{Allowable stress} = \frac{\text{Yield stress}}{\text{F.o.s.}}$$
$$\text{Margin of safety} = (\text{F.o.s.} - 1)$$

- For ductile materials Factor of safety is applied on **yield stress.**
- For brittle material F.o.s. is applied on **ultimate stress.**

Strength of material is not the only criterion that must be considered is designing structures. Stiffness hardness, toughness and ductility are the other properties which helps in chosing the type of material. Thus, various tests are carried out and one such test is the tension test which helps in developing some important basic concepts. In tension test, stress and strain are plotted for different value of loads applied on the test specimen. We have already described stresses, we now try to define strains.

NORMAL STRAIN

Normal strain is defined as deformation per unit length.

$$\text{Normal strain } \varepsilon = \frac{\delta}{L}$$

where, δ = Change in length and L = Actual length.

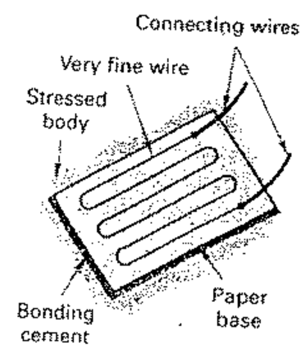
The above value gives only the average value of strain. The correct value of strain at any position is

$$\varepsilon = \frac{d\delta}{dL}$$

$d\delta$ = differential elongation (small elongation)

dL = differential length (small length)

- Normal strain is measured using extensometer.
- Normal strain on the surface of an element is measured using wire strain gauge. Figure shows that on a body a loop of wire is pasted and then it is stressed. Due to stress, strain occurs and length of the wire as well as dia of wire changes.



Note: *Because of increase in length of wire and decrease in diameter, electrical resistivity of wire changes and if current is passed through the wire, its value changes on straining. By correlating the change in current with change in length, strain can be measured.*

MATHEMATICAL DEFINITION OF STRAIN

If u, v, w are the three displacement components occurring respectively in x, y and z direction of co-ordinate axis, the basic definition of strains will be

$$\varepsilon_x = \frac{\partial u}{\partial x} = \text{Normal strain in x-direction} \qquad \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = \text{shearing strain}$$

$$\varepsilon_y = \frac{\partial v}{\partial y} = \text{Normal strain in y-direction} \qquad \gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = \text{shearing strain}$$

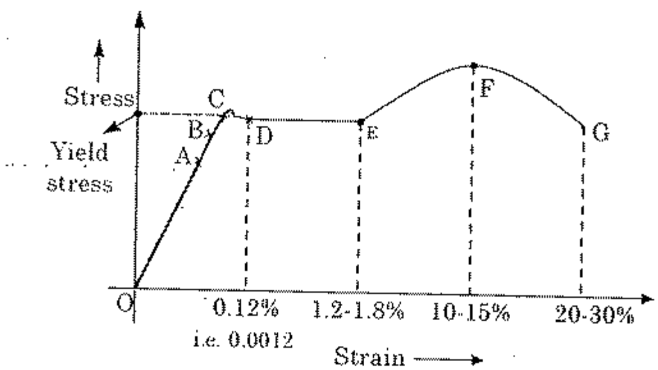
$$\varepsilon_z = \frac{\partial w}{\partial z} = \text{Normal strain in z-direction} \qquad \gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} = \text{shearing strain}$$

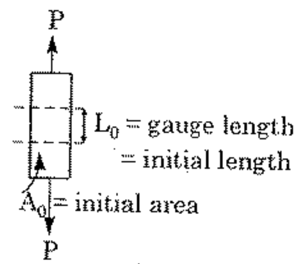
STRESS STRAIN CURVE

If we a take prismatic bars (bar of constant x-section throughout the length) of different lengths and areas of x-section and plot load-deformation curve, the load-deformation curve will be different for different bars. However if stress-strain curve is plotted it will be almost same for all the bars. Thus, **stress-strain curve is a characteristic property of material** and does not depend upon the dimension of particular specimen used.

Hence stress strain curve is used to study the properties of a material but not the load deformation curve.

Stress-Strain Curve for Mild Steel (Low Carbon Steel-Ductile Steel)





$$\text{Stress} = \frac{P}{A_0}$$

$$\text{Strain} = \frac{\delta}{L_0}$$

OA	=	Linear curve
A	=	Proportional limit
B	=	Elastic limit
C	=	Upper yield point
D	=	Lower yield point
DE	=	Plastic region
EF	=	Strain hardening region
FG	=	Necking region
F	=	Ultimate stress point
G	=	Fracture point

SOME IMPORTANT POINTS ABOUT STRESS-STRAIN CURVE OF MILD STEEL

Region OA

- From O to A stress is proportional to strain.

$$\text{i.e. } \frac{\text{stress}}{\text{strain}} = \text{constant} = E = \text{modulus of elasticity} = \text{slope of OA}$$

- Strains are infinitesimal.
- Volume of specimen increases due to tension.

Region AD

- Beyond A, strain increases more rapidly as compared to stress.
- If specimen is unloaded at point B, the unloading curve will be B–A–0.
- Upper yield point corresponds to the load reached just before yield starts.
- Lower yield point corresponds to the load required to maintain yield.

- Since upper yield point is transient, the lower yield point should be used to determine the yield strength of material.
- Volume of specimen increases.

Region DE

- Once yield stress is reached, the specimen undergoes *large deformation* with a relatively small increase in applied load.
- This deformation is caused by slippage of the material along oblique surfaces and is due therefore primarily to shearing stresses. The deformations are parmanent. Volume of specimen does not change.
- Note that under uniaxial tensile stress, max shear stress is at 45° (oblique) angle to normal stress.

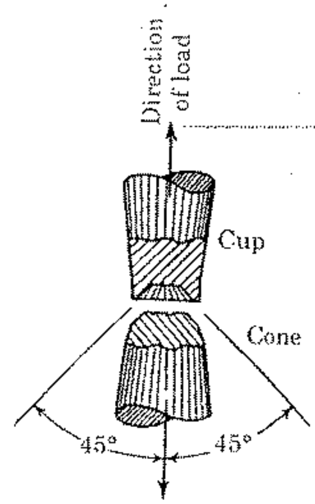
Region EF

- After undergoing large strains that occur during yielding in region DE, the steel begins to strain harden.
- During strain hardening, the material undergoes *changes in its crystalline structure*, resulting in increased resistance of the material to further deformation.

Region FG

- After point F (ultimate stress), the *diameter of a portion* of the specimen begins to decrease, because of local instability. This phenomenon is called necking.
- After necking has begun, somewhat lower loads are sufficient to keep the specimen elongating further untill it finally ruptures.
- Rupture occurs along a cone shaped surface that forms an angle of approx. 45° with the original surface of the specimen **(cup-cone failure)**.
- This indicates that *shear* is primarily responsible for the *failure of ductile materials* and confirms the fact that, under an axial load, shearing stresses are largest on surfaces forming an angle of 45° with the load.
- Percentage reduction in area at the time of fracture is approx. 50%

$$50 = \frac{A_0 - A_{fracture}}{A_0} \times 100$$



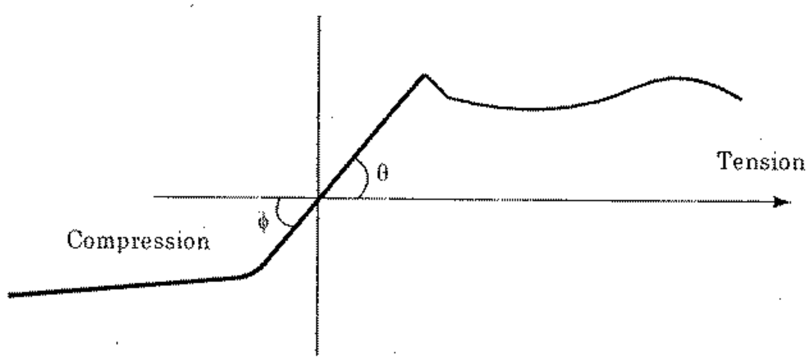
Test speciment at the time of tensile failure (Cup-Cone Failure)

For all practical purposes, proportional limit, elastic limit and yield point are assumed to be same. Stress-strain curve may vary depending upon the *temperature of specimen*, *rate of loading* (static load does not affect the properties of material. but dynamic load affects the properties of material), *manufacturing processes* such as rolling and on the fact that whether the specimen is being loaded for the 1st times or as it is a reloading case etc.

- Rate of stressing for tension test is 6–60 N/mm²/sec to determine yield stress or proof stress.
- Rate of loading actually depends on what strength is to be found out. (i.e. yield strength or ultimate strength).
- Length dimension of specimen to be tested depends on the product for which tensile test is to carried out i.e. for sheet or wire or tube etc.

Mild Steel in Compression

- If *structural steel (mild steel)* is subjected to compression instead of tension, *the stress strain curve will essentially be same through its initial straight line portion and through the beginning of the portion corresponding to yield and strain hardening.* For larger value of strain, stress-strain curve will diverge.
- In compression no necking occurs.

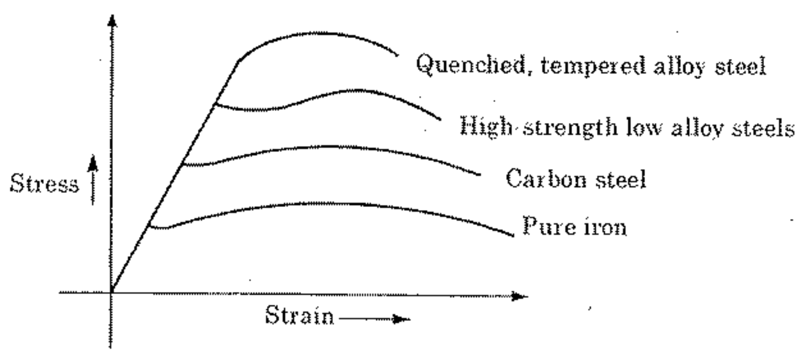


- **Note that modulus of elasticity in tension = modular of elasticity in compression.**

i.e. $\theta = \phi$

Stress Strain Curve for Other Materials

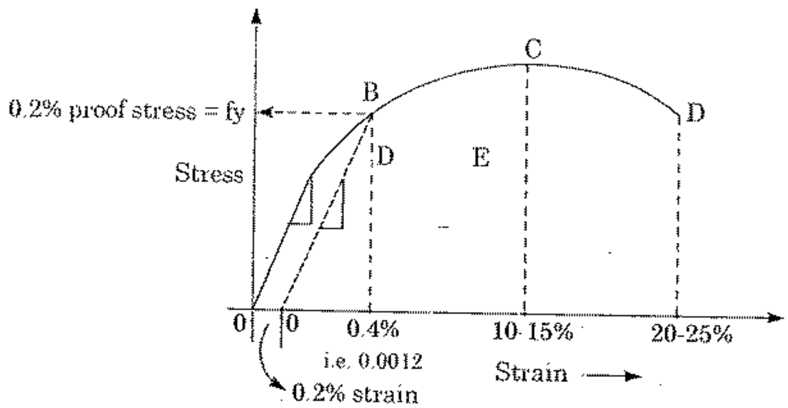
- Some of the physical properties of structural metals, such as *strength, ductility* and corrosion resistance, can be greately affected by alloying, *heat treatment* and the *manufacturing processes used.*



- For these 4-different grades of steel, *yield stress, ultimate stress* and fracture strain (ductility) differ *greatly.*
- All of them possess the same *modulus of elasticity* [i.e. stiffness with in linear range is same].
- Note that, as yield strength increases, ductility falls.

Aluminium and Copper

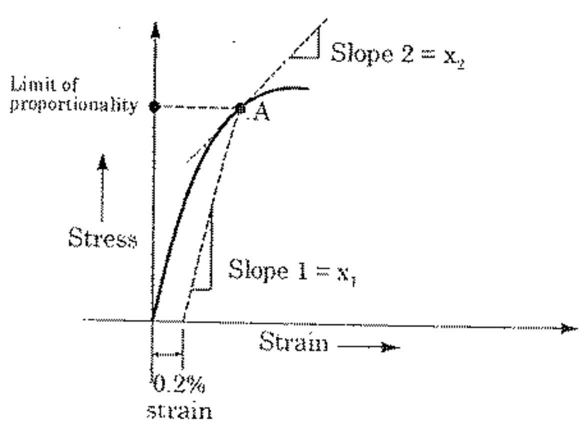
- *For ductile materials* which do not have clarify defined yield point, *yield strength* is defined by offset method. (Al, Cu).



Stress strain curve the aluminium

- A line parallel to the initial straight line portion of stress-strain diagram is drawn through a point of 0.2% strain. The point where it cuts the stress-strain curve, gives the yield strength or proof stress at 0.2% strain.
- Offset yield stress or 0.2% proof stress is not a material property. It is used only for calculation purposes.
- $\boxed{E_{Al} = \frac{1}{3} E_{St}}$

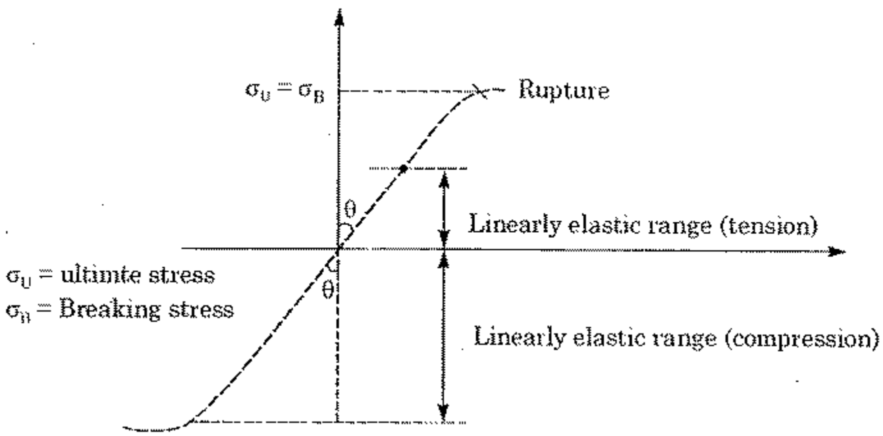
Note: *The stress strain diagram of many materials is actually a curve on which there is no definite proportional limit. In such cases stress-strain proportionality is assumed to exist upto a stress at which the strain increases at a rate 50% greater than that shown by initial tangent to stress-strain diagram. If a line is drawn parallel to the initial straight line portion of stess strain curve from 0.2% strain, it cuts the stress-strain curve at the point where strain rate increases at a rate 50% more than that shown by initial straight line. This is the concept behing chosing 0.2% proof stress as the yield stress.* **The point A, as shown below, is choosen as the yield stress.**



$$\frac{1}{x_2} = 1.5\left(\frac{1}{x_1}\right)$$

Various ductile materials are Al, Cu, Mg, Pb, Nickle, Boron, Bronze, Nylon, Teflon etc.

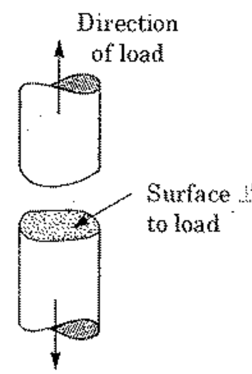
Stress Strain Diagram for Brittle Material (Tension Test)



- Brittle materials which comprises cast iron, glass, stons, high carbon steels etc. are characterised by the fact that rupture occurs without any noticeable prior change in the rate of elongation.
- Strain at rupture is much smaller as compared to ductile material (rupture strain is elastic).

  Rupture stress = ultimate stress
- No necking occurs and *rupture occurs along a surface perpendicular to load.*

  Thus *Normal stresses are primarily responsible for the failure of brittle materials.*



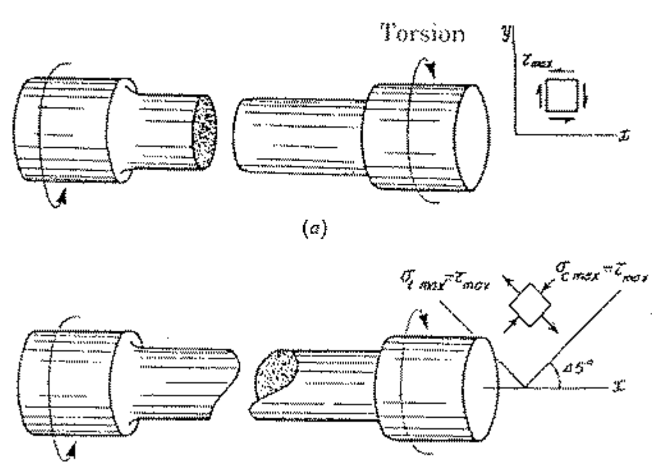
Type of Failure in Torsion Test:

- Brittle material fails at 45°
- Ductile material fails at 90°

  [Because brittle material failes due to normal stress which is max at 45° to the axis in case of torsion, while in ductile material failure is due to shear which, in case of tonsion, occurs at 90° to the axis.]

However, in Tension test

- Brittle material fails at 90°
- Ductile material fails at 45°



Fracture of (a) mild steel and (b) cast iron in torsion.

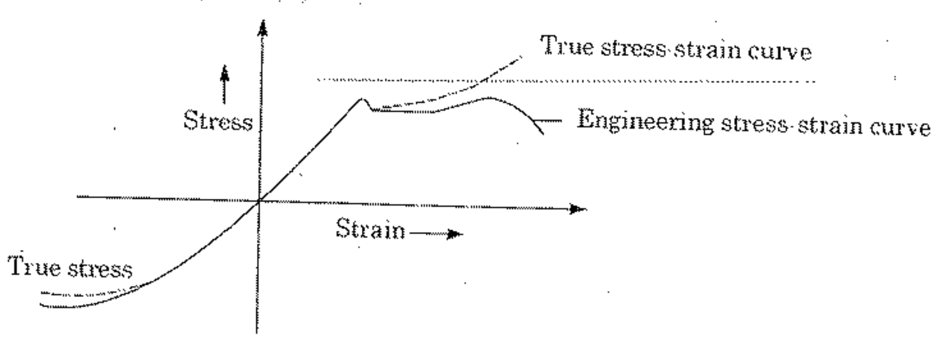
Brittle Materials in Compression

- For most brittle material, the ultimate strength in compression is much larger than ultimate strength in tension.
- This is due to the presence of flaws, such as microscopic cracks as cavities, which tend to weaken the materials in tension, while not appreciably affecting its resistance to compressive failure.
- *Linear elastic range in compression is larger as compared to that in tension.*
- Modulus of elasticity is same in tension and compression.

BRITTLE FRACTURE

- A material which is ductile at normal temperature may become brittle at very low temperature. Similarly, a material which is brittle at normal temperature may became ductile at very high temperature.
- At sub-zero temperature, a structural steel may fail by *brittle-fracture*. In this case modulus of elasticity (E) remains same but ductility reduces greately. Thereby, leading to failure without sufficient prior warning (due to large scale deformation). Brittle fracture will also occur at sections where due to irregularity, stress concentration occur.

TRUE STRESS AND TRUE STRAIN



Note: *True stress curve is below engineering stress curve in compression because resisting area in compression increases.*

- As elongation or contraction takes place, x-sectional area and length of the test specimen also changes. Till now we have been plotting stress-strain curve using original length and original area. If stress and strain are plotted corresponding to actual area of x-section and actual length at any time, during straining, we get true stress-strain diagram.
- It should be noted that there is no decrease in true stress during necking (although applied load value decreases).
- The results obtained from tensile and compressive tests will yield essentially the same plot when true stresses and true strains are used. This is not the case for large values of strains, when engineering stress is plotted against engineering strain.
- In practical cases we will use only engineering stress and engineering strain curve (if we are with in proportional limit the two curves will essentially be same).

$$\text{Engineering stress} = \frac{P}{A_0}$$

$$\text{Engineering strain} = \frac{\delta}{L_0}$$

$$\text{True stress} = \frac{P}{A}$$

$$\text{True strain} = \frac{\Delta L}{L}$$

where A_0 = original area of specimen
L_0 = original length of specimen
A and L are area of x-section and length at any point during straining.

- True strain for finite increment of loading such that length changes from L_0 to L is given by

$$\boxed{\varepsilon_t = \int_{L_0}^{L} \frac{dL}{L} = \ln \frac{L}{L_0}}$$

RELATION BETWEEN TRUE STRESS AND ENGINEERING STRESS

- Volume of the test specimen in elastic range increases during tension and decreases during compression. But in plastic zone, volume is constant. However if volume changes are not considered in elastic zone than

$$A_0L_0 = AL$$

In tension

$$A = \frac{A_0L_0}{L} = \frac{A_0}{\frac{L_0 + \delta L}{L_0}} = \frac{A_0}{1+\varepsilon}$$

$$\Rightarrow \quad \sigma = \frac{P}{A} = \frac{P}{A_0 \times \frac{A}{A_0}} = \frac{P}{A_0\left[\frac{1}{1+\varepsilon}\right]} = \frac{P}{A_0}(1+\varepsilon)$$

$$\Rightarrow \quad \boxed{\sigma = \sigma_0 (1+\varepsilon)}$$

In compression

$$A = \frac{A_0L_0}{L} = \frac{A_0}{\frac{L_0 - \delta L}{L_0}} = \frac{A_0}{1-\varepsilon}$$

$$\Rightarrow \quad \boxed{\sigma = \sigma_0 (1-\varepsilon)}$$

Example 1

The volume of a tensile specimen is essentially constant while plastic deformation occurs. If the initial diameter of the specimen is d_1, show that when the diameter is d, the true strains is $\epsilon_t = 2 \ln (d_1/d)$.

Sol: If the volume is constant then

$$\frac{\pi}{4}d^2L = \frac{\pi}{4}d_1^2 L_0$$

$$\frac{L}{L_0} = \frac{d_1^2}{d^2} = \left(\frac{d_1}{d}\right)^2$$

$$\epsilon_t = \ln\frac{L}{L_0} = \ln\left(\frac{d_1}{d}\right)^2 = 2\ln\frac{d_1}{d}$$

Example 2

Denoting by ϵ the "engineering strain" in a tensile specimen, show that the true strain is $\epsilon_t = \ln(1+\epsilon)$.

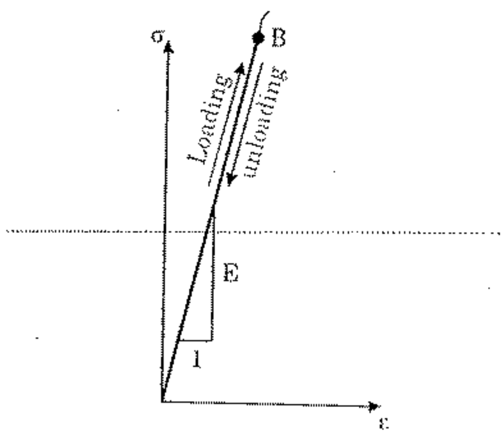
Sol:
$$\epsilon = \ln\frac{L}{L_0} = \ln\frac{L_0+\delta}{L_0} = \ln\left(1+\frac{\delta}{L_0}\right) = \ln(1+\epsilon)$$

Thus
$$\epsilon_t = \ln(1+\epsilon)$$

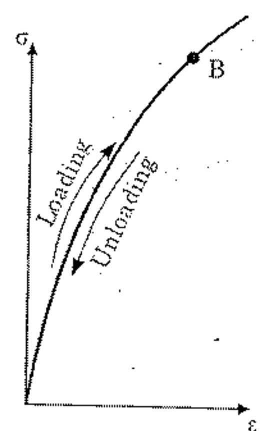
PROPERTIES OF MATERIALS

Elasticity

The property of a material by virtue of which, it returns to its original dimension during unloading is called elasticity and the material is called elastic. If material is unloaded before elastic limit (B) is reached, the unloading curve will follow the original curve. When material is unloaded before elastic limit, the original dimension of the member is regained instantly

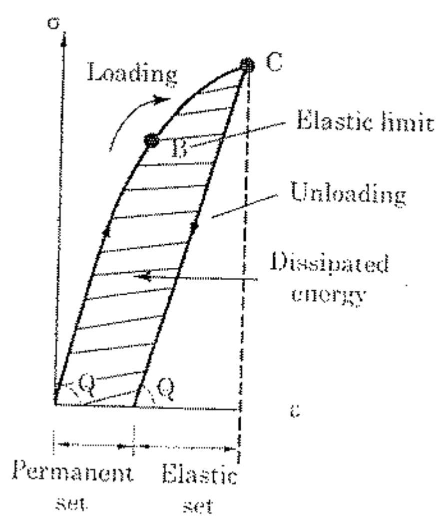


(a) linearly elastic material



(b) non linearly elastic material.

- If however, material is stressed beyond elastic limit and than unloaded, it will have a residual strain and the unloading curve will be different from original loading curve. The unloading curve will be parallel to the initial portion of the loading curve.



- Residual elongation of bar is called permanent set.

Plasticity

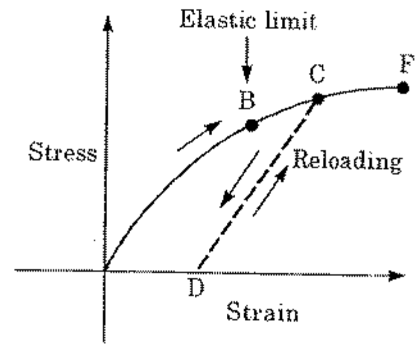
The characteristic of a material by which it undergoes inelastic strain beyond the strain at the elastic limit is known as *plasticity*.

Reloading

- During reloading material behaves as linearly elastic from D to C.
- C is the new proportional limit.
- Thus, proportional limit is increased.
- But ductility decreases, because yielding zone reduces from

  B → F to

  C → F



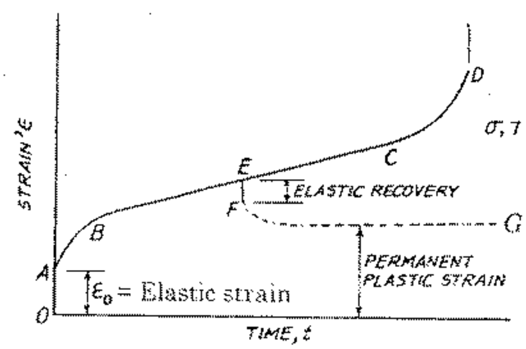
Creep

The property by virtue of which a material undergoes additional deformation (over and above that due to applied load) with passage of time under sustained loading within elastic limit is called creep.

- Rate of creep decrease with time (because as stress increases strain hardening takes place).
- Creep is usually more important at high temperature and higher stresses.
- It depends on temperature level, stress level, time, type of loading (static or dynamic).
- Generally, effect of creep becomes noticeable at approx 30% of melting point (in degree kelvin) for metals.
- Moderate creep in concrete is sometimes welcomed because, it relieves tensile stress that might otherwise lead to cracking.

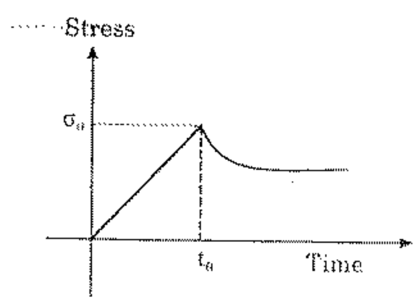
  OA → elastic strain

  BD → creep region. If member is unloaded at point *E*, the strain will follow path *EFG*.



Relaxation

- The decrease in stress in steel as a result of creep within steel under prolonged strain is called relaxation.



- If bar is stretched to σ_0 stress in time t_0 and thereafter left to bear that stress, then the stress will go on reducing and ultimately becomes constant.

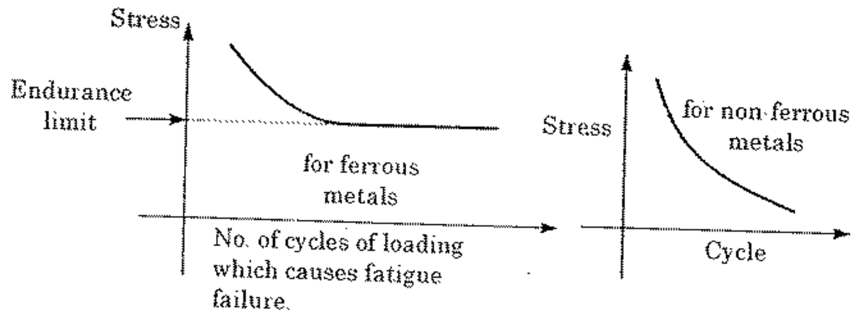
Fatigue

Deterioration of a material *under repeated cycles of stress or strain* resulting in progressive cracking that eventually produces fracture is called *Fatigue*.

Note: *In a typical fatigue failure, a microscopic crack form at a point of high stress (usually at stress concentration) and gradually enlarges as the loads are applied repeatedly. When crack becomes large, sudden failure occurs.*

Magnitude of load causing fatigue failure is less than the load that can be sustained statically. [Failure can occur at static stress less than σ_y].

Fatigue failure depends on magnitude of loading and number of cycles of loading.



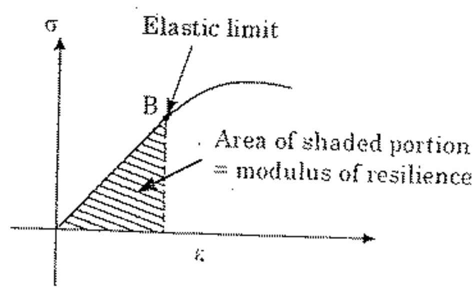
Endurance limit is the stress level below which even large no of stress cycle cannot produce fatigue failure.

For structural steel endurance limit = $\frac{1}{2}$ × ultimate strength

- For non ferrous metal stress at failure continues to decrease. Hence, we define *fatigue limit* as the stress corresponding to failure after a specified number of loading cycles.
- Due to corrosion effect, endurance limit is reduced to upto 50% of that under normal condition.

Resilience

- It is the property of a material to absorb energy when it is deformed elastically and then, upon unloading to have this energy recovered. Hence greater the resilience more desirable is the material for spring action.
- The area under stress strain curve with in elastic limit is called modulus of resilience.



- For a linearly elastic material strain energy stored per unit volume

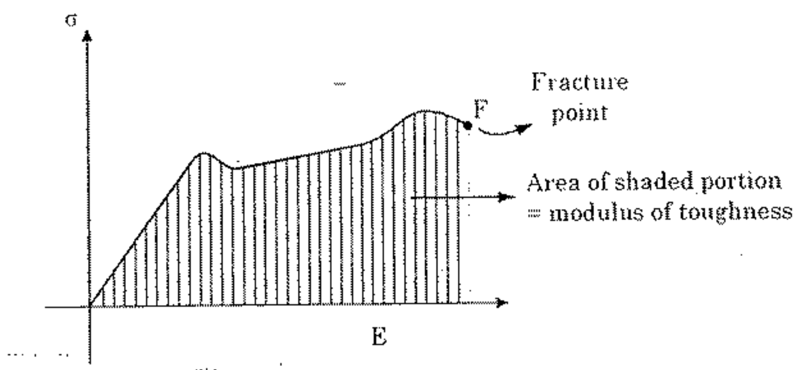
$$= \frac{1}{2}\sigma_y \cdot \frac{\sigma_y}{E}$$

$$= \frac{\sigma_y^2}{2E} = \text{modulus of resilience}$$

- Area under load-deformation curve within elastic limit is called resilience.

Toughness

- Ability to absorb mechanical energy upto failure is called toughness.
- Area under stress strain curve upto fracture is called modulus of toughness.



- Modulus of toughness $= \dfrac{\text{Strain energy stored upto fracture}}{\text{Volume of material}}$
- Toughness is desirable against impact loading.
- As failure strain is more in ductile material. Mild steel is more tough than cast iron.

Note: Toughness → ability to resist fracture.
Hardness → ability to resist scratch or abration.
(The higher the yield stress, the higher is the hardness).

Tenacity

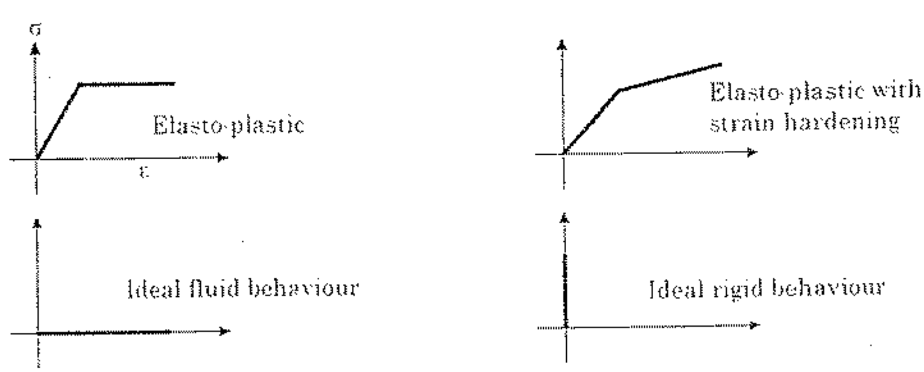
Property of metal to resist fracture when under the action of tensile load is called tenacity.

VISCOELASTIC MATERIAL

Viscoelastic material have both *viscous and elastic property* and exibits time dependent strain. Elasticity is the result of bond stretching along crystallographic planes in an ordered solid. Viscosity is the result of diffusion of atoms or molecules inside an amorphous material.

- Viscoelastic materials have following property:
  - (a) hysteresis is seen in stress strain curve.
  - (b) stress relaxation occurs
  - (c) creep occurs.

APPROXIMATE STRESS-STRAIN CURVES



HOOKE'S LAW

Stress is proportional to strain (with in proportional limit).

$$\sigma = E \cdot \varepsilon$$

σ → Stress, E → Modulus of elasticity, ε → Strain

Hooke's law is valid for *homogeneous isotropic and linearly elastic material.*

DEFORMATION OF MEMBER UNDER AXIAL LOAD

Case I: ***Bar of uniform section***

1. If $\sigma = \dfrac{P}{A}$, does not exceed proportional limit then

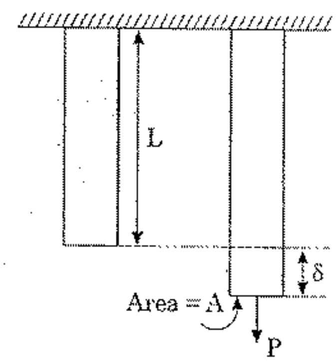
   Stress (σ) = Modulus of elasticity × Strain = $E \cdot \varepsilon$

$$\Rightarrow \quad \varepsilon = \frac{\sigma}{E} = \frac{P}{AE}$$

$$\text{But} \quad \varepsilon = \frac{\delta}{L}$$

$$\Rightarrow \quad \frac{\delta}{L} = \frac{P}{AE}$$

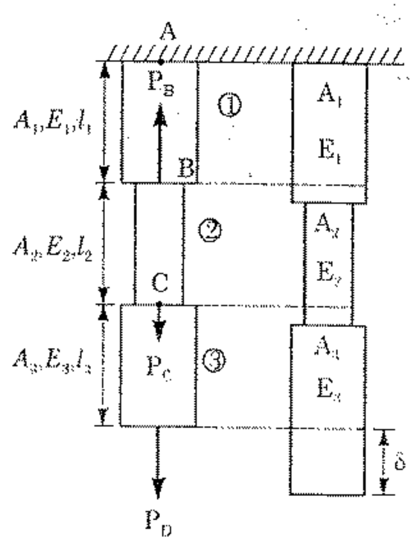
$$\Rightarrow \quad \delta = \frac{PL}{AE}$$

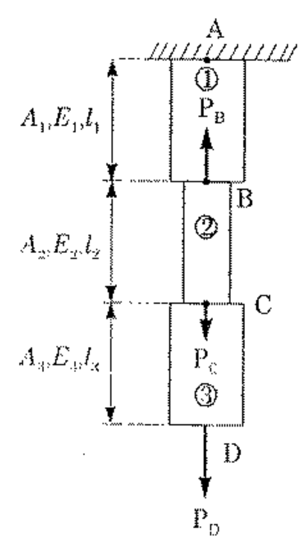


Note: $P = \left(\dfrac{AE}{L}\right) \cdot \delta = K.\delta$, K = Stiffness of the prismatic bar

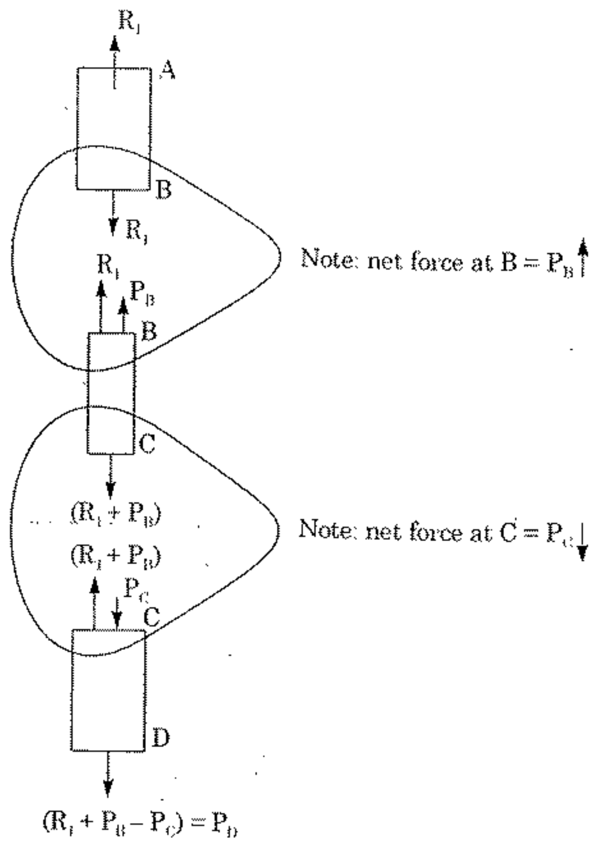
$\delta = \left(\dfrac{L}{AE}\right) P = f \cdot P$, f = Flexibility of the bar

Case II: ***Stepped Bars***





For three bars as shown above forces in the bars will be calculated as follows:



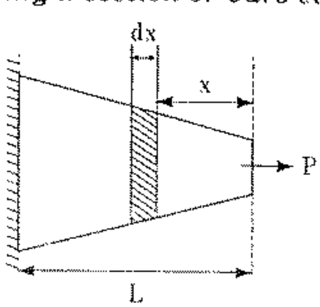
For equilibrium of CD	$R_1 + P_3 - P_C = P_D$	$\Rightarrow$	$R_1 = P_D + P_C - P_B$
Force in AB = N_1	$N_1 = P_D + P_C - P_B$		
Force in BC	$N_2 = R_1 + P_B = P_D + P_C$		
Force in CD	$N_3 = P_D$		

Total elongation of the bars = elongation of AB + elongation of BC + elongation of CD

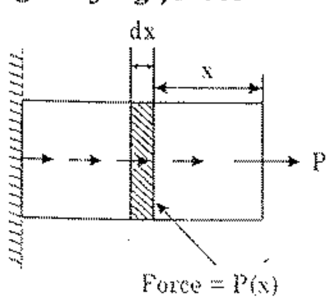
$$= \delta_{AB} + \delta_{BC} + \delta_{CD}$$

$$= \frac{N_1 l_1}{A_1 E_1} + \frac{N_2 l_2}{A_2 E_2} + \frac{N_3 l_3}{A_3 E_3}$$

Case III: *Bars of varying x-section or bars carrying varying forces*



(1) Area varying



(2) Force varying

$A(x)$ = Area at a distance (x) from the right end.

$P(x)$ = Force at a distance x from right end

$$\delta = \int_0^L \frac{P(x)\cdot dx}{A(x)\cdot E}$$

- If force is constant as in Fig. (1),

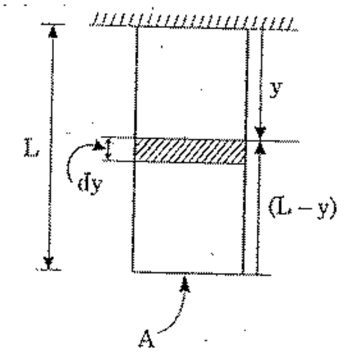
$$\delta = \int \frac{Pdx}{A(x)\,E}$$

- If x-sec area is constant as in Fig. (2)

$$\delta = \int_0^L \frac{P(x)\,dx}{AE}$$

Case IV: ***Deflection due to self wt***

Force on differential element of length 'dy' and x-section area A, is the wt of bar below the differential element Elongation of this differential element $= d\delta = \frac{\gamma A(L-y)\,dy}{AE}$



$$\text{Total elongation, } \delta = \int_0^L \frac{\gamma A(L-y)\,dy}{AE}$$

$$\boxed{\delta = \frac{\gamma L^2}{2E}}$$

where, unit wt = γ; weight of rod = W = γAL

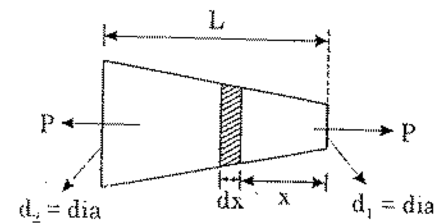
$$\Rightarrow \delta = \frac{WL}{2AE} = \frac{(W/2)\,L}{AE}$$

$\Rightarrow$ Deflection $= \frac{1}{2}$ × that due to load equal to wt of rod

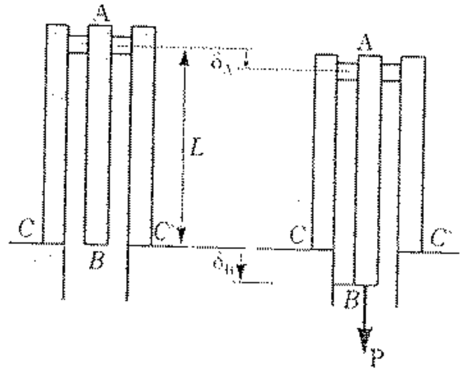
Circular Bar of Tapering x-sec

$$\delta = \int_0^L \frac{Pdx}{\frac{\pi}{4}\left[d_1 + \frac{d_2 - d_1}{L}x\right]^2 E}$$

$$\boxed{\delta = \frac{4PL}{\pi d_1 d_2 E} = \frac{PL}{\left(\frac{\pi d_1 d_2}{4}\right)E}}$$



Effect of Rigid Body Movement



(a) (b)

Area of AB = A_1

Area of bar C = Area of bar C′ = A_2

E_1 = modulus of elasticity of AB

E_2 = modulus of elasticity of C and C′

If a force 'P' is applied to AB, force on C and C′ will be $\frac{P}{2}$ each.

C and C′ will be compressed by $\frac{(P/2)L}{A_2E_2} = \delta_A$

- Thus AB will have rigid body movement of δ_A.

$$\text{Elongation of bar AB} \quad \delta_{AB} = \frac{PL}{A_1E_1}$$

Note that rigid body movement will not have any effect on elongation of a body.

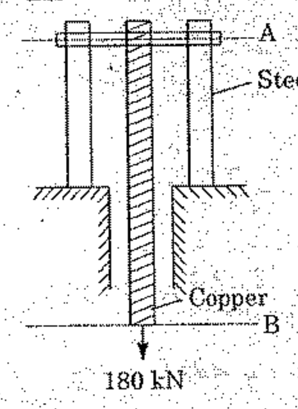
- Point B will move down by $(\delta_A + \delta_{AB})$.

Example 3

A long rectangular copper bar under a tensile load 'P' hangs from a pin that is supported by two steel posts. The copper bar has a length of 2.0 m, x-section-area = 4800 mm², E_{copper} = 120 GPa. For each steel post, height = 0.5 m, x-sec. area = 4500 mm² and E_{steel} = 200 GPa.

(a) Determine the downward displacement 'δ' of the lower end of the copper bar due to load P = 180 kN.

(b) What is the max permissible load (P_{max}) if the displacement 'δ' is limited to 1.0 mm.



Sol:

$$\delta_B - \delta_A = \frac{Pl}{AE} \qquad \left[\text{where } \frac{Pl}{AE} = \text{elongation of copper bar}\right]$$

$$\delta_B - \delta_A = \frac{180\times10^3\,\text{N}\times2\times10^3\text{ mm}}{4800\times1.2\times10^5\text{ N/mm}^2}$$

$$= 0.625 \text{ mm}$$

$$\delta_A = \frac{\left(\frac{P}{2}\right)l}{AE} = \frac{\frac{180}{2}\times10^3\times0.5\times10^3}{4500\times2\times10^5}$$

$$= 0.05 \text{ mm}$$

$\Rightarrow \quad \delta_B = 0.625 + 0.05 = 0.675$ mm = downward displacement of lower end of copper bar

We know that $\quad \delta_B - \delta_A \propto P$

and $\quad \delta_A \propto P$

$\Rightarrow \quad \delta_B \propto P$

$$\Rightarrow \quad \frac{\delta_{B\max}}{P_{\max}} = \frac{\delta_B}{P}$$

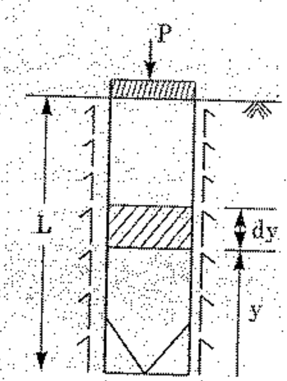
$$\Rightarrow \quad P_{\max} = \frac{\delta_{B\max} \times P}{\delta_B} = \frac{180 \times 1.0}{0.675} = 266.67 \text{ kN}$$

Example 4

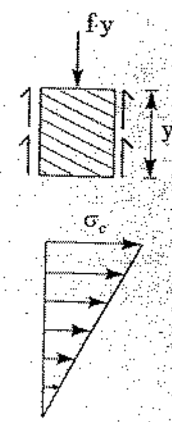
A wooden pile, driven into the earth, supports a load 'P' entirely by friction along its sides. The friction force 'f' per unit length of the pile is assumed to be uniformly distributed over the surface of pile. The pile has length 'L', x-sec. area 'A' and modulus of elasticity 'E'.

(a) Derive formula for shortening of the pile in term of P, L, A, E.

(b) Show the variation of compressive stress σ_c throughout the length of pile.



Sol: Since frictional force per unit length is uniformly distributed.



Hence, $\boxed{P = fL}$

$$\text{Shortening of pile } \delta = \int d\delta = \int_0^L \frac{f \cdot y \cdot dy}{AE} = \frac{fL^2}{2AE}$$

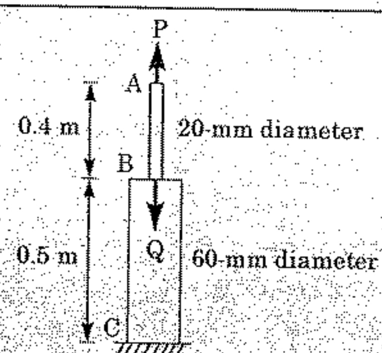
$$\Rightarrow \quad \boxed{\delta = \frac{PL}{2AE}}$$

Compressive stress in pile

$$\sigma_c = \frac{f \cdot y}{A} = \frac{Py}{LA}$$

Example 5

The rod ABC is made up of aluminum for which E = 70 GPa. Knowing that P = 6 kN and Q = 42 kN, determine the deflection of (a) point A, (b) point B.



Sol: (a) $A_{AB} = \frac{\pi}{4}d_{AB}^2 = \frac{\pi}{4}(0.020)^2 = 314.16\times10^{-6}\text{m}^2$

$A_{BC} = \frac{\pi}{4}d_{BC}^2 = \frac{\pi}{4}(0.060)^2 = 2.8274\times10^{-3}\text{m}^2$

$P_{AB} = P = 6 \times 10^3$ N

$P_{BC} = P - Q = 6 \times 10^3 - 42 \times 10^3 = -36 \times 10^3$ N

$L_{AB} = 0.4$ m $\quad L_{BC} = 0.5$ m

$$\delta_{AB} = \frac{P_{AB}L_{AB}}{A_{AB}E_B} = \frac{(6\times10^3)(0.4)}{(314.16\times10^{-6})(70\times10^9)}$$

$= 109.135 \times 10^{-6}$ m

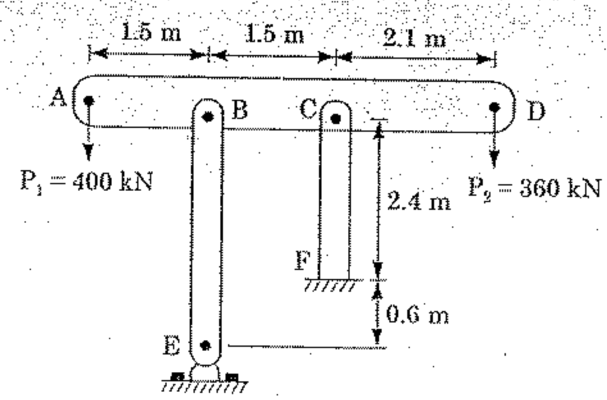
$$\delta_{BC} = \frac{P_{BC}L_{BC}}{A_{BC}E} = \frac{(-36\times10^3)(0.5)}{(2.8274\times10^{-3})(70\times10^9)} = -90.947 \times 10^{-6} \text{ m}$$

$\delta_A = \delta_{AB} + \delta_{BC} = 109.135 \times 10^{-6} - 90.947 \times 10^{-6}$ m $= 18.19 \times 10^{-6}$ m $= 0.01819$ mm

(b) $\delta_B = \delta_{BC} = -90.9 \times 10^{-6}$ m $= $ **-0.0909 mm**

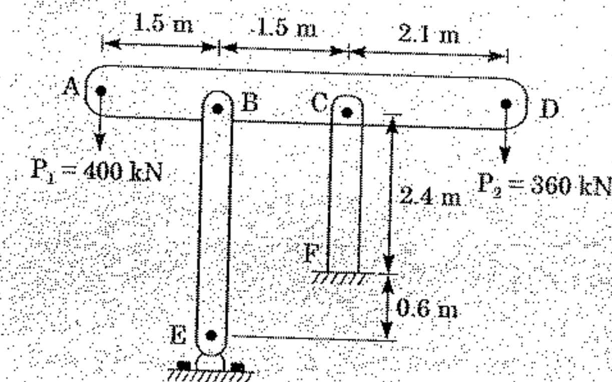
Example 6

The horizontal rigid beam ABCD is supported by vertical bars BE and CF and is loaded by vertical forces $P_1 = 400$ kN and $P_2 = 360$ kN acting at points A and D, respectively (see figure). Bars BE and CF are made of steel (E = 200 GPa) and have cross-sectional areas $A_{BE} = 11{,}000$ mm², $A_{CF} = 9280$ mm² and bars are shown in the figure.



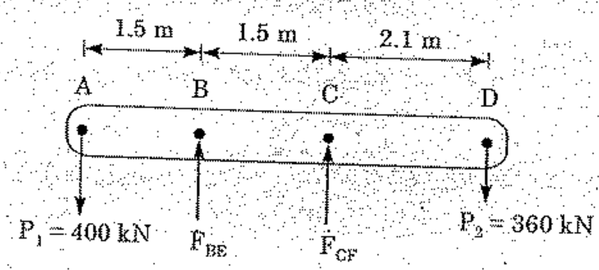
Determine the vertical displacements δ_A and δ_D of points A and D, respectively.

Sol: Rigid beam supported by vertical bars



$$A_{BE} = 11,000 \text{ mm}^2$$
$$A_{CF} = 9,280 \text{ mm}^2$$
$$E = 200 \text{ GPa}$$
$$L_{BE} = 3.0 \text{ m}$$
$$L_{CF} = 2.4 \text{ m}$$
$$P_1 = 400 \text{ kN};\ P_2 = 360 \text{ kN}$$

Free body diagram of bar ABCD:



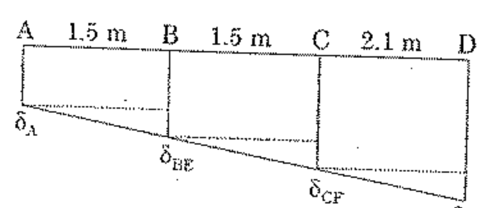
$$\Sigma M_B = 0$$
$$(400 \text{ kN})(1.5 \text{ m}) + F_{CF}(1.5 \text{ m}) - (360 \text{ kN})(3.6 \text{ m}) = 0$$
$$F_{CF} = 464 \text{ kN}$$
$$\Sigma M_C = 0$$
$$(400 \text{ kN})(3.0 \text{ m}) - F_{BE}(1.5 \text{ m}) - (360 \text{ kN})(2.1 \text{ m}) = 0$$
$$F_{BE} = 296 \text{ kN}$$

Shortening of Bar BE

Force F_{BE} and F_{CF} are applied by bars BE and CF on ABCD. Hence equal and opposite force is applied by ABCD on BE and CF but the force will be compressive.

$$\delta_{BE} = \frac{F_{BE}L_{BE}}{EA_{BE}} = \frac{(296 \text{ kN})(3.0 \text{ m})}{(200 \text{ GPa})(11000 \text{ mm}^2)}$$
$$= 0.400 \text{ mm}$$

Shortening of Bar CF



$$\delta_{CF} = \frac{F_{CF}L_{CF}}{EA_{CF}} = \frac{(464\text{kN})(2.4\text{m})}{(200\text{ GPa})(9.280\text{ mm}^2)}$$

$$= 0.600 \text{ mm}$$

$$\delta_{BE} - \delta_A = \delta_{CF} - \delta_{BE} \text{ or } \delta_A = 2\delta_{BE} - \delta_{CF}$$

$$\delta_A = 2(0.400 \text{ mm}) - 0.600 \text{ m}$$

$$= 0.200 \text{ mm}$$

(Downward)

$$\delta_D - \delta_{CF} = \frac{2.1}{1.5}(\delta_{CF} - \delta_{BE})$$

$$\text{or} \quad \delta_D = \frac{12}{15}\delta_{CF} - \frac{7}{5}\delta_{BE}$$

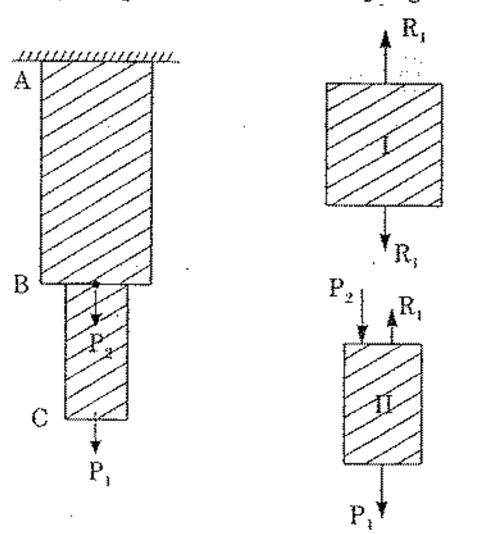
$$= \frac{12}{5}(0.600 \text{ mm}) - \frac{7}{5}(0.400 \text{ mm})$$

$$= 0.880 \text{ mm} \quad \text{(Downward)}$$

STATICALLY INDETERMINATE PROBLEM

If forces in component members can be determined simply by the use of *equation of statics* (i.e. $\Sigma F = 0$, $\Sigma M = 0$), the structure as a whole is called statically determinate.

For the figure shown below free body diagram is as shown by figure on the right side.



Free body diagram

$$\text{From} \quad \Sigma F = 0 \text{ on element II}$$

$$P_2 + P_1 - R_1 = 0$$

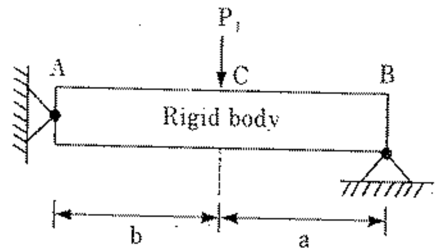
$$\Rightarrow \quad R_1 = P_1 + P_2$$

$$\Rightarrow \quad \text{Force on AB} = P_1 + P_2$$

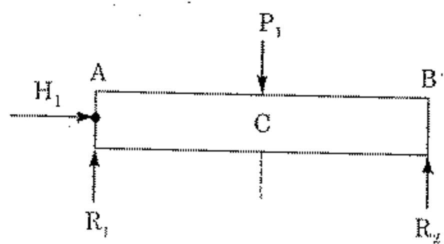
$$\text{Force on BC} = P_1$$

Thus forces on component members has been determined using only equation of statics. Hence, the member ABC in this case is statically determinate.

Similarly, for the condition shown below.



Free body diagram is



$$\text{From} \quad \Sigma F_V = 0$$

$$\Rightarrow \quad R_1 + R_2 = P_1$$

$$\Sigma F_H = 0 \quad \Rightarrow \quad H_1 = 0$$

Summation of moment about A, $\Sigma M_A = 0$

$$\Rightarrow \quad R_2 (a + b) - P_1 b = 0$$

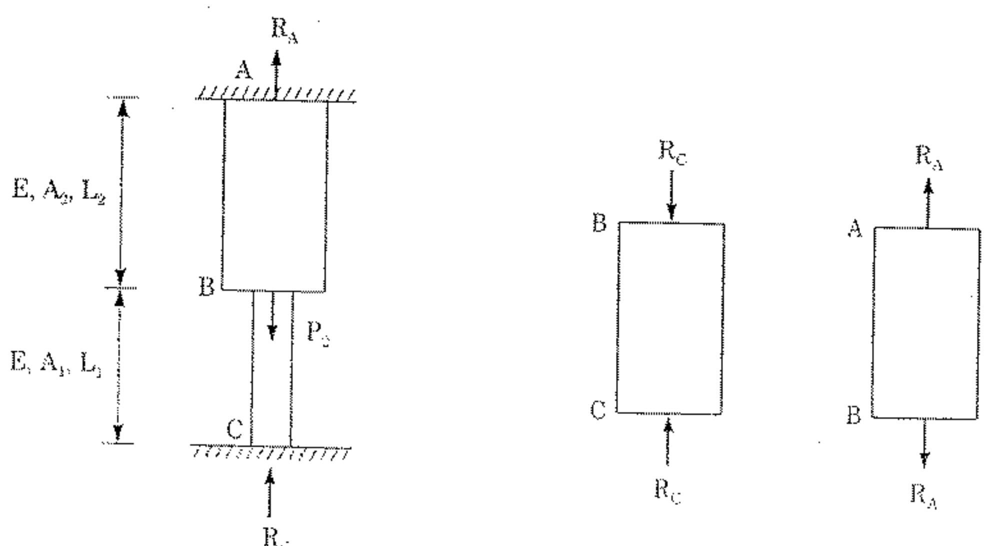
$$\Rightarrow \quad \boxed{R_2 = \frac{P_1 b}{a+b}}$$

$$\boxed{R_1 = P - R_2 = \frac{P_1 a}{a+b}}$$

Thus, as the reactions are known, member forces can be calculated. Thus it is also a case of statically determinate structure.

If, however, forces cannot be determined simply by equation of statics, the structure is called redundant or statically indeterminate structure.

Analysis of Statically Indeterminate Case



Forces in component members can be known only when R_A and R_C are known.

From equation of statics ($\Sigma F = 0$)

$$R_A + R_C - P_2 = 0 \quad \text{(i)}$$

No. of unknown are two (R_A & R_C) but, no. of eq. of statics is only one (i.e. $\Sigma F = 0$). Hence, we require one more equation to find R_A and R_C.

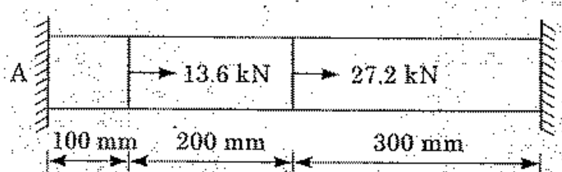
- The other equation is obtained from the equation of compatibility or eq. of consistant deformation.
- If supports are unyielding then total change in length of member, will be zero. Condition of compatibility

$$\Rightarrow \quad \frac{R_A L_2}{A_2 E} - \frac{R_C L_1}{A_1 E} = 0 \quad \text{(ii)}$$

From (i) and (ii) R_A and R_C can be calculate [note that elongation has been taken as (+) ve and contraction has been taken as (–) ve]

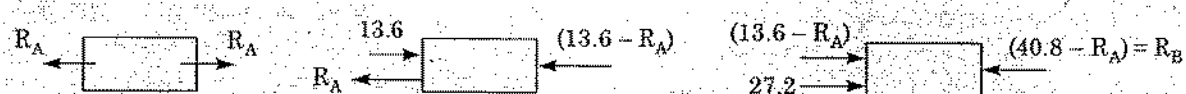
Example 7

A prismatic bar is fastened between two rigid walls at A and B and subjected to loads as shown in figure below. Determine the reactions at the supports.



Sol:

The free body diagram is as shown below.



$$\boxed{R_B = 40.8 - R_A}$$

Total change in length = 0

$$\Rightarrow \quad \frac{R_A l_1}{AE} - \frac{(13.6 - R_A)\, l_2}{AE} - \frac{R_B l_3}{AE} = 0$$

$$R_A \times 100 - (13.6 - R_A) \times 200 - (40.8 - R_A) \times 300 = 0$$

$$100\, R_A - 2720 + 200\, R_A - 12240 + 300\, R_A = 0$$

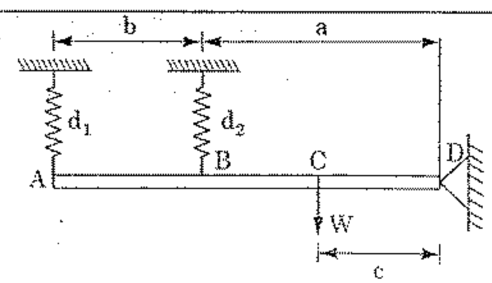
$$600\, R_A = 14960$$

$$R_A = 24.93 \text{ kN}$$

$$R_B = 15.87 \text{ kN}$$

Example 8

A rigid beam ABCD is hinged at D and supported by two springs at A and B as shown. The beam carries a vertical load W at the point C at a distance of c from the hinged end. The flexibilities (deflection/unit load) of springs at A and B are d_1 and d_2 respectively. Determine the forces in the springs and also the reaction at the hinged support.

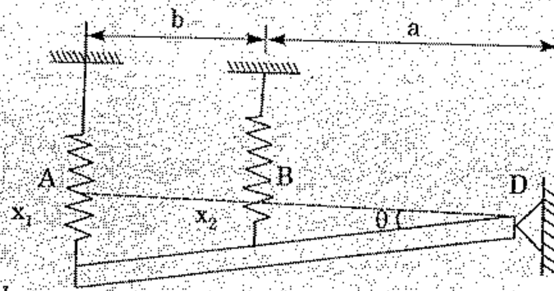


Sol. The above problem is indeterminate to one degree. So a compatibility equation apart from simple statics equation is also to be created. Since rod AD is rigid so it will under go rigid body rotation as shown

Since, slope θ is constant

$$\frac{x_2}{a} = \frac{x_1}{(a+b)} \quad \text{.........(i)}$$

where,



x_1 and x_2 are deflection at A and B.

Let the forces in springs be R_A & R_B

So deflection in springs will be

$x_1 = R_A d_1$

$x_2 = R_B d_2$

Putting above values in equation (i) we get,

$$\frac{R_B d_2}{a} = \frac{R_A d_1}{(a+b)}$$

$$\Rightarrow \quad R_B d_2(a+b) = R_A d_1(a) \quad \text{.............(ii)}$$

Second equation can be formed by taking moments about hinge D.

$$(a+b)R_A + aR_B = WC \quad \text{............(iii)}$$

Solving equation (ii) and (iii) we get

$$R_A = \frac{Wc\, d_2(a+b)}{a^2 d_1 + d_2(a+b)^2}$$

$$R_B = \frac{Wc\, ad_1}{a^2 d_1 + d_2(a+b)^2}$$

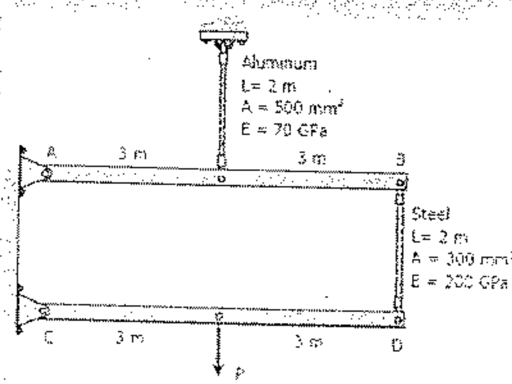
Now by simple statics, $\Sigma V = 0$

$\Rightarrow$ R_D = Reaction at hinged support

$$= W - \frac{Wc\, d_2(a+b)}{a^2 d_1 + d_2(a+b)^2} - \frac{Wc\, ad_1}{a^2 d_1 + d_2(a+b)^2}$$

Example 9

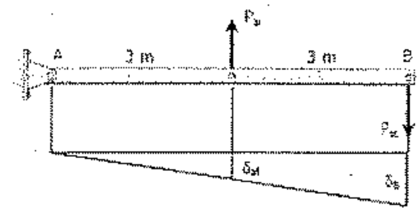
The rigid bars AB and CD shown in Figure are supported by pins at A and C and the two rods. Determine the maximum force P that can be applied as shown if its vertical movement is limited to 5 mm. Neglect the weights of all members.



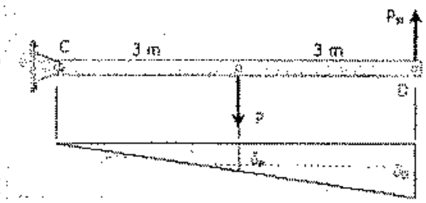
Sol: $[\Sigma M_A = 0]$ $3P_{al} = 6P_{st}$

$$P_{al} = 2P_{st}$$

By ratio and proportion:



FBD and movement diagram of bar AB



FBD and movement diagram of bar CD

$$\frac{\delta_B}{6} = \frac{\delta_{al}}{3}$$

$$\delta_B = 2\delta_{al} = 2\left[\frac{PL}{AE}\right]_{al}$$

$$\delta_B = 2\left[\frac{P_{al}(2000)}{500\,(70000)}\right]$$

$$\delta_B = \frac{1}{8750}P_{al} = \frac{1}{8750}(2P_{st})$$

$$\delta_B = \frac{1}{4375}P_{st} \rightarrow \text{movement of B}$$

Movement of D:

$$\delta_D = \delta_{st} + \delta_B = \left[\frac{PL}{AE}\right]_{st} + \frac{1}{4375}P_{st}$$

$$\delta_D = \frac{P_{st}(2000)}{300\,(200000)} + \frac{1}{4375}P_{st}$$

$$\delta_D = \frac{11}{42000}P_{st}$$

$[\Sigma M_C = 0]$ $6P_{st} = 3P$

$$P_{st} = \frac{1}{2}P$$

By ratio and proportion:

$$\frac{\delta_P}{3} = \frac{\delta_D}{6}$$

$$\delta_P = \frac{1}{2}\delta_D = \frac{1}{2}\left(\frac{11}{42000}P_{st}\right)$$

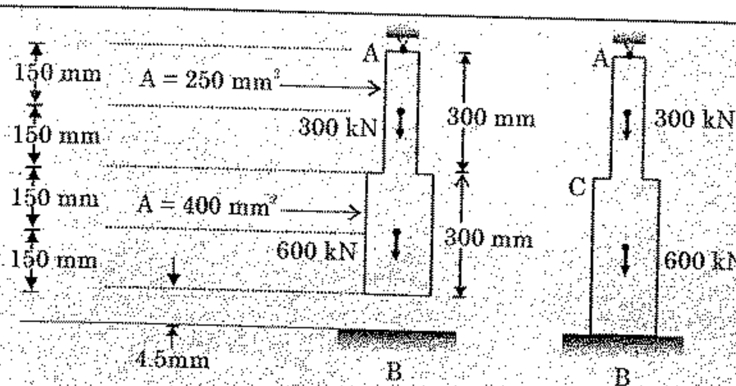
$$\delta_P = \frac{11}{84000}P_{st}$$

$$5 = \frac{11}{84000}P_{st}$$

$$P = 76363.64 \text{ N} = 76.4 \text{ kN}$$

Example 10

Determine the reactions at A and B for the steel bar and loading as shown below, assuming that a 4.50 mm clearance exists between the bar and the ground before the loads are applied. Assume E = 200 GPa.



Sol: Elongation due to loading 300 and 600 kN is

$$\delta_L = \sum \frac{P_i L_i}{A_i E} = \left(0 + \frac{600\times10^3\ \text{N}}{400\times10^{-6}\text{m}^2} + \frac{600\times10^3\text{N}}{250\times10^{-6}\text{m}^2} + \frac{900\times10^3\text{N}}{250\times10^{-6}\text{m}^2}\right)\frac{0.15\text{m}}{E}$$

$$\delta_L = \frac{1.125\times10^9}{E}$$

Contraction due to reaction at B

$$\delta_R = (-)\left(\frac{R_B L_1}{A_1 E} + \frac{R_B L_2}{A_2 E}\right) = (-)\frac{(1.95\times10^3)R_B}{E}$$

$$\delta = \delta_L + \delta_R = 4.5\times10^{-3}\text{m}$$

$$\delta = \frac{1.125\times10^9}{200\times10^9} - \frac{(1.95\times10^3)R_B}{200\times10^9} = 4.5\times10^{-3}\text{m}$$

Solving for R_B, we obtain

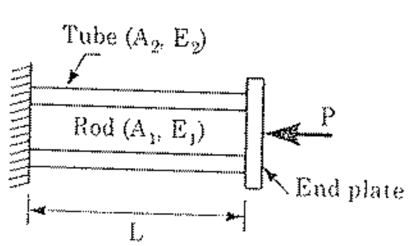
$R_B = 115.4 \times 10^3$ N = 115.4 kN

The reaction at A is obtained from the free-body diagram of the bar:

$\Sigma F_y = 0$ $\qquad R_A - 300\text{ kN} - 600\text{ kN} + R_B = 0$

$R_A = 900\text{ kN} - R_B = 900\text{ kN} - 115.4\text{ kN} = 784.6\text{ kN}$

COMPOSITE BARS



We have to find forces in two members 1 and 2. Let force in the two members be P_1 and P_2

$$\Rightarrow \qquad P = P_1 + P_2 \quad \text{.............(i)}$$

The other eq. needed to find out P_1 and P_2 is obtained from compatibility condition. **The compatibility condition is that change in length of the two members will be same.**

$$\Rightarrow \quad \delta_1 = \frac{P_1 L}{A_1 E_1} = \frac{P_2 L}{A_2 E_2} = \delta_2 \quad \text{------(ii)}$$

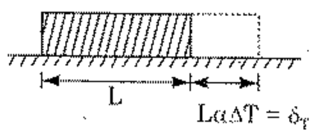
By solving (i) and (ii) P_1 and P_2 can be foundout

$$P_1 = \frac{A_1 E_1 P}{A_1 E_1 + A_2 E_2}$$

$$P_2 = \frac{A_2 E_2 P}{A_1 E_1 + A_2 E_2}$$

PROBLEM INVOLVING TEMPERATURE CHANGE

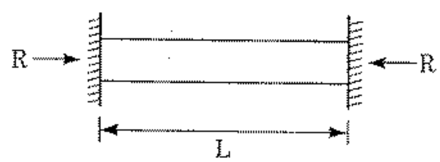
If a body is free and temperature is raised then elongation will be $L\,\alpha\,\Delta T$, where 'α' is the coefficient of thermal expansion and ΔT = changing temperature of bar.



Thus, strain will be, $\text{Strain} = \frac{\delta_T}{L}$

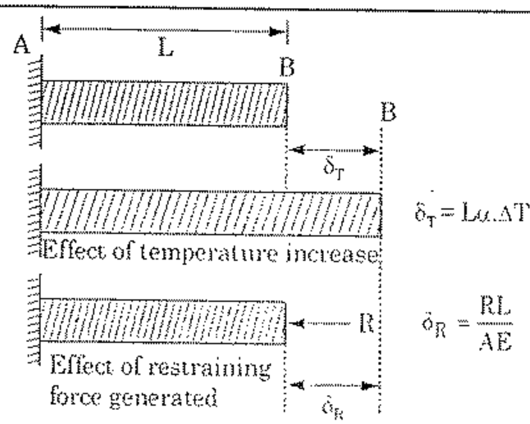
and the stress generated in bar = Stress = 0.

If, however, the body is restrained and temperature is changed, then stress occurs but strain = 0.



To find out the stress generated, the principal of superposition is used. Effect due to temperature and due to support restraint is calculated separately and combined to meet the condition of compatibility.

Note: *Principal of superposition is applicable only when stress is with in proportional limit.*



From capatibility $\delta_T = \delta_R = 0$

$$\Rightarrow \quad L\alpha\Delta T = \frac{RL}{AE}$$

$$\Rightarrow \quad \boxed{\frac{R}{A} = \text{Stress} = E\alpha\Delta T}$$

Note: *If temperature is increased and member is restrained, then, force produced is compressive. However, if temperature is decreased, the force produced is tensile.*

TEMPERATURE CHANGE WITH YIELDING SUPPORT

If the temperature changes in a restrained bar and support yields by amount δ.

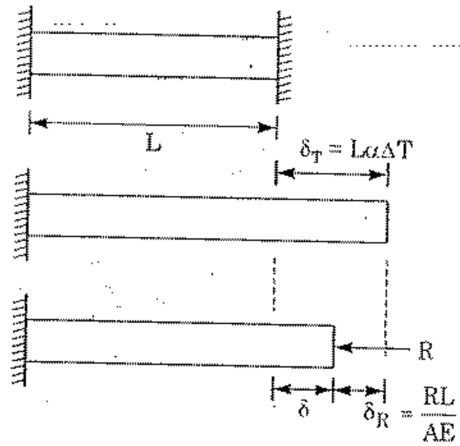
then, $\delta_T - \delta_R = \delta$ = compatibility condition

$$L\alpha\Delta T - \frac{RL}{AE} = \delta$$

$$R = \left[L\alpha\Delta T - \delta\right]\frac{EA}{L}$$

$$\boxed{\text{Stress} = \frac{R}{A} = \frac{E}{L}[L\alpha\Delta T - \delta]}$$

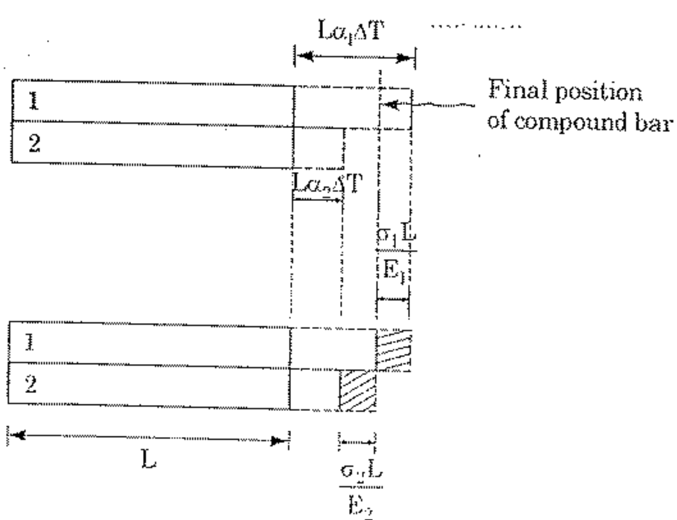
$$\boxed{\text{Stress} = \frac{E(L\alpha\Delta T - \delta)}{L}}$$



Note: Stress $= \left[\frac{(\text{Expansion prevented})}{L}\right]E$

COMPOSITE BAR (TEMPERATURE CHANGE CASE)

(i) Temperature Rise Case



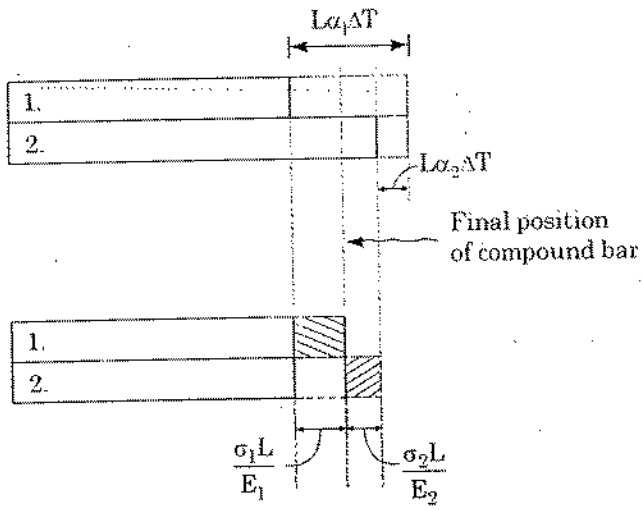
As the ends of bar are free, net force on the composite bar even after temperature rise must be zero. Thus, tensile force in bar 2 must be equal to compressive force in bar 1.

$$\boxed{\sigma_1 A_1 = \sigma_2 A_2}$$

$$\text{also,} \quad \left(\frac{\sigma_1}{E_1} + \frac{\sigma_2}{E_2}\right) L = L(\alpha_1 - \alpha_2)\Delta T$$

$$\Rightarrow \quad \boxed{\frac{\sigma_1}{E} + \frac{\sigma_2}{E} = (\alpha_1 - \alpha_2)\,\Delta T}$$

(ii) Temperature Fall Case



As the ends of bar are free, net force on the composite bar even after temperature rise must be zero. Thus, compressive force in bar 2 must be equal to tensile force in bar 1.

$$\boxed{\sigma_1 A_1 = \sigma_2 A_2}$$

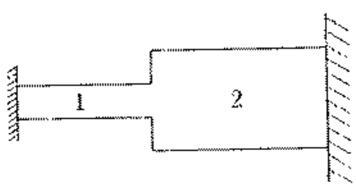
$$\text{also,} \quad \left(\frac{\sigma_1}{E_1} + \frac{\sigma_2}{E_2}\right) L = L(\alpha_1 - \alpha_2)\Delta T$$

$$\Rightarrow \quad \boxed{\frac{\sigma_1}{E} + \frac{\sigma_2}{E} = (\alpha_1 - \alpha_2)\,\Delta T}$$

Note: (a) Temperature ↑ → The material for which α ↑ → compression
(b) Temperature fall ↓ → The material for which α ↑ → tension

SERIES CONNECTION WITH END SUPPORTED

In this case expansion/contraction will be prevented by the supports hence compressive/tensile force will be exerted in both components 1 and 2.



If temperature is increased free extension $= L_1\alpha_1\Delta T + L_2\alpha_2\Delta T$

But this expansion has to be restrained. Hence compression develops in bars.

$$\frac{\sigma_1 L_1}{E_1} + \frac{\sigma_2 L_2}{E_2} = \text{compression}$$

\# $$\boxed{\frac{\sigma_1 L_1}{E_1} + \frac{\sigma_2 L_2}{E_2} = L_1\alpha_1\Delta T + L_2\alpha_2\Delta T} \quad \text{.......... (i)}$$

σ_1 and σ_2 are compressive stress.

\# **If the support yields by amount 'δ'**

$\Rightarrow$ $$\boxed{\frac{\sigma_1 L_1}{E_1} + \frac{\sigma_2 L_2}{E_2} = \left[L_1\alpha_1\Delta T + L_2\alpha_2\Delta T - \delta\right]} \quad \text{.......... (iA)}$$

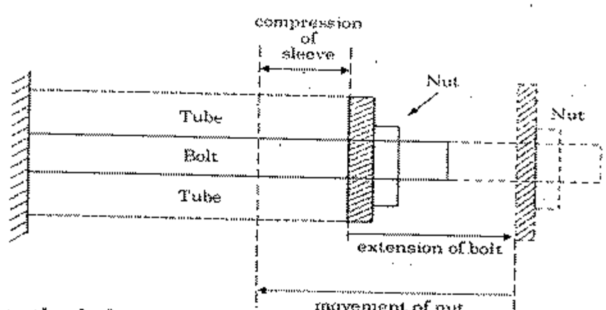
Note that expansion prevented is $= L_1\alpha_1\Delta T + L_2\alpha_2\Delta T - \delta$

$$\boxed{\sigma_1 A_1 = \sigma_2 A_2} \quad \text{.......... (ii)}$$

From (i) and (ii) find σ_1 and σ_2

Note that formula (i) and (ii) will still be valid if temperature falls down.

NUT AND BOLT PROBLEM



because of tightening of nut, the bolt comes under tension and the tube comes under compression.

Extension of bolt + contraction of tube = movement of nut

$$\boxed{\frac{\sigma_b L}{E_b} + \frac{\sigma_T L}{E_c} = np}$$

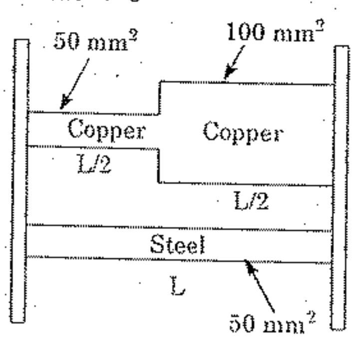
where, n = no. of rotation of bolt; p = pitch of thread.

Example 11

Following figure shows copper and steel bars fixed at their ends using plates. If temperature of assembly is raised by 30°C find the stresses in the rod.

$E_s = 2 \times 10^5$ N/mm², $\alpha_s = 1.2 \times 10^{-5}$/°C

$E_c = 10^5$ N/mm², $\alpha_c = 1.5 \times 10^{-5}$/°C



Sol: Free expansion of copper $= \frac{1}{2}[\alpha_c \Delta T + \alpha_c \Delta T] = L\alpha_c \Delta T$

Free expansion of steel $= L\alpha_s \Delta T$

Since, $L\alpha_s \Delta T < L\alpha_c \Delta T$ hence, C_u will come under compression and steel will come under tension.

Force in steel and copper will be same

$$\Rightarrow \quad \boxed{P_c = P_s}$$

Compression of copper + extension of steel $= \delta = L(\alpha_c - \alpha_s)\Delta T$

$$\frac{\sigma_s L}{E_s} + \frac{\sigma_{c_1} L}{2E_c} + \frac{\sigma_{c_2} L}{2E_c} = L(\alpha_c - \alpha_s)\Delta T$$

$$\frac{P_s L}{A_s E_s} + \frac{P_c L}{2 \times A_{c_1} E_c} + \frac{P_c L}{2 \times A_{c_2} E_c} = L(\alpha_c - \alpha_s)\Delta T$$

$$\text{Let } P_s = P_c = P \quad \text{----------- (i)}$$

$$A = A_s = A_{c_1} = \frac{A_{c_2}}{2} \quad \text{----------- (ii)}$$

$$\frac{PL}{AE_s} + \frac{PL}{2AE_c} + \frac{PL}{2 \times 2AE_c} = L(\alpha_c - \alpha_s)\Delta T$$

$$\frac{PL}{A}\left[\frac{1}{E_s} + \frac{1}{2E_c} + \frac{1}{4E_c}\right] = L(\alpha_c - \alpha_s)\Delta T$$

$$\boxed{P = \frac{A(\alpha_c - \alpha_s)\Delta T}{\left(\frac{1}{E_s} + \frac{3}{4E_c}\right)}}$$

Once 'P' is know, stresses can be calculated

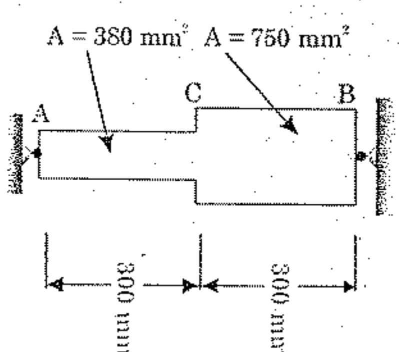
$\sigma_s = 7.2 \text{ N/mm}^2$ tensile

$\sigma_{c_1} = 7.2 \text{ N/mm}^2$ compressive

$\sigma_{c_2} = 3.6 \text{ N/mm}^2$ compressive

Example 12

Determine the values of the stress in portions AC and CB of the steel bar as shown in Fig. when the temperature of the bar is –45°C. A close fit exists at both of the rigid supports when the temperature is +24°C. Use the values E = 200 GPa and $\alpha = 11.7 \times 10^{-6}/°C$ for steel. Also calculate the deformation in portion AC & CB.



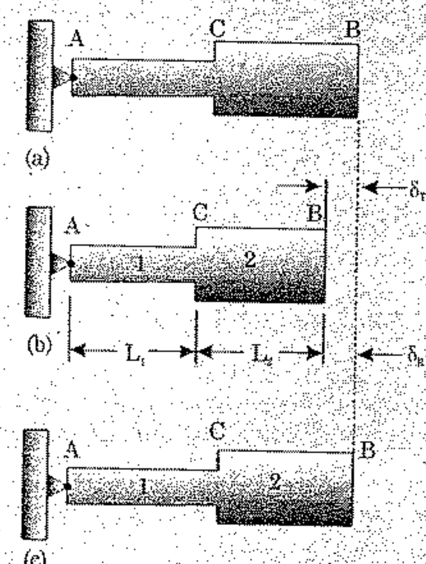
Sol. We first determine the reactions at the supports. Since the problem is statically indeterminate, we detach the bar from its support at B and let it undergo the temperature change

$$\Delta T = (-45^\circ C) - (24^\circ C) = -69^\circ C$$

The corresponding deformation is

$$\delta_T = \alpha(\Delta T)L = (11.7\times10^{-6}/^\circ C)(-69^\circ C)(600\text{ mm})$$
$$= -0.484\text{ mm}$$

Applying now the unknown force R_B at end B (Fig. c), the bar will be extended to its original position.



$$\Rightarrow \quad \delta_R = \frac{P_1L_1}{A_1E} + \frac{P_2L_2}{A_2E} = \frac{R_B}{E}\left(\frac{L_1}{A_1}+\frac{L_2}{A_2}\right)$$

$$= \frac{R_B}{200\text{ GPa}}\left(\frac{300\text{ mm}}{380\text{ mm}^2}+\frac{300\text{ mm}}{750\text{ mm}^2}\right)$$

Expressing that the total deformation of the bar must be zero as a result of the imposed constraints, we write

$\delta = \delta_T + \delta_R = 0$

$= -0.484\text{ mm} + (5.95\times10^{-6}\text{mm/N})R_B = 0$

from which we obtain

$$R_B = 81.34\times10^3\text{ N} = 81.34\text{ kN}$$

The reaction at A is equal and opposite.

knowing that the forces in the two portions of the bar are $P_1 = P_2 = 81.34$ kN, we obtain the following values of the stress in portions AC and CB of the bar:

$$\sigma_1 = \frac{P_1}{A_1} = \frac{81.34\text{ kN}}{380\text{ mm}^2} = 214.1\text{ MPa}$$

$$\sigma_2 = \frac{P_2}{A_2} = \frac{81.34\text{ kN}}{750\text{ mm}^2} = 108.5\text{ MPa}$$

Calculation of deformation in AC and CB

While the total deformation of the bar must be zero, the deformations of the portions AC and CB are not zero. Let us determine the strain ϵ_{AC} in portion AC of the bar. The strain ϵ_{AC} can be divided into two component parts; one is the thermal strain ϵ_T produced in the unrestrained bar by the temperature change ΔT (Fig. b).

$$\epsilon_T = \alpha\Delta T = (11.7\times10^{-6}/^\circ C)(-69^\circ C)$$
$$= -807.3\times10^{-6}\text{ mm/mm}$$

The other component of ϵ_{AC} is associated with the stress σ_1 due to the force R_B applied to the bar (Fig. b). From Hooke's law, we express this component of the strain as

$$\frac{\sigma_1}{E} = \frac{+214.1\text{ MPa}}{200\text{ GPa}} = +1070.5\times10^{-6}\text{ mm/mm}$$

Adding the two components of the strain in AC, we obtain

$$\epsilon_{AC} = \epsilon_T + \frac{\sigma_1}{E} = -807.3\times10^{-6} + 1070.5\times10^{-6}$$

$$= 263.2 \times 10^{-6} \text{ mm/mm}$$

A similar computation yields the strain in portion CB of the bar:

$$\epsilon_{CB} = \epsilon_T + \frac{\sigma_2}{E} = -807.3\times10^{-6} + 542.5\times10^{-6}$$

$$= -264.8 \times 10^{-6} \text{ mm/mm}$$

The deformations δ_{AC} and δ_{CB} of the two portions of the bar are expressed respectively as

$$\delta_{AC} = \epsilon_{AC}(AC) = \left(263.2\times10^{-6}\right)(300 \text{ mm})$$

$$= \mathbf{0.079 \text{ mm}}$$

$$\delta_{CB} = \epsilon_{CB}(CB) = \left(-264.8\times10^{-6}\right)(300 \text{ mm})$$

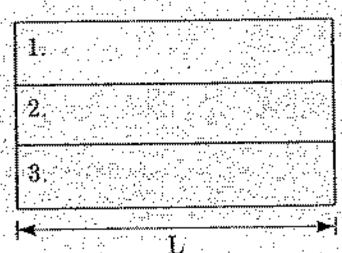
$$= \mathbf{-0.079 \text{ mm}}$$

We thus check that, while the sum $\delta = \delta_{AC} + \delta_{CB}$ of the two deformations is zero, neither of the deformations is zero.

Example 13

Three bars of same length 'L', section areas A_1, A_2, A_3 and Elastic modulus E_1, E_2, E_3 and coefficient of linear expansion α_1, α_2, α_3 are connected to each other at their ends. If the assembly is subjected to rise in temperature by t°C, calculate the actual change in length of the assembly.

If now the assembly is subjected to a pull 'P' what will be the actual change in length of assembly.



Sol: Let the final change in length of assembly be 'δ'.

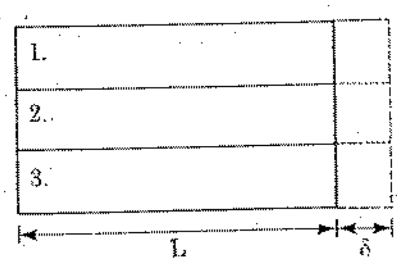
Hence any bar having $L\alpha\Delta T$ greater than 'δ', will have compression and those having $L\alpha\Delta T$ smaller than δ will have tension.

Taking tension (+) ve and compression (–) ve

$(\delta - L\alpha_1\Delta T) \rightarrow$ change in length of bar 1.

$(\delta - L\alpha_2\Delta T) \rightarrow$ change in length of bar 2.

$(\delta - L\alpha_3\Delta T) \rightarrow$ change in length of bar 3.



Resultant force on the complete assembly will be zero

$$\Rightarrow \quad P_1 + P_2 + P_3 = 0$$

$$\Rightarrow \quad \frac{(\delta - L\alpha_1\Delta T)}{L}E_1A_1 + \frac{(\delta - L\alpha_2\Delta T)}{L}E_2A_2 + \frac{(\delta - L\alpha_3\Delta T)}{L}E_3A_3 = 0$$

$$\delta\left(\frac{E_1A_1}{L} + \frac{E_2A_2}{L} + \frac{E_3A_3}{L}\right) = \left(E_1A_1\alpha_1 + E_2A_2\alpha_2 + E_3A_3\alpha_3\right)\Delta T$$

$$\boxed{\delta = \frac{(E_1A_1\alpha_1 + E_2A_2\alpha_2 + E_3A_3\alpha_3)\Delta T \times L}{(E_1A_1 + E_2A_2 + E_3A_3)}}$$

If a pull force is applied, resultant force $P = P_1 + P_2 + P_3$

$$\Rightarrow \quad \boxed{\frac{(\delta - L\alpha_1\Delta T)}{L}E_1A_1 + \frac{(\delta - L\alpha_2\Delta T)}{L}E_2A_2 + \frac{(\delta - L\alpha_3\Delta T)}{L}E_3A_3 = P}$$

Example 14

At a temperature of 80°C, a steel tyre 12 mm thick and 90 mm wide that is to be shrunk onto a locomotive driving wheel 2 m in diameter just fits over the wheel, which is at a temperature of 25°C. Determine the contact pressure between the tyre and wheel after the assembly cools to 25°C. Neglect the deformation of the wheel caused by the pressure of the tyre. Assume α = 11.7 μm/(m·°C) and E = 200 GPa.

Sol: $\delta = \delta_T$ (Since length of tire will not change)

$$\Rightarrow \quad \frac{PL}{AE} = \alpha L \Delta T$$

$$P = \alpha\Delta TAE$$

$$P = (11.7 \times 10^{-6})(80 - 25)(90 \times 12)(200 \times 1000)$$

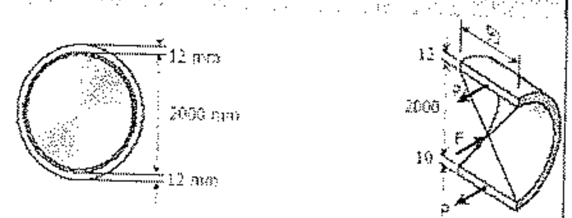
$$P = 138996 \text{ N}$$

$$F = 2P$$

$$\Rightarrow \quad pDL = 2P \text{ (where } p = \text{pressure)}$$

$$p(2000)(90) = 2(138996)$$

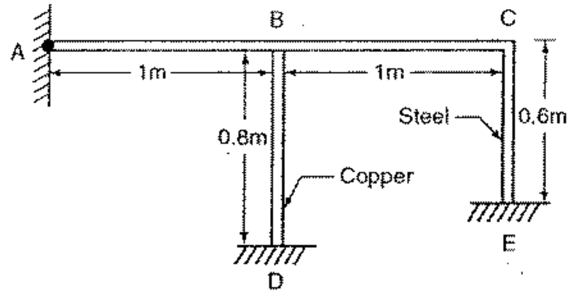
$$p = 1.5444 \text{ MPa}$$



Example 15

A rigid bar ABC is pinned at A and is connected by a steel bar CE and a copper bar BD as shown in Fig. (a). If the temperature of the whole assembly is raised by 40°C, find the stresses induced in steel and copper rods. Given:

	For steel bar	For copper bar
Area	400 mm^2	600 mm^2
Modulus of elasticity	2×10^5 N/mm^2	1×10^5 N/mm^2
Coefficient of thermal expansion	12×10^{-6}/°C	18×10^{-6}/°C



(a)

Sol. Since $\alpha_c > \alpha_s$, free expansion of copper is more than free expansion of steel as shown in Fig.b. But the bar ABC is rigid and is hinged. Hence, its final position will be as shown in Fig. b. This restriction on free expansion introduce compressive force P_c in copper bar and tension P_s in steel bar to cause change in lengths Δ_c and Δ_s respectively.

The static equilibrium condition is obtained by taking moment about A as

$P_s \times 2 = P_c \times 1$

or $\quad P_c = 2P_s$

From property of similar triangles we get,

$$\frac{\alpha_c t L_c - \Delta_c}{1} = \frac{\alpha_s t L_s + \Delta_s}{2}$$

$$2(\alpha_c t L_c - \Delta_c) = \alpha_s t L_s + \Delta_s$$

or $(2\alpha_c L_c - \alpha_s L_s)t = \dfrac{P_s L_s}{A_s E_s} + 2\dfrac{P_c L_c}{A_c E_c}$

Substituting the values $P_c = 2P_s$

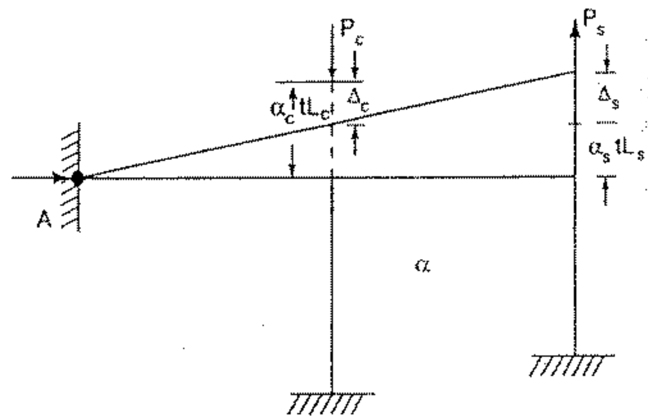
we get, $(2 \times 18 \times 10^{-6} \times 800 - 12 \times 10^{-6} \times 600)40$

$$= \frac{P_s \times 600}{400 \times 2 \times 10^5} + \frac{4P_s \times 800}{600 \times 1 \times 10^5}$$

$$= \frac{P_s}{10^5}\left(\frac{600}{800} + \frac{3200}{600}\right)$$

$P_s = 14202.74$ N

Hence, $P_c = 2P_s = 28405.48$ N



(b)

Stress in steel rod $= \dfrac{P_s}{A_s} = \dfrac{14202.74}{400}$

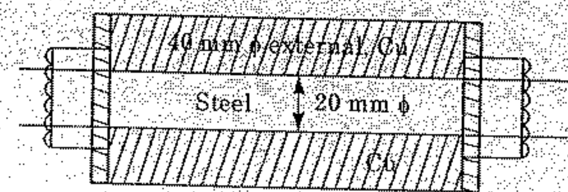
$= 35.507$ N/mm² (Tensile)

Stress in copper rod $= \dfrac{P_c}{A_c} = \dfrac{28405.48}{600}$

$= 47.342$ N/mm² (Compressive)

Example 16

A steel Rod of 20 mm diameter passes centrally through a tight fitting copper tube of 40 mm external diameter. The tube is closed with the help of a rigid washer of negligible thickness and nuts threaded on the rod. The nuts are tightened, till the compressive load on the tube is 50 kN. Determine the stresses in the rod and tube when the temperature of the assembly falls by 50°C. Take E for steel = 200×10^3 N/mm² and E for copper = 100×10^3 N/mm² and coefficient of expansion, α for steel = 12×10^{-6}/°C and copper = 18×10^{-6}/°C.



Sol: Nut tightened till comp. load on Cu = 50 kN

$\sigma_{St} = ?$ $\qquad$ $\sigma_{Cu} = ?$

$E_{St} = 200 \times 10^3$ N/mm² $\qquad$ $E_{Cu} = 100 \times 10^3$ N/mm²

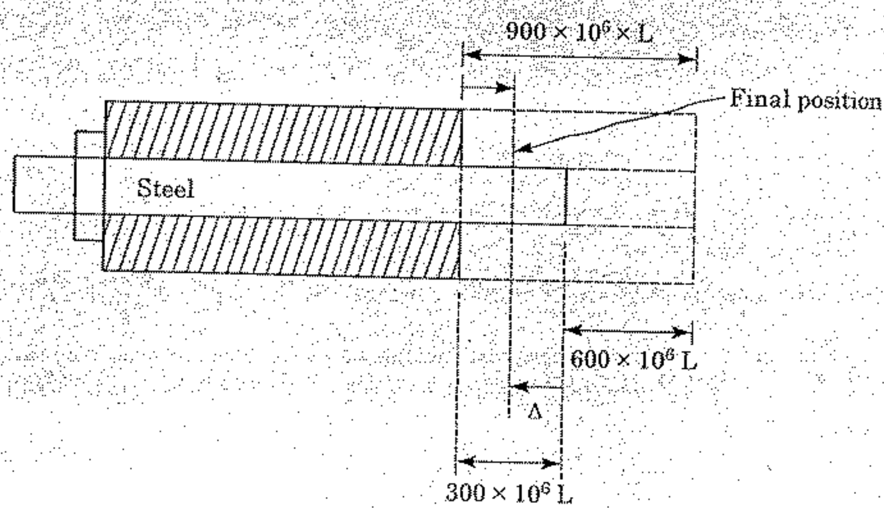
$\alpha_{St} = 12 \times 10^{-6}$/°C $\qquad$ $\alpha_{Cu} = 18 \times 10^{-6}$/°C

When temperature falls, contraction of steel assuming it to be free is given by

$$= L \times \alpha\Delta T = L \times 12 \times 10^{-6} \times 50$$
$$= L \times 600 \times 10^{-6}$$

Contraction of Cu assuming it to be free is given by

$$= L \times \alpha\Delta T$$
$$= L \times 18 \times 10^{-6} \times 50$$
$$= L \times 900 \times 10^{-6}$$



Let due to temperature fall, force induced in steel = P (compressive)

$$\Rightarrow \quad \frac{Pl}{A_{St}E_{St}} = \Delta$$

Force induced in Cu (tensile) = P

$$\frac{Pl}{E_{Cu}.A_{Cu}} = (300 \times 10^{-6}\ L - \Delta)$$

$$\Rightarrow \quad \frac{Pl}{E_{Cu}.A_{Cu}} = 300 \times 10^{-6}\ L - \frac{Pl}{A_{St}E_{St}}$$

$$\left(\frac{P}{E_{Cu}A_{Cu}}+\frac{P}{A_{St}E_{St}}\right) = 300 \times 10^{-6}$$

$$\Rightarrow P = \frac{300\times10^{-6}\times E_{Cu}A_{Cu}\times E_{St}A_{St}}{E_{St}A_{St}+E_{Cu}A_{Cu}}$$

$$P = \frac{300\times10^{-6}}{\left(\frac{1}{E_{Cu}A_{Cu}}+\frac{1}{E_{St}A_{St}}\right)}$$

$$= \left[\frac{300\times10^{-6}}{\left(\frac{4}{100\times10^3\times\pi(40^2-20^2)}\right)+\frac{4}{200\times10^3\times\pi\times(20)^2}}\right]$$

$$= \frac{300\times10^{-6}}{\frac{4}{\pi\times10^5}\left(\frac{1}{1200}+\frac{1}{800}\right)}$$

$$= \frac{300\times10^{-1}\times\pi\times1200\times800}{4\,(800+1200)}\ \text{N}$$

$$= 11304 \text{ N}$$

$$= \mathbf{11.304\ kN}$$

Hence, Final force in Cu after temperature fall = **50 kN (compressive) + 11.304 kN (tensile)**

= **38.696 kN compressive**

Final force in steel = **50 kN (Tensile) + 11.304 kN (compressive)**

= **38.696 kN (tensile)**

Example 17

A rigid block ABC weighing 200 kN is supported by three rods symmetrically placed as shown in figure given below. The lower ends of the rods are at the same level before the block is attached and temperature is changed. Assuming the block to remain horizontal, determine the stresses in each rod after a temperature rise of 20°C.

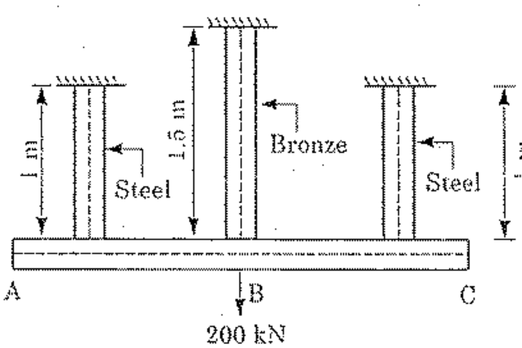
Given:

$A_s = 900\ mm^2$ $\quad A_b = 1500\ mm^2$

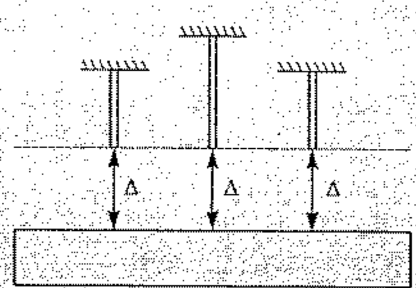
$L_s = 1\ m$ $\quad L_b = 1.5\ m$

$E_s = 205\ kN/mm^2$ $\quad E_b = 82\ kN/mm^2$

$\alpha_s = 12 \times 10^{-6}/°C$ $\quad \alpha_b = 20 \times 10^{-6}/°C$



Sol: Total elongation in three bars must be same if the bar ABC is to remain horizontal.



$$\Delta = \Delta_T + \Delta_{load}$$

Δ_T = deflection due to temperature

Δ_{load} = deflection due to load

Let force in each steel bar = F_s

Let force in each bronze bar = F_b

$$\Rightarrow \quad L_s \alpha_s \Delta T + \frac{F_s l_s}{A_s E_s} = L_b \alpha_b \Delta T + \frac{F_b l_b}{A_b E_b} \text{ ------------(i)}$$

Also from equillibrium of forces

$$\boxed{2F_s + F_b = 200 \text{ kN}} \text{ ------------------ (ii)}$$

$$1 \times 12 \times 10^{-6} \times 20 + \frac{F_s \times 1 \text{ m}}{205 \times 900} = 1.5 \times 20 \times 10^{-6} \times 20 + \frac{F_b \times 1.5}{82 \times 1500}$$

where F_s and F_b are in kN

$$2.4 \times 10^{-4} + F_s \times 5.42 \times 10^{-6} = 6 \times 10^{-4} + 1.2195 \times 10^{-5} F_b$$

$$2.4 + F_s \times 5.42 \times 10^{-2} = 6 + 1.2195 \times 10^{-1} F_b$$

$$2.4 + F_s \times 5.42 \times 10^{-2} = 6 + 1.2195 \times 10^{-1} (200 - 2F_s)$$

$$2.4 + \frac{5.42 F_s}{100} = 6 + \frac{1.2195}{10}(200) - \frac{1.2195 F_s \times 2}{10}$$

$$\frac{2 \times 1.2195 + 0.542}{10} F_s = 27.99$$

Hence

$$F_s = 93.89 \text{ kN}$$

$$F_b = 12.214 \text{ kN}$$

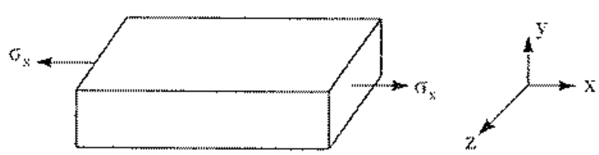
$$\text{Stress } \sigma_s = \frac{93.89 \times 10^3 \text{ N}}{900 \text{ mm}^2} = 104.32 \text{ N/mm}^2$$

$$\sigma_b = \frac{12.214 \times 10^3 \text{N}}{1500 \text{ mm}^2} = 8.14 \text{ N/mm}^2$$

POISSON'S RATIO

For Homogeneous and isotropic material (i.e. engineering materials normally used), elongation/(contraction) produced by any axial force in the direction of force is accompanied by contraction/(elongation) in all transverse direction and all such contractions/(elongations) are same.

If σ_x is the stress in x-direction as shown below, then



$\varepsilon_y = \varepsilon_z$ for stress applied in x-direction. ε_y and ε_z are called lateral strain.

$$\boxed{\text{Poisson's ratio } (\mu) = -\frac{\text{Lateral strain}}{\text{Axial strain}}}$$

$$\mu = -\frac{\varepsilon_y}{\varepsilon_x} = \frac{-\varepsilon_z}{\varepsilon_x}$$

$$\varepsilon_x = \frac{\sigma_x}{E} \quad (+)\text{ ve means elongation}$$

$$\varepsilon_y = \frac{-\mu\sigma_x}{E} \quad (-)\text{ ve means contraction}$$

$$\varepsilon_z = \frac{-\mu\sigma_x}{E}$$

Note: *Volume of rod does not remain unchanged as a result of combined effect of elongation and transverse contraction.*

Note: *Polymer foams expand laterally when stretched. Thus axial and lateral strains have same sign. Hence Poisson's ratio is negative.*

- For engineering materials, the value of Poisson's ratio ranges between 0.0 to 0.50

$\mu_{cork} = 0.0$ $\mu_{\text{perfectly elastic rubber}} = 0.5$

$\mu_{concrete} = 0.1 - 0.2$ $\mu_{metal} = 0.25 - 0.4$

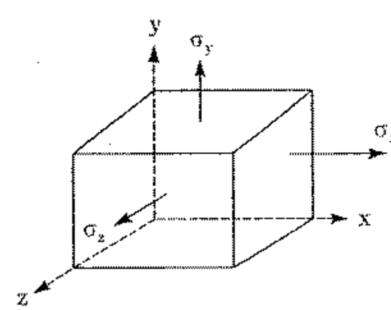
$\mu_{Aluminium} = 0.33$ $\mu_{\text{cast iron}} = 0.2 - 0.3$

$\mu_{steel} = 0.27 - 0.3$ $\mu_{rubber} = 0.45 - 0.5$

MULTI AXIAL LOADING: (SHEARING STRESSES NOT CONSIDERED)

For homogeneous and isotropic material, the strains are as shown below

$$\left\{\begin{aligned} \varepsilon_x &= \frac{\sigma_x}{E} - \mu\frac{\sigma_y}{E} - \frac{\mu\sigma_z}{E} \\ \varepsilon_y &= \frac{\sigma_y}{E} - \mu\frac{\sigma_z}{E} - \frac{\mu\sigma_x}{E} \\ \varepsilon_z &= \frac{\sigma_z}{E} - \mu\frac{\sigma_x}{E} - \frac{\mu\sigma_y}{E} \end{aligned}\right\}$$

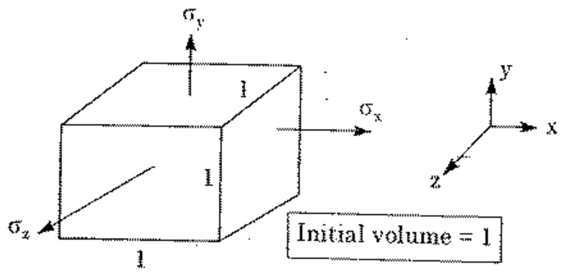


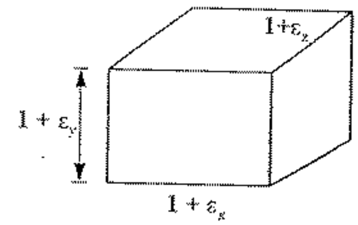
The above relationship is called generalised Hooke's law. The sign convention used is:

(+) stress → Tensile
(–) stress → compressive
(+) strain → elongation
(–) strain → contraction

- The above result has been obtained by considering the effect of each stress component separately and combining their effects.
- This is based on the principle of superposition. For superposition principle to hold, the following conditions must be satisfied.
  (i) Each effect is linearly related to the load that produces it. (i.e. Hooke's law is valid).
  (ii) The deformation resulting from any given load is small and does not affect the condition of applications of other loads.

DILATATION; BULK MODULUS





Final volume = $(1+\varepsilon_x)(1+\varepsilon_y)(1+\varepsilon_z) = (1+\varepsilon_x+\varepsilon_y+\varepsilon_z+\varepsilon_x\varepsilon_y+\varepsilon_y\varepsilon_z+\varepsilon_z\varepsilon_x)$

Final volume = $(1+\varepsilon_x+\varepsilon_y+\varepsilon_z)$ [After neglecting product of strains, being small].

$$\Rightarrow \quad \text{Change in volume} = \varepsilon_x+\varepsilon_y+\varepsilon_z$$

$$\Rightarrow \quad \text{Volumetric strain } \varepsilon_v = \left(\frac{\Delta V}{V}\right) = \frac{(\varepsilon_x+\varepsilon_y+\varepsilon_z)}{1}$$

$$\Rightarrow \quad \boxed{\varepsilon_v = \varepsilon_x+\varepsilon_y+\varepsilon_z}$$

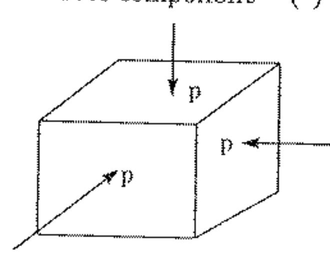
Volumetric strain is also known as dilatation.

$$\varepsilon_v = \left(\frac{\sigma_x}{E}-\frac{\mu\sigma_y}{E}-\frac{\mu\sigma_z}{E}\right)+\left(\frac{\sigma_y}{E}-\frac{\mu\sigma_z}{E}-\frac{\mu\sigma_x}{E}\right)+\left(\frac{\sigma_z}{E}-\frac{\mu\sigma_x}{E}-\frac{\mu\sigma_y}{E}\right)$$

$$\boxed{\varepsilon_v = \frac{(\sigma_x+\sigma_y+\sigma_z)}{E}(1-2\mu)} \quad \text{------------ (A)}$$

Note: *This equation is valid even if material does not follow Hooke's law.*
When strain varies throughout the volume, the value of volume change can be obatained by the applying $\left(\frac{dV}{V}=\varepsilon_x+\varepsilon_y+\varepsilon_z\right)$ *to a differential element and then integrating this expression to obtain the total volume change.*

For hydrostatic pressure condition, each stress component = (–) p, where p = pressure.



$$\Rightarrow \quad \varepsilon_v = -\frac{3p}{E}(1-2\mu)$$

$$\Rightarrow \quad \boxed{\varepsilon_v = -\frac{p}{K}}$$

where $\boxed{K = \frac{E}{3(1-2\mu)}}$ = Bulk modulus or modulus of compression of material.

$$\varepsilon_v = \frac{-p \times 3(1-2\mu)}{E} = \frac{(\sigma_x + \sigma_y + \sigma_z)(1-2\mu)}{E}$$

$$\Rightarrow \quad \boxed{\textbf{Hydrostatic pressure} = \frac{(\sigma_x + \sigma_y + \sigma_z)}{3}}$$

For hydrostatic pressure condition, volume will decrease

$$\Rightarrow \quad \varepsilon_v \text{ is negative}$$

$$\Rightarrow \quad \frac{E}{3(1-2\mu)} > 0$$

$$\Rightarrow \quad \boxed{\mu < \frac{1}{2}}$$

For all engineering material μ is (+) ve

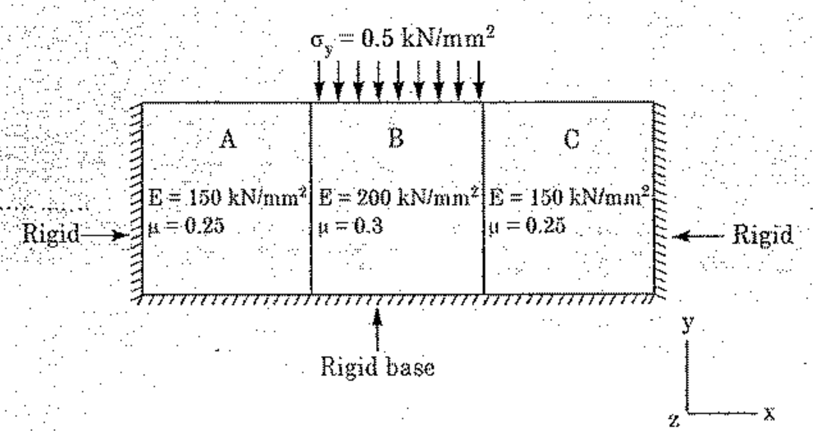
Hence $\boxed{0 < \mu < \frac{1}{2}}$

Note: *Equation 'A' shows that stretching of material in one direction i.e. due to σ_x ($\sigma_y = \sigma_z = 0$) will lead to increase in volume.*

Note: *During plastic deformation, volume of specimen remains constant.*

Example 18

(i) The figure given below shows three metal cubes A, B and C of side 100 mm in direct contact, resting on a rigid base and confined in the x-co-ordinate direction between two rigid end plates. If the upper face of the centre cube B is subjected to uniform compressive stress of 0.5 kN/mm², compute for cube B, the following:



(a) The direct stress in the x-direction (σ_x).

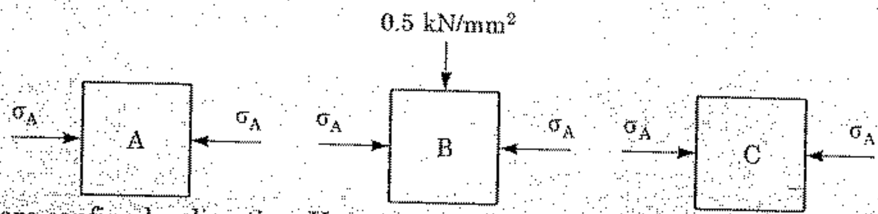
(b) The direct strains in the three coordinate direction x, y and z.

(c) The volumetric strain.

The elastic properties for the three cubes A, B and C are given in figure.

(ii) State all assumptions made.

Sol: The free body diagrams are shown below.



Since the cubes are confined x-direction. Hence,

$$\boxed{\Delta_A + \Delta_B + \Delta_C = 0} \text{ in x-direction}$$

$$\Rightarrow \left(-\frac{\sigma_A \times l_A}{E_A}\right) + \left(-\frac{\sigma_A}{E_B} - \frac{\mu_B(-0.5)}{E_B}\right) \times l_B + \left(-\frac{\sigma_A l_C}{E_C}\right) = 0$$

$$\frac{\sigma_A}{150} + \frac{\sigma_A}{200} - \frac{0.5 \times 0.3}{200} + \frac{\sigma_A}{150} = 0$$

$$\Rightarrow \quad \sigma_A = \frac{\frac{0.15}{200}}{\left(\frac{2}{150} + \frac{1}{200}\right)} = \mathbf{0.041\ kN/mm^2} \quad \textbf{(compressive)}$$

Direct strain in B

$$\text{Strain in x-direction } (\varepsilon_x) = \frac{-\sigma_x}{E} - \frac{\mu(+\sigma_y)}{E}$$

$$\varepsilon_x = \frac{-0.041}{200} + \frac{0.3 \times 0.5}{200}$$

$$\varepsilon_x = +5.45 \times 10^{-4}$$

Similarly,

$$\varepsilon_y = \frac{\sigma_y}{E} - \frac{\mu\sigma_x}{E} = \frac{-0.5}{200} - \frac{0.3(-0.041)}{200}$$

$$\varepsilon_y = -2.44 \times 10^{-3}$$

$$\varepsilon_z = 0 - \frac{\mu\sigma_y}{E} - \frac{\mu\sigma_x}{E}$$

$$= -\frac{0.3}{200}\left[\sigma_y + \sigma_x\right] = -\frac{0.3}{200}\left[-0.5 - 0.041\right]$$

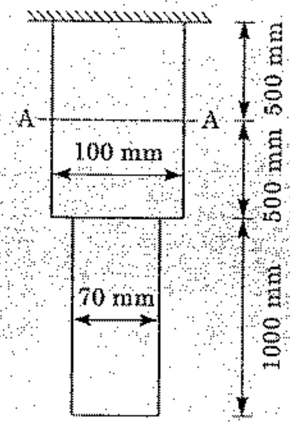
$$\varepsilon_z = 8.115 \times 10^{-4}$$

$$\text{Volumetric strain} = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

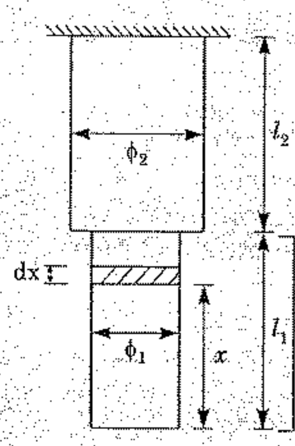
$$= -1.0835 \times 10^{-3}$$

Example 19

A stepped bar with circular cross-section and supported at top, hangs vertically under its own weight. Dimensions of the bar are shown in the figure below. Calculate the elongation of the bar under its own weight. What is the change in diameter of the bar at section AA shown in the figure? $E = 2 \times 10^5$ N/mm², density $\gamma = 8 \times 10^{-5}$ N/mm³ and Poisson's ratio $\mu = 0.2$.

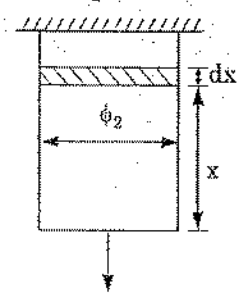


Sol:



$$\text{Elongation of smaller dia bar} = \int_0^{l_1} \frac{\frac{\pi\phi_1^2}{4} x\,\gamma\, dx}{\frac{\pi\phi_1^2}{4} \times E} = \int_0^{l_1} \frac{\gamma x\, dx}{E} = \frac{\gamma}{E}\frac{x^2}{2}\bigg|_0^{l} = \frac{\gamma l_1^2}{2E}$$

$$\text{Elongation of larger dia bar} = \int_0^{l_2} \frac{\left(\frac{\pi\phi_1^2}{4}\gamma l_1 + \frac{\pi\phi_2^2}{4} x\gamma\right) dx}{\frac{\pi\phi_2^2}{4} E} = \frac{1}{E}\int_0^{l_2}\left(\frac{\gamma\phi_1^2 l_1}{\phi_2^2} + \gamma x\right) dx = \frac{\gamma\phi_1^2 l_1 l_2}{\phi_2^2 E} + \frac{\gamma l_2^2}{2E}$$



$$\left(\frac{\pi\phi_1^2}{4} l_1 \gamma\right)$$

$$\text{Total elongation} = \frac{\gamma l_1^2}{2E} + \frac{\gamma l_2^2}{2E} + \frac{\gamma \phi_1^2 l_1 l_2}{\phi_2^2 E}$$

$$= \frac{8.0\times10^{-5}}{2\times10^5\times2}\left[(1000)^2 + (1000)^2 + 2\times(1000)(1000)\times\left(\frac{70}{100}\right)^2\right]$$

$$= 5.96 \times 10^{-4} \text{ mm}$$

Change in dia at A-A

$$\text{Lateral strain} = \frac{-\mu\sigma}{E} = \frac{\Delta\phi}{\phi_2}$$

$$\Delta\phi = \frac{-\mu\sigma}{E}\phi_2$$

Stress at A-A

$$\sigma = \frac{\gamma\frac{\pi}{4}\left(\phi_1^2 l_1 + \phi_2^2\frac{l_2}{2}\right)}{\frac{\pi\phi_2^2}{4}}$$

$$\sigma = \gamma\left[\frac{\phi_1^2}{\phi_2^2}l_1 + \frac{l_2}{2}\right]$$

$$\sigma = 8\times10^{-5}\left[\left(\frac{70}{100}\right)^2\times1000 + 500\right]$$

$$\sigma = 0.0792 \text{ N/mm}^2$$

$$\Rightarrow \quad \Delta\phi = \frac{0.2\times0.0792}{2\times10^5}\times100$$

$$\Delta\phi = 7.92 \times 10^{-6} \text{ mm (decrease)}$$

Example 20

A cylindrical piece of steel of 80 mm dia and 120 mm long is subjected to an axial compressive force of 50,000 kg. Calculate the change in volume of the piece if bulk modulus = 1.7×10^6 kg/cm^2 and Poisson's ratio = 0.3.

Sol:

$$K = 1.7 \times 10^6 \text{ kg/cm}^2$$

$$\mu = 0.3$$

Change in volume = ΔV (say)

$$\text{Volumetric strain } \left(\frac{\Delta V}{V}\right) = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

$$\frac{\Delta V}{V} = \frac{(\sigma_x + \sigma_y + \sigma_z)(1-2\mu)}{E}$$

$$\text{Bulk modulus} = \frac{E}{3(1-2\mu)} = 1.7 \times 10^6$$

$$\Rightarrow \quad \frac{E}{(1-2\mu)} = 5.1 \times 10^6 \text{ kg/cm}^2$$

$$\Rightarrow \quad \frac{\Delta V}{V} = \frac{50000 \text{ kg}}{5.1\times10^6 \text{ kg/cm}^2 \times \frac{\pi}{4}(8)^2 \text{cm}^2}$$

$$= \frac{50000\times4}{5.1\times10^6\times\pi\times64}$$

$$= 1.9514 \times 10^{-4}$$

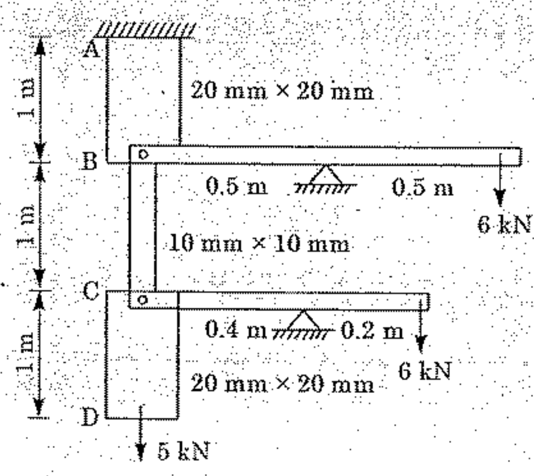
$$\Delta V = \frac{\pi}{4}(8)^2\times12\times1.9514\times10^{-4} \text{ cm}^3$$

$$\boxed{\Delta V = 0.118 \text{ cm}^3}$$

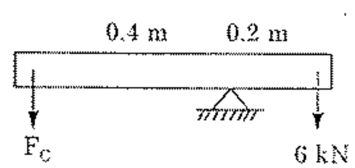
Example 21

A stepped vertical steel bar ABCD is fixed at the top end A. Each segment of the bar AB, BC and CD is 1 m long and has cross-sections 20 mm × 20 mm, 10 mm × 10 mm and 20 m,m × 20 mm respectively. A 5 kN load is applied directly at D and 6 kN loads are applied on the levers attached to the stepped bar at B and C as shown in figure. Find the vertical displacement of D and the change in volume of the bar.

$E = 2 \times 10^5$ MPa and Poisson's ratio $\mu = 0.25$. Connections between the levers and bar at B and C are hinged.



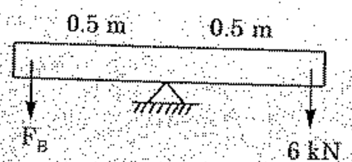
Sol: Let F_C and F_B are the forces acting on lever at C and B. These forces on levers will be applied by stepped bars. Hence, forces equal and opposite to F_C and F_B will be applied by the lever on the stepped bar.



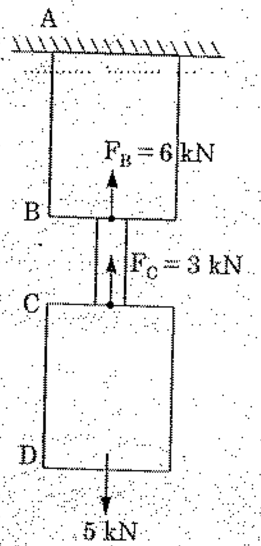
Taking moment of all forces in the figure shown below about hinge we get,

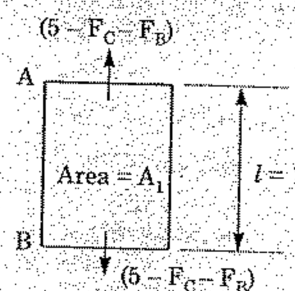
$$6 \times 0.2 = F_C \times 0.4$$

$$\Rightarrow \quad F_C = \frac{6 \times 10.2}{0.4} = 3\text{ kN}$$

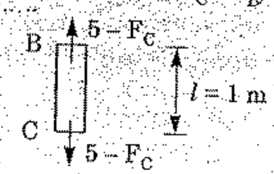


$$\boxed{F_B = 6\text{ kN}}$$

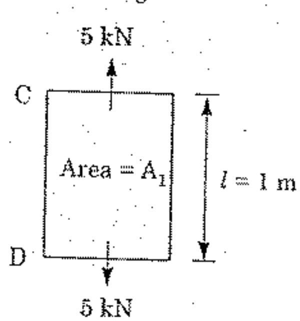




$$\Delta_{AB} = \frac{(5 - F_C - F_B) \times l}{A_1 E} = \frac{(5 - 3 - 6)\, l}{A_1 E}$$



$$\Delta_{BC} = \frac{(5 - F_C)\, l}{A_2 E} = \frac{(5 - 3)\, l}{A_2 E}$$



$$\Delta_{CD} = \frac{5\, l}{A_1 E} = \frac{5l}{A_1 E}$$

Total deflection at D:

$$= \Delta_{AB} + \Delta_{BC} + \Delta_{CD}$$

$$= \frac{l}{A_1E} + \frac{2l}{A_2E}$$

$$= \frac{1\,\text{kN} \times 1\,\text{m}}{2\times10^5 \times 400\times10^{-3}} + \frac{2\times1}{2\times10^5\times100\times10^{-3}}\,\text{m}$$

$$= \left(\frac{1000}{8\times10^7} + \frac{2000}{2\times10^7}\right)m$$

$$= \frac{9000}{8\times10^7}\,\text{m} = \frac{9}{80}\,\text{mm} = \mathbf{0.1125\ mm}$$

Change in Volume:

$$\Delta V_{AB} + \Delta V_{BC} + \Delta V_{CD} = \Delta V$$

$$\frac{\Delta V_{AB}}{V_{AB}} = \frac{\sigma_{AB}}{E} - \mu\frac{\sigma_{AB}}{E} - \mu\frac{\sigma_{AB}}{E} = \frac{\sigma_{AB}(1-2\mu)}{E}$$

$$\sigma_{AB} = \frac{(5-9)\,\text{kN}}{20\times20\,\text{mm}^2} = -\frac{4000}{400} = -10\ \text{N/mm}^2$$

$$\sigma_{BC} = \frac{(5-3)\times1000}{10\times10} = 20\ \text{N/mm}^2$$

$$\sigma_{CD} = \frac{5\times1000}{20\times20} = \frac{50}{4} = 12.5\ \text{N/mm}^2$$

$$\frac{\Delta V_{AB}}{V_{AB}} = \frac{-10\,(1-0.25\times2)}{2\times10^5}$$

$$\Delta V_{AB} = \frac{-10\times0.5}{2\times10^5}\times20\times20\times1000\ \text{mm}^3$$

$$= -10\ \text{mm}^3$$

$$\Delta V_{BC} = \frac{20\times(1-0.5)}{2\times10^5}\times10\times10\times1000$$

$$= 10\times0.5 = 5\ \text{mm}^3$$

$$\Delta V_{CD} = \frac{12.5\times(1-0.5)}{2\times10^5}\times20\times20\times1000$$

$$= 12.5\times2\times0.5 = 12.5\ \text{mm}^2$$

Total change in volume $= -10 + 5 + 12.5 = \mathbf{7.5\ mm^3}$

Example 22

Determine the perçentage change in volume of a steel bar 40 mm square in section and 1 m long when subjected to an axial compressive load of 15 kN.

What change in volume would a 100 mm cube of steel suffer at a depth of 4 km in sea water?

Take $E = 2\times10^5$ N/mm² and $G = 0.81\times10^5$ N/mm², $\gamma_{\text{sea water}} = 10080$ N/m³

Sol.

(a) Volume of bar, V = $b^2.L$

$$\therefore \quad \frac{\delta V}{V} = 2\frac{\delta b}{b} + \frac{\delta L}{L} = 2e_b + e_L$$

or $$\frac{\delta V}{V} = -\frac{2p\mu}{E} + \frac{p}{E} = \frac{p}{E}(1-2\mu) \quad \text{---} (1)$$

Now, $$E = 2G(1+\mu)$$

$$\therefore \quad \mu = \frac{E}{2G} - 1 = \frac{2\times 10^5}{2\times 0.81\times 10^5} - 1 = 0.2346$$

Hence from (1),

$$\frac{\partial V}{V} = \frac{p}{E}(1-2\times 0.2346) = \frac{-15000}{1600\left(2\times 10^5\right)}\times 0.5309 = -2.488\times 10^{-5}$$

$\therefore$ % reduction in volume $= \frac{\partial V}{V}\times 100 = 2.488\times 10^{-5}\times 100 = 0.00249$

(b) On the cube, p = γh

Here, γ = 10080 N/m³ (for sea water) and h = 4 km = 4000 m.

$\therefore$ p = 10080 × 4000 = 40.32 × 10^6 N/ m²

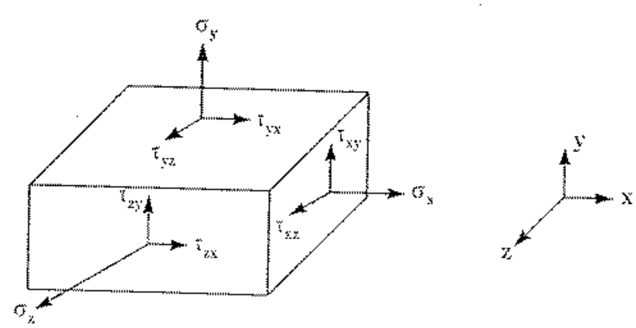
$$K = \frac{E}{3(1-2\mu)} = \frac{2\times 10^5}{3(1-2\times 0.2346)} = 1.256\times 10^5 \text{ N/mm}^2$$

Now, $$e_v = \frac{\partial V}{V} = \frac{p}{K} \quad \text{(By definition)}$$

$$\therefore \quad \partial V = \frac{p}{K}\times V = \frac{40.32}{1.256\times 10^5}\times (100)^3 = 321 \text{ mm}^3$$

SHEARING STRAIN

General stress situation is represented as below:



The various stresses are σ_x, σ_y, σ_z, τ_{xy}, τ_{xz}, τ_{yx}, τ_{yz}, τ_{zx}, τ_{zy}

But

$$\tau_{xy} = \tau_{yx}$$
$$\tau_{xz} = \tau_{zx}$$
$$\tau_{yz} = \tau_{zy}$$

Hence no. of stress elements at a point to describe general stress situation are

$$\sigma_x, \sigma_y, \sigma_z \Rightarrow 3 \text{ normal stresses}$$

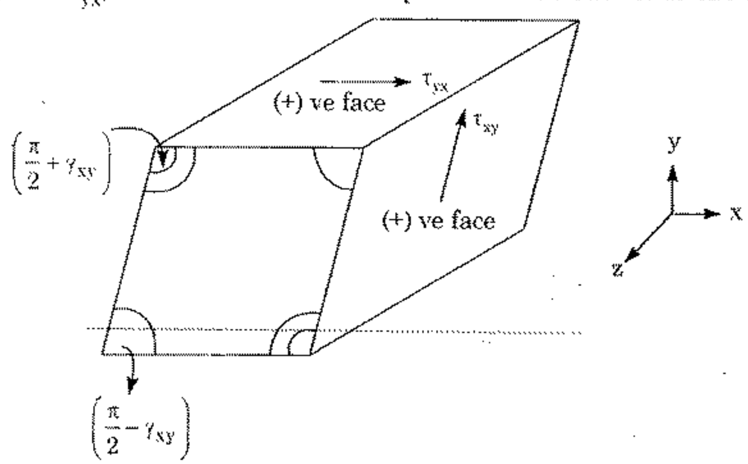
$$\tau_{xy}, \tau_{yz}, \tau_{zx} \Rightarrow 3 \text{ shear stresses}$$

For homogeneous isotropic materials, shear stresses have no direct effect on the normal strain and as long as all the deformations involved remains small, they will not affect the derivation and validity of eq. 'α' i.e.

$\varepsilon_x = \frac{\sigma_x}{E} - \frac{\mu\sigma_y}{E} - \frac{\mu\sigma_z}{E}$. Thus, equation of generalised Hooke's law derived earlier will still be valid even if shear stresses are acting.

However, shear stresses will tend to deform the cubic element of material into an oblique parallelepiped. Deformation without change in volume is called distortion. If only shearing stresses are acting then volume of the specimen does not change.

If we consider only τ_{xy} and τ_{yx}, then the deformed shape will be as shown in the figure.



γ_{xy} = shearing strain corresponding to x and y direction

- When angle between two (+) ve faces as described earlier reduces, like from $\frac{\pi}{2}$ to $\left(\frac{\pi}{2} - \gamma_{xy}\right)$, γ_{xy} is (+) ve otherwise negative.
- Shear stress (τ_{xy}) and shear strain (γ_{xy}) curve is obtained from *Torsion Test*.
- The curve is similar to normal stress-strain curve except that values obtained for yield strength, ultimate strength etc. are *only about half as that in tension*.
- Shear stress-strain diagram also have initial linear portion.
- For values of shearing stress that does not exceed the *proportional limit in shear*, we can write for homogeneous, isotropic material

$$\boxed{\tau_{xy} = G\gamma_{xy}} \text{ Hooke's law for shearing-stress and strain}$$

where G = Modulus of rigidity or shear modulus

where

$$\boxed{G = \frac{E}{2(1+\mu)}}$$

$$\boxed{\frac{E}{3} < G < \frac{E}{2}}$$

Note: When $\mu = 0$, $G = \frac{E}{2}$, when $\mu = 0.5$, $G = \frac{E}{3}$.

Thus for homogeneous and isotropic material, generalised Hooke's law is decribed as follows:

$$\varepsilon_x = \frac{\sigma_x}{E} - \frac{\mu\sigma_y}{E} - \frac{\mu\sigma_z}{E} \qquad \gamma_{xy} = \frac{\tau_{xy}}{G}$$

$$\varepsilon_y = \frac{\sigma_y}{E} - \frac{\mu\sigma_z}{E} - \frac{\mu\sigma_x}{E} \qquad \gamma_{yz} = \frac{\tau_{yz}}{G} \quad \text{.......... (B)}$$

$$\varepsilon_z = \frac{\sigma_z}{E} - \frac{\mu\sigma_x}{E} - \frac{\mu\sigma_y}{E} \qquad \gamma_{zx} = \frac{\tau_{zx}}{G}$$

The above relation **B** can be written in matrix form as

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix} = \begin{bmatrix} \frac{1}{E} & -\frac{\mu}{E} & -\frac{\mu}{E} & 0 & 0 & 0 \\ \frac{-\mu}{E} & \frac{1}{E} & -\frac{\mu}{E} & 0 & 0 & 0 \\ \frac{-\mu}{E} & \frac{-\mu}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix}$$

RELATION BETWEEN ELASTIC CONSTANTS

$$\boxed{G = \frac{E}{2(1+\mu)}} \qquad \boxed{K = \frac{E}{3(1-2\mu)}} \qquad \boxed{E = \frac{9KG}{3K+G}} \qquad \boxed{\mu = \frac{3K-2G}{6K+2G}}$$

If E and G are known, μ can be calculated. Hence for homogeneous and isotropic material there are only two independent and distinct elastic constants. (E and G, or, E and μ, or G and μ).

- For isotropic material, as discussed earlier, normal strain does not depend on shear strain.
- **For orthotropic material** (like one laminate placed over other, ex. wood), *Normal strain doesnot depend on shear strain.* But *Poisson's ratio* and E are different in different direction. Thus,

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_x} & -\frac{\nu_{yx}}{E_y} & -\frac{\nu_{zx}}{E_z} & 0 & 0 & 0 \\ \frac{-\nu_{xy}}{E_x} & \frac{1}{E_y} & -\frac{\nu_{zy}}{E_z} & 0 & 0 & 0 \\ \frac{-\nu_{xz}}{E_x} & \frac{-\nu_{yx}}{E_y} & \frac{1}{E_z} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{xy}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{yz}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{zx}} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix}$$

Thus, 9—independent and distinct elastic constants are required to describe the state of stress at a point in orthotropic material.

- For *Anisotropic material* normal strain will depend on shear strain also.
- Hence zero terms in the above matrix will not be zero.
- Thus, *no. of independent and distinct elastic constant* will be $\left(\frac{36-6}{2}\right)+6 = 21$

  36 = Total no. of elements in the matrix

  6 = no. of elements along the diagonal

Example 23

In a tensile test, a test piece of 25 mm diameter is tested over a gauge length of 125 mm. The elongation over this length is 0.0875 mm under a pull of 68725 N. In a torsion test, a test piece was made of the same material and of same diameter, and it twisted 0.025 radians over a length of 250 mm at a torque of 0.3068 kN-m. Find Poisson's ratio and the three elastic moduli of the test piece material.

Sol: Elongation = 0.0875 mm; Under P = 68725 N

$$\boxed{\frac{Pl}{AE} = \Delta} \quad \text{------------ (i)}$$

Torsion

$$\theta = 0.25 \text{ radian}; \qquad l = 250 \text{ mm}$$

$$\boxed{\frac{Tl}{GJ} = \theta} \quad \text{------------ (ii)}$$

From 1:

$$E = \frac{Pl}{A\Delta} = \frac{68725 \times 125 \text{ Nmm}}{\frac{\pi}{4}(25)^2 \text{mm}^2 \times 0.0875 \text{ mm}}$$

$$= 2.0011 \times 10^5 \text{ N/mm}^2$$

From 2:

$$G = \frac{Tl}{J\theta} = \frac{0.3068 \times 10^6 \times 250 \text{ mm} \cdot \text{Nmm}}{\frac{\pi(25)^4}{32} \times 0.025 \text{ mm}^4}$$

$$\Rightarrow \quad G = 0.8004 \times 10^5 \text{ N/mm}^2$$

$$G = \frac{E}{2(1+\mu)} = 0.8 \times 10^5$$

$$\frac{2 \times 10^5}{2(1+\mu)} = 0.8 \times 10^5$$

$$(1+\mu) = \frac{1}{0.8} = 1.25$$

$$\Rightarrow \quad \boxed{\mu = 0.25}$$

Thus,

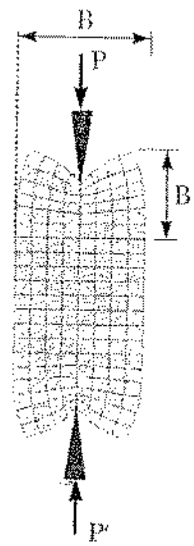
$$\boxed{E = 2 \times 10^5 \text{ N/mm}^2}$$

$$\boxed{G = 0.8 \times 10^5 \text{ N/mm}^2}$$

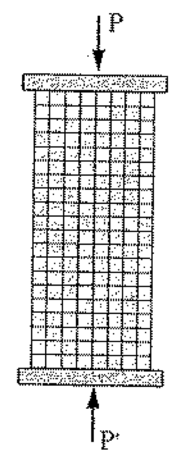
$$K = \frac{E}{3(1-2\mu)} = \frac{2 \times 10^5}{3 \times (1-0.5)} = \frac{2}{1.5} \times 10^5$$

$$\boxed{K = 1.33 \times 10^5 \text{ N/mm}^2}$$

SAINT-VENANT PRINCIPLE



(a)



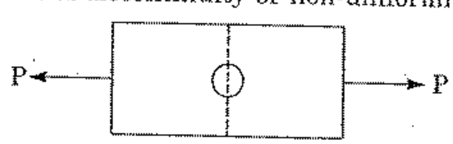
(b)

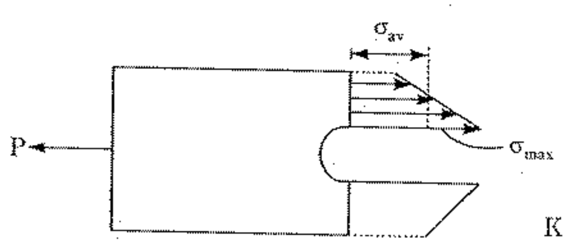
Except in the immediate vicinity of point of application of loads, the stress distribution may be assumed independent of the actual mode of application of loads. Thus, if B is the width of the member, stress becomes almost constant beyond a depth of B from one end.

However if the looding is as shown in the figure (b), stress will be constant throughout.

STRESS CONCENTRATION

It occurs at the point of discontinuity or non-uniformity.





$$\frac{\sigma_{max}}{\sigma_{av}} = K$$

K = stress concentration factor

When load is increased such that

$\sigma_{max} = \sigma_y$ (yield stress), then at that point
$P = P_y$ (i.e. load corresponding to 1st yield)

$$\Rightarrow \quad P_y = \sigma_{av}.A = \frac{\sigma_y}{K} \cdot A$$

$$\boxed{P_y = \frac{\sigma_y A}{K}}$$

When complete section has yielded $P = P_u$ = ultimate load

$$\boxed{P_u = \sigma_y A}$$

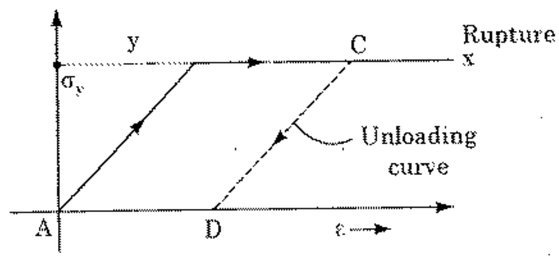
$$\Rightarrow \quad \frac{P_u}{P_y} = \frac{\sigma_y A}{\sigma_y A/K} = K$$

$$\boxed{K = \frac{P_u}{P_y}} = \text{Stress concentration factor.}$$

PLASTIC DEFORMATION

When the yield stress in a material is exceeded, plastic flow occurs. To get a considerable insight into the plastic behaviour of the material an idealised curve is studied.

For elastoplastic material idealised curve is as shown below.



RESIDUAL STRESS

When some part of an indeterminate structure undergoes plastic deformation, or different part undergoes different plastic deformation the *stress* in various parts of the structure will not return to zero after the load has been removed. Thus stresses, called residual stress will remain in various parts of structure.

OBJECTIVE QUESTIONS

1. Match **List-I** with **List-II** and select the correct answer using the codes given below the lists:

List-I

A. Ductility
B. Brittleness
C. Tenacity
D. Toughness

List-II

1. Failure without warning
2. Drawn permanently over great changes of shape without rupture
3. Absorption of energy at high stress without rupture
4. High tensile strength

Codes:

	A	B	C	D
(a)	1	2	4	3
(b)	1	2	3	4
(c)	2	3	4	1
(d)	2	1	4	3

2. Match **List-I** (Material) with **List-II** (Characteristic) and select the correct answer using the codes given below the lists:

List-I

A. Inelastic material
B. Rigid plastic material
C. Ductile material
D. Brittle material

List-II

1. No plastic zone
2. Large plastic zone
3. Strain is not recovered after unloading
4. Strain is zero upto a stress level and then stress remains constant.

Codes:

	A	B	C	D
(a)	3	4	2	1
(b)	3	4	1	2
(c)	4	3	2	1
(d)	4	3	1	2

3. Match **List-I** (Property) with **List-II** (Characteristic) and select the correct answer using the codes given below the lists:

List-I

A. Fatigue
B. Creep
C. Plasticity
D. Endurance limit

List-II

1. Material continues to deform with time under sustained loading

2. Decreased resistance of material to repeated reversals of stress
3. Material has a high probability of not failing under reversals of stress of magnitude below this level
4. Material continues to deform without any further increase in stress

Codes:

	A	B	C	D
(a)	2	1	4	3
(b)	2	1	3	4
(c)	1	2	4	3
(d)	1	2	3	4

4. For a linear, elastic, isotropic material; the number of independent elastic constants is
   (a) 1 (b) 2
   (c) 3 (d) 4

5. Creep of a material is a property indicated by
   (a) a time dependent strain of the material
   (b) elongation of the material due to changes in the material properties
   (c) shortening caused by shrinkage of the member
   (d) the decrease in the volume of the material affected by the weather conditions

6. Elastic limit is the point
   (a) up to which stress is proportional to strain
   (b) at which elongation takes place without application of additional load
   (c) up to which if the load is removed, original volume and shape are regained
   (d) at which the toughness is maximum

7. Match **List-I** (Material) with **List-II** (Properties) and select the correct answer using the codes given below the lists:

List-I
A. Isotropic
B. Homogeneous
C. Visco-elastic
D. Brittle

List-II
1. Time dependent stress-strain relation
2. No plastic zone
3. Identical properties in all directions
4. Similar properties throughout the volume

Codes:

	A	B	C	D
(a)	3	1	2	4
(b)	4	1	2	3
(c)	3	4	1	2
(d)	4	3	2	1

8. What is the correct sequence of the following metals in the decreasing order of their ...

?

1. Aluminium
2. Cast iron
3. Steel

Select the correct answer using the codes given below ;

(a) 1-2-3 (b) 2-1-3
(c) 1-3-2 (d) 3-1-2

9. **Assertion (A):** The principle of superposition is valid whenever the strain or stress to be obtained is directly proportional to the applied loads.
Reason (R): Strain energy depends on the product of stress and strain.
Of these statements
(a) both A and R are true and R is the correct explanation of A
(b) both A and R are true but R is not a correct explanation of A
(c) A is true but R is false
(d) A is false but R is true

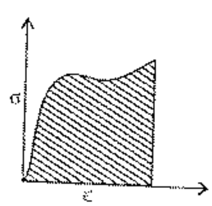
10. Which one of the following favours brittle fracture in a ductile material?
(a) Elevated temperature
(b) Slow rate of straining
(c) Presence of notch
(d) Circular cross-section

11. Consider the following statements:
Assertion (A): Cast iron torsion specimen has a helicoidal fracture at failure.
Reason (R): Cast iron is weak in tension and failure of the specimen due to torsion takes place at 45° to the axis of the specimen.
Of these statements
(a) both A and R are true and R is the correct explanation of A
(b) both A and R are true but R is not a correct explanation of A
(c) A is true but R is false
(d) A is false but R is true

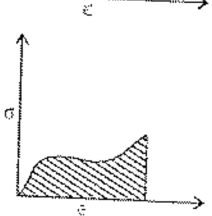
12. Match **List-I** with **List-II** and select the correct answer using the codes given below the lists:

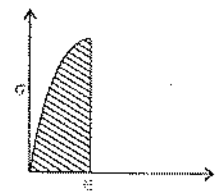
List-I **List-II**

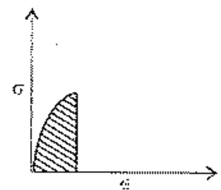
A. Soft & Weak

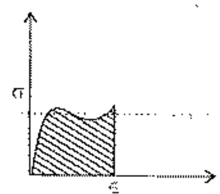
1. 

B. Hard brittle

2. 

C. Hard strong 3. 

D. Soft tough 4. 

E. Hard tough 5. 

Codes:

	A	B	C	D	E
(a)	5	4	3	2	1
(b)	1	2	3	4	5
(c)	5	3	2	4	1
(d)	1	4	3	2	5

13. **Assertion (A):** The ultimate load of a structure made of ductile material, subjected to reversible repeating loads and plastic deformation, is lowered with each reversal of load.
Reason (R): When subjected to repeated reversal of loads and plastic deformation, the structure made of a ductile material accumulates residual strains.
Of these statements
(a) both A and R are true and R is the correct explanation of A
(b) both A and R are true but R is not a correct explanation of A
(c) A is true but R is false
(d) A is false but R is true

14. Match **List-I** (Various test stages) with **List-II** (Observation) and select the correct answer using the codes given below the lists:

List-I	**List-II**
A. I-Stage	**1.** Yield point
B. II-Stage	**2.** Limit of proportionality
C. III-Stage	**3.** Breaking stress
D. IV-Stage	**4.** Ultimate stress

Codes:

	A	B	C	D
(a)	2	1	3	4
(b)	2	1	4	3
(c)	1	2	4	3
(d)	1	2	3	4

15. **Assertion (A):** When a material is subjected to repeated tensile stress within elastic range, it is found that the material deteriorates and fractures after many but finite number of repeated application of stress.

Reason (R): The critical stress below which fluctuating stresses cannot cause a fatigue failure is termed as 'endurance limit'.

Of these statements

(a) both A and R are true and R is the correct explanation of A

(b) both A and R are true but R is not a correct explanation of A

(c) A is true but R is false

(d) A is false but R is true

16. Match **List-I** (Material) with **List-II** (Modulus of elasticity, N/mm^2) and select the correct answer using the codes given below the lists:

List-I	**List-II**
A. Steel	1. 0.6×10^5
B. Cast Iron	2. 1×10^5
C. Aluminium	3. 2×10^5
D. Timber	4. 0.1×10^5

Codes:

	A	B	C	D
(a)	3	2	1	4
(b)	2	3	1	4
(c)	3	2	4	1
(d)	2	3	4	1

17. As soon as the external forces causing deformation in a perfectly elastic body, are withdrawn, the elastic deformation disappears

(a) only partially

(b) completely over a prolonged period of time

(c) completely and instantaneously

(d) completely after an initial period of rest

18. Consider the following statements regarding tensile test diagrams for carbon steels with varying carbon contents:

As the carbon content increases

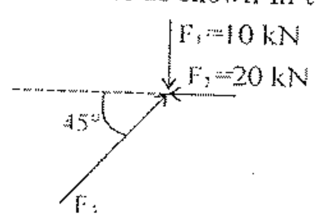
1. the ultimate strength of steel decreases
2. the elongation before fracture increases
3. the ductility of the metal decreases
4. the ultimate strength of steel increases

Which of these statements are correct?

(a) 3 and 4 (b) 1 and 3

(c) 1, 2 and 3 (d) 1 and 2

19. For the coplanar concurrent system of forces as shown in the given figure, the system will be



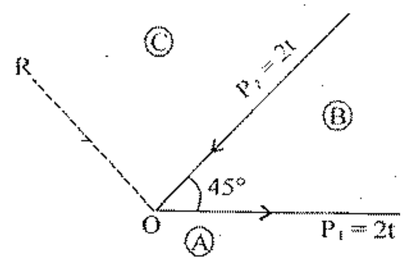
(a) in equilibrium if $|F_3| = 10$ kN

(b) in equilibrium if $|F_3| = 10\sqrt{2}$ kN

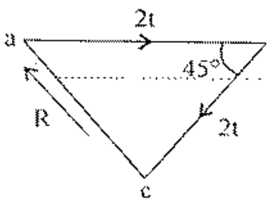
(c) in equilibrium if $|F_3| = 20$ kN

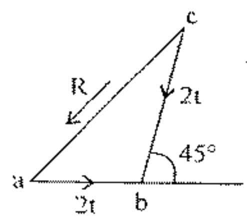
(d) will not be in equilibrium whatever be the magnitude of F_3

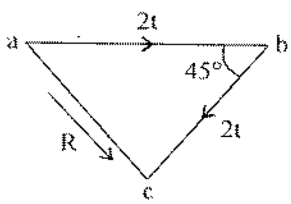
20. Two coplanar concurrent forces $P_1 = 2$ t and $P_2 = 2$ t meeting at O act on a lamina at 45° as shown in figure.

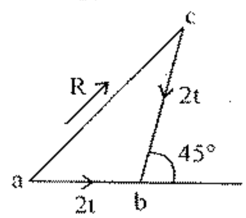


From the force diagram the force R to be applied at O in order to keep the body in equilibrium is given by

(a) 

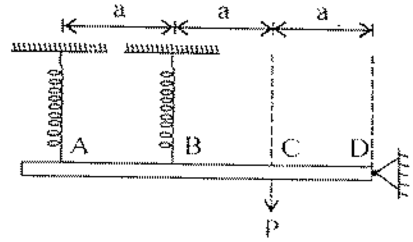
(b) 

(c) 

(d) 

21. Parallelogram law of forces states that if two forces acting simultaneously at a point be represented in magnitude and direction by two adjacent sides of a parallelogram, their resultant may be represented in magnitude and direction by
(a) longer side of the other two sides
(b) shorter side of the other two sides
(c) diagonal of the parallelogram which does not pass through their point of intersection
(d) diagonal for the parallelogram which passes through their point of intersection

22. A rigid beam ABCD is hinged at D and supported by two springs at A and B as shown in the given figure. The beam carries a vertical load P at C. The stiffness of spring at A is 2k and that of B is k.



The ratio of forces of spring at A and that of spring at B is
(a) 1 (b) 2
(c) 3 (d) 4

23. A bar of circular cross-section varies uniformly from a cross-section 2D to D. If extension of the bar is calculated treating it as a bar of average diameter, then the percentage error will be
(a) 10 (b) 25
(c) 33.33 (d) 50

DIRECTIONS :

The following items consists of two statements; one labelled as **'Assertion (A)'** and the other as **'Reason (R)'**. You are to examine these two statements carefully and select the answers to these items using the codes given below:

Codes:

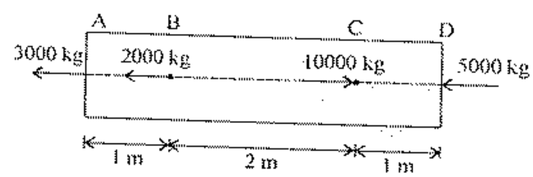
(a) Both A and R are true and R is the correct explanation of A
(b) Both A and R are true but R is not a correct explanation of A
(c) A is true but R is false
(d) A is false but R is true

24. **Assertion (A):** Strain is a fundamental behaviour of the material, while the stress is a derived concept.
Reason (R): Strain does not have a unit while the stress has a unit.

25. **Assertion (A):** The amount of elastic deformation at a certain point, which an elastic body undergoes, under given stress is the same irrespective of the stresses being tensile or compressive.
Reason (R): The modulus of elasticity and Poisson's ratio are assumed to be the same in tension as well as compression.

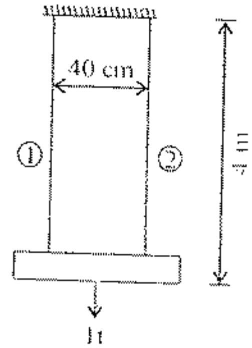
26. **Assertion (A):** A mild steel tension specimen has a cup and cone fracture at failure.
Reason (R): Mild steel is weak in shear and failure of the specimen in shear takes place at 45° to the direction of the applied tensile force.

27. A prismatic bar of uniform cross-sectional area of 5 cm^2 is subjected to axial loads as shown in the given figure.



Portion BC is subjected to an axial stress of
(a) 400 kg/cm^2 tension
(b) 2000 kg/cm^2 compression
(c) 1000 kg/cm^2 tension
(d) 600 kg/cm^2 tension

28. Two wires of equal length are suspended vertically at a distance of 40 cm as shown in the figure below. Their upper ends are fixed to the ceiling while their lower ends support a rigid horizontal bar which carries a central load of 1t midway between the wires. Details of the two wires are given below:



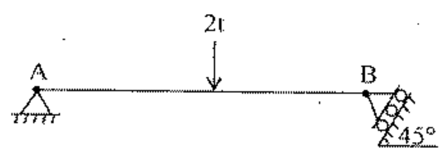
Wire No.	Area (cm^2)	Material	Modulus of Elasticity (kg/cm^2)	Elongation
1	4	Copper	1×10^6	Δ_c
2	2	Steel	2×10^6	Δ_s

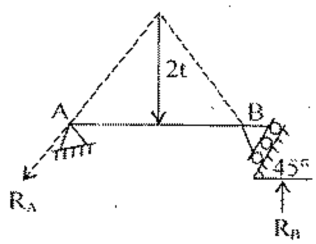
The ratio of the elongation of the two wires, Δ_c/Δ_s is

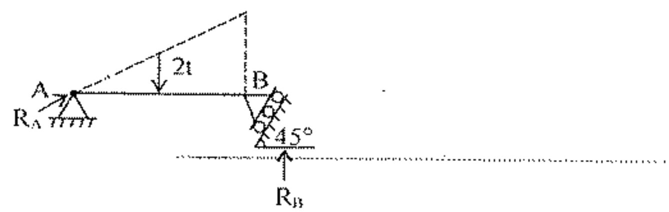
(a) 0.25 (b) 0.5

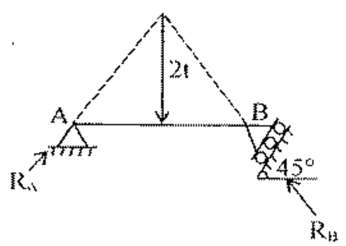
(c) 2 (d) 1

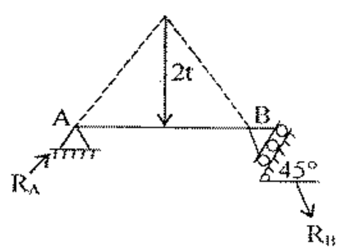
29. In a beam AB, support A is hinged and support B is on rollers as shown below. The directions of the reactions at A and B will be as in



(a)


(b)


(c)


(d)


DIRECTIONS :

The following items consists of two statements; one labelled as **'Assertion (A)'** and the other as **'Reason (R)'**. You are to examine these two statements carefully and select the answers to these items using the codes given below:

Codes:

(a) Both A and R are true and R is the correct explanation of A

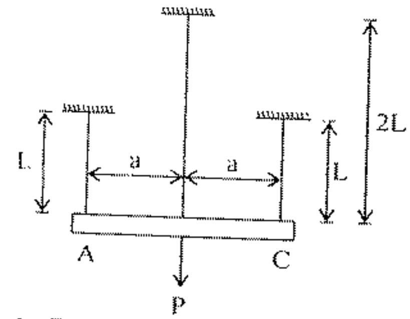
(b) Both A and R are true but R is not a correct explanation of A

(c) A is true but R is false

(d) A is false but R is true

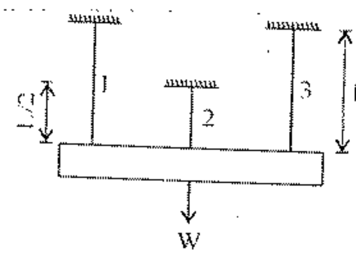
30. **Assertion (A):** The strain and stress system for a bar of length l subjected to an axial pull will be the same whether both ends of the bar are free or one end is fixed and the other end is free.
Reason (R): Rigid body displacements have no effect on the elastic deformations.

31. A rigid bar AC is supported by three rods of same material and of equal diameter. The bar AC is initially horizontal. A force P is applied such that the bar AC continues to remain horizontal. Forces in each of the shorter bars and in the longer bar are, respectively



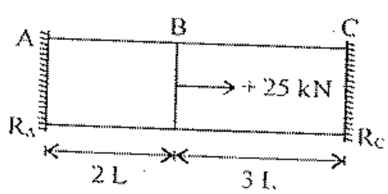
(a) 0.4P, 0.2P (b) 0.3P, 0.4P
(c) 0.2P, 0.6P (d) 0.5P, zero

32. Three wires of steel 1, 2 and 3, each having area A support a load W as shown in the figure below. What is the ratio between collapse load and the load corresponding to yielding of one of the wires?



(a) 3 : 1 (b) 3 : 2
(c) 3 : 2.5 (d) 3 : 3

33. A prismatic bar ABC is subjected to an axial load of 25 kN; the reactions R_A and R_C will be



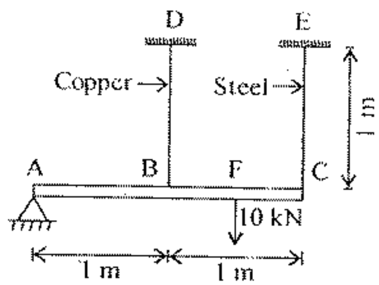
(a) $R_A = -10$ kN and $R_C = -15$ kN
(b) $R_A = 10$ kN and $R_C = -35$ kN
(c) $R_A = -15$ kN and $R_C = -10$ kN
(d) $R_A = 15$ kN and $R_C = -40$ kN

34. A mild steel rod tapers uniformly from 30 mm diameter to 12 mm diameter in a length of 300 mm. The rod is subjected to an axial load of 12 kN. $E = 2 \times 10^5$ N/mm². What is the extension of the rod in mm ?

(a) $\frac{4\pi}{5}$ (b) $\frac{2}{5\pi}$
(c) $\frac{\pi}{5}$ (d) $\frac{1}{5\pi}$

35. A short bar element of uniform cross-section subjected to concentrated axial forces at its two ends. The longitudinal stress distribution on the cross-section is uniform at
(a) all sections
(b) the two ends only
(c) the mid-section only
(d) sections reasonably away from the two ends of the bar

36. ABC is a rigid bar. It is hinged at A and suspended at B and C by two wires, BD and CE made of copper and steel respectively, as shown in the given figure. The bar carries a load of 10 kN at F, midway between B and C.



Given that

$A_c = 4\ cm^2$, $A_s = 2\ cm^2$, $E_c = 1 \times 10^5\ N/mm^2$,

$E_s = 2 \times 10^5\ N/mm^2$

Subscript 'c' and 's' stands for copper and steel. If the extensions in the steel and copper wires are Δ_s and Δ_c respectively, the ratio $\frac{\Delta_s}{\Delta_c}$ would be

(a) 1/4 (b) 4

(c) 2 (d) 1/2

37. The symmetry of stress tensor at a point in the body under equilibrium is obtained from

(a) conservation of mass

(b) force equilibrium equations

(c) moment equilibrium equations

(d) conservation of energy

38. Two bars one of material A and the other of material B of same length are tightly secured between two unyielding walls. Coefficient of thermal expansion of bar A is more than that of B. When temperature rises, the stresses induced are

(a) tension in both the materials

(b) tension in material A and compression in material B

(c) compression in material A and tension in material B

(d) compression in both the materials

39. The length, coefficient of thermal expansion and Young's modulus of bar A are twice that of bar B. If the temperature of both bars is increased by the same amount while preventing any expansion, then the ratio of stress developed in bar A to that in bar B will be

(a) 2 (b) 4

(c) 8 (d) 16

40. A copper bar of 25 cm length is fixed by means of supports at its ends. Supports can yield (total) by 0.01 cm. If the temperature of the bar is raised by 100°C, then the stress induced in the bar for α_c = 20×10^{-6} °C and $E_c = 1 \times 10^6\ kg/cm^2$ will be

(a) $2 \times 10^2\ kg/cm^2$ (b) $4 \times 10^2\ kg/cm^2$

(c) $8 \times 10^2\ kg/cm^2$ (d) $16 \times 10^2\ kg/cm^2$

41. A steel bar 300 mm long and having 24 mm diameter, is turned down to 18 mm diameter for one third of its length. It is heated 30°C above room temperature, clamped at both ends and then allowed to cool to room temperature. If the distance between the clamps is unchanged, the maximum stress in the bar ($\alpha = 12.5 \times 10^{-6}$ per °C and E = 200 GN/m^2) is

(a) 25 MN/m^2 (b) 50 MN/m^2
(c) 75 MN/m^2 (d) 105 MN/m^2

42. A square plate (a × a) rigidly held at three edges is free to move along the fourth edge. If temperature of the plate is raised by temperature 't', then the free expansion at the fourth edge will be (coefficient of thermal expansion of the material = α, modulus of elasticity of the material = E and its Poisson's ratio = μ)

(a) $a\alpha t(1 + \mu)$ (b) $a\alpha t$

(c) $a\left(\alpha t + \frac{\alpha t \mu}{E}\right)$ (d) $a\alpha t(1 - \mu)$

43. A straight wire 15 m long is subjected to tensile stress of 2000 kgf/cm^2. Coefficient of linear expansion for the material is 16.66×10^{-6}/°F. What is the temperature change to produce the same elongation as due to the 2000 kgf/cm^2 tensile stress in the material ($E = 1.5 \times 10^6$ kgf/cm^2)?

(a) 40°F (b) 80°F
(c) 120°F (d) 160°F

44. Match **List-I** (Elastic constant) with **List-II** (Definition) and select the correct answer using the codes given below the lists:

List-I

A. Young's modulus
B. Poisson's ratio
C. Bulk modulus
D. Rigidity modulus

List-II

1. Lateral strain to linear strain within elastic limit
2. Stress to strain within elastic limit
3. Shear stress to shear strain within elastic limit
4. Direct stress to corresponding volumetric strain

Codes:

	A	B	C	D
(a)	3	1	4	2
(b)	2	1	4	3
(c)	2	4	1	3
(d)	3	4	1	2

45. The bulk modulus of elasticity of a material is twice its modulus of rigidity. The Poisson's ratio of the material is

(a) 1/7 (b) 2/7
(c) 3/7 (d) 4/7

46. Given E as the Young's modulus of elasticity of a material, what can be the minimum value of its bulk modulus of elasticity ?

(a) $\frac{E}{2}$ (b) $\frac{E}{3}$

(c) $\frac{E}{4}$ (d) $\frac{E}{5}$

47. A mild steel bar of square cross-section 40 mm × 40 mm is 400 mm long. It is subjected to a longitudinal tensile stress of 440 N/mm^2 and lateral compressive stress of 200 N/mm^2 in perpendicular direc...

$E = 2 \times 10^5$ N/mm², $\mu = 0.3$. What is the approximate elongation of the bar in the longitudinal direction?

(a) 0.44 mm (b) 0.88 mm
(c) 0.22 mm (d) 1 mm

48. A bar of 40 mm diameter and 400 mm length is subjected to an axial load of 100 kN. It elongates by 0.150 mm and the diameter decreases by 0.005 mm. What is the Poisson's ratio of the material of the bar?

(a) 0.25 (b) 0.28
(c) 0.33 (d) 0.37

49. The material of a rubber balloon has a Poisson's ratio of 0.5. If uniform pressure is applied to blow the balloon, the volumetric strain of the material will be

(a) 0.50 (b) 0.25
(c) 0.20 (d) zero

50. A bar of elastic material is subjected to a direct compressive stress σ_1 in the longitudinal direction. Suitable lateral compressive stress σ_2 is applied along each of the other two lateral directions to limit the net strain in each of the lateral directions to half the magnitude of what it could be under σ_1 acting alone. If μ is the Poisson's ratio of the material, then the magnitude of σ_2 is

(a) $\frac{2(1-\mu)}{\mu}\sigma_1$ (b) $\frac{1}{2}\frac{(1-\mu)}{\mu}\sigma_1$

(c) $\frac{1}{2}\frac{\mu}{(1-\mu)}\sigma_1$ (d) $\frac{1}{2}\frac{\mu}{(1-\mu^2)}\sigma_1$

51. A bolt is threaded through a tubular sleeve and the nut is turned up just tight. Then the nut is turned further, the bolt being put in tension and the sleeve in compression.
The distance by which the nut is turned is equal to

(a) deformation in the bolt
(b) deformation in the sleeve
(c) difference in deformations in the bolt and the sleeve
(d) summation of deformations in the bolt and the sleeve

CHAPTER – 1
ANSWERS

1. (d)	14. (b)	27. (c)	40. (d)
2. (a)	15. (b)	28. (d)	41. (d)
3. (a)	16. (a)	29. (c)	42. (a)
4. (b)	17. (c)	30. (a)	43. (b)
5. (a)	18. (a)	31. (a)	44. (b)
6. (c)	19. (d)	32. (b)	45. (b)
7. (c)	20. (a)	33. (c)	46. (b)
8. (c)	21. (d)	34. (d)	47. (d)
9. (b)	22. (c)	35. (d)	48. (c)
10. (c)	23. (a)	36. (c)	49. (d)
11. (a)	24. (b)	37. (c)	50. (c)
12. (a)	25. (a)	38. (d)	51. (d)
13. (a)	26. (a)	39. (b)	

SOLUTION...

4. (b) For linear elastic and isotropic material the value of elastic constants E μ G are same in all directions. Hence no. of variables are 3(E, μ, G) but $G = \frac{E}{2(1+\mu)}$. Hence only two independent constants are there.

8. (c)

$$\mu_{steel} = 0.27 - 0.3$$
$$\mu_{Al} = 0.33$$
$$\mu_{cast\,iron} = 02 - 0.3$$

9. (b) Assertion is correct, however in a material even if stress is not directly proportional to strain, the strain energy will be obtained from the product of stress and strain. Hence (R) is also correct but not the explanation for (A).

11. (a) When torsion is applied to a specimen, max normal tensile stress develops at 45° to the axis of shaft. Thus as brittle material is weak in tension, the failure occurs on a plane at 45° to the axis of shaft

12. (a) Hard materials have high yield stress value.

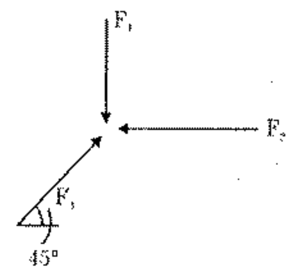
Soft material have low yield stress value. Tough material will absorb more energy before fracture *i.e.*, area under stress-strain curve upto fracture is more.

Looking at the options,

A will be 5
E will be 1
B seems to be 3
C seems to be 3
D will be 2

Confusion is between B & C, but as we are 100% sure about A, D and E. Hence we will go with option (a).

19. (d)



For equilibrium

$\sum F = 0$

$\sum M = 0$

as all the forces are passing through one point

$\sum M = 0$ satisfied. But from force equilibrium

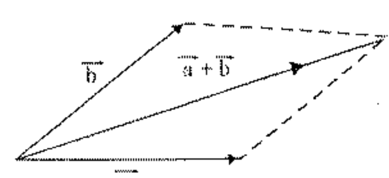
$$F_3 \sin 45 - F_1 = 0$$
$$F_3 \cos 5 - F_2 = 0$$
$$\frac{F_1}{F_2} = \tan 45° = 1$$

Which is not possible with F_1 = 10kN and F_2 = 20 kN for any value of F_3.

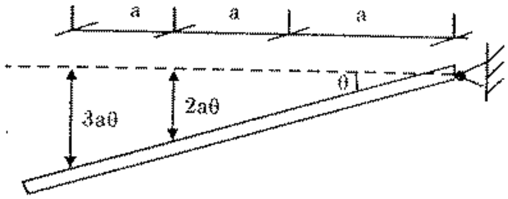
20. (a) To keep the body in equilibrium the force required to be applied should be equal and opposite to the resultant of $\overline{P_1}$ and $\overline{P_2}$.

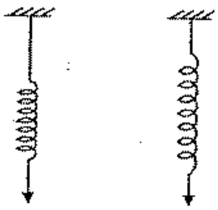
Hence $\overline{R}$ should be opposite to ($\overline{P_1} + \overline{P_2}$)

21. (d)



22. (c) The rigid beam will have rigid body movement (i.e. movement without bending).

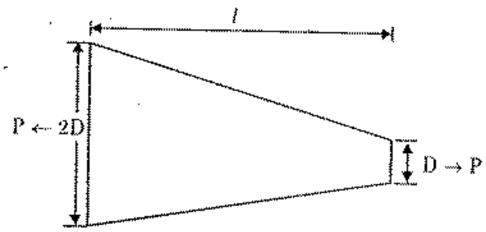




$R_A = K_A.3a\theta \quad R_B = K_B \times 2a\theta$

$$\Rightarrow \quad \frac{R_A}{R_B} = \frac{3K_A}{2K_B} = \frac{3 \times 2K}{2 \times K} = 3 = 3$$

23. (a)



$$\text{Actual extension of bar} = \frac{Pl}{\frac{\pi D \times 2D}{4} \times E}$$

$$= \frac{4Pl}{2\pi D^2 E}$$

By treating the bar as a bar of uniform diameter equal to

$$\frac{D+2D}{2} = 1.5\,D$$

$$\text{Extension} = \frac{Pl}{\frac{\pi}{4}(1.5D)^2 E} = \frac{4Pl}{2.25\pi D^2 E}$$

$$\Rightarrow \quad \%\text{ error} = \frac{\frac{4Pl}{\pi D^2 E}\left(\frac{1}{2} - \frac{1}{2.5}\right)}{\frac{1}{2} \times \frac{4Pl}{\pi D^2 E}} \times 100$$

$$= \left(1 - \frac{2}{2.5}\right) \times 100 = 11.4\%$$

24. (b) Fundamental quantities are those quantities which can measured by some instrument. Hence, strain is a fundamental quantity where as stress can only be derived as stress $= \frac{\text{Force}}{\text{Area}}$.

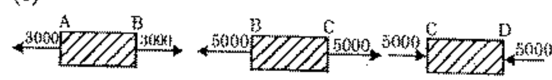
25. (a) We know that elastic deformation is given by strain (ε)

$$(\varepsilon_x) = \frac{\sigma_x}{E} - \frac{\mu\sigma_y}{E} - \frac{\mu\sigma_z}{E}$$

If all the stress are either tensile or compressive the magnitude of ε_x will not change if E and μ are assumed to be same in tension and compression.

26. (a) In mild steel tensile specimen the max shear stress occurs at 45° to the direction of applied tensile force.

27. (c)



Free body diagram is as shown above.

Check that net force at B = $\underline{-3000} + \underline{5000}$

$= \underline{2000}$

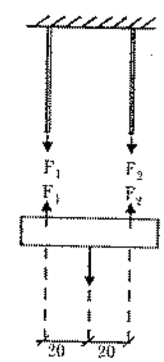
Check that net force at C = $\underline{5000} + \underline{5000}$

$= \underline{10000}$

Hence BC is subjected to axial stress of

$$\frac{5000\,\text{kg}}{5\,\text{cm}^2} = 1000\ \text{kg/cm}^2\ \text{(Tensile)}$$

28. (d) From equilibrium of forces.



$$SF = 0$$

$$F_1 + F_2 = 1$$

$$\Sigma M = 0$$

Taking moment about point of application of unit load

$$F_1 = F_2$$

$$\Delta_1 = \text{elongation of } 1 = \Delta_C$$

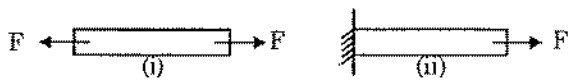
$$\Delta_2 = \text{elongation of } 2 = \Delta_S$$

$$\Rightarrow \quad \frac{\Delta_1}{\Delta_2} = \frac{\Delta_C}{\Delta_S} = \frac{\frac{F_1 l}{A_1 E_1}}{\frac{F_2 l}{A_2 E_2}} = \frac{A_2 E_2}{A_1 E_1}$$

$$\Rightarrow \quad \frac{\Delta_C}{\Delta_S} = \frac{2}{4} \times \frac{2 \times 10^6}{1 \times 10^6} = 1$$

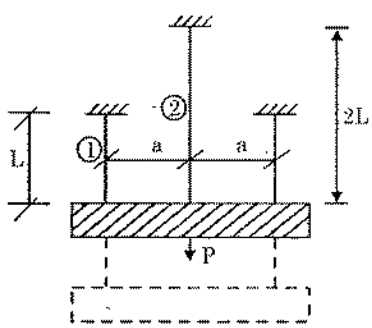
29. (c) Reaction at B will always be ⊥ to the support. Hence, (a) and (b) are eliminated $\overline{R_A} + \overline{R_B}$ will be equal and opposite to 2t force (from equilibrium of force). This is possible only in (c).

30. (a)



These two conditions will lead to same elastic deformation even if (i) is moving with uniform velocity. Thus rigid body displacement will have no effect on the elastic deformation.

31. (a)



Rigid bar will remain horizontal if elongation in all bars are same.

Let F_1 and F_2 are the forces in bars (1) and (2) respectively.

$$\Rightarrow \quad 2F_1 + F_2 = P \qquad \text{... (i)}$$

$$\text{also} \quad \frac{F_1 L}{AE} = \frac{F_2 \times 2L}{AE} \qquad \text{... (ii)}$$

$$\Rightarrow \quad F_1 = 2F_2$$

From (i) and (ii)

$$4F_2 + F_2 = P \Rightarrow F_2 = 0.2\,P$$

$$F_1 = 0.4\,P$$

Force in shorter bar = 0.4 P

Force in longer bar = 0.2 P

32. (b) When the system is in collapse stage, forces in each bar will be $F_y A$

$$\Rightarrow \quad W_{collapse} = 3F_y A$$

As there is symmetry about bar (2) *i.e.* bar (1) and (3) are similar and loading in these bars are also similar hence elongation of these bars will be same.

⇒ Bar (1), (2) and (3) will extend by the same amount.

$$\Rightarrow \quad 2F_1 + F_2 = W \qquad \text{... (i)}$$

$$\text{and} \quad \frac{F_1 l}{AE} = \frac{F_2 \times \frac{L}{2}}{AE}$$

$$\Rightarrow \quad F_2 = 2F_1 \qquad \text{... (ii)}$$

⇒ Force is more in bar (2).

⇒ Let it reaches the yield stress.

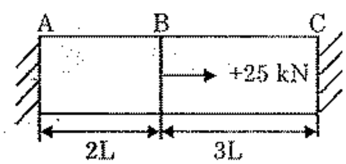
$$F_2 = f_y A$$

$$\Rightarrow \quad F_1 = f_y A/2$$

$$\Rightarrow \quad W_{yield} = 2F_1 + F_2 = 2f_y A$$

$$\Rightarrow \quad \frac{W_{collapse}}{W_{yield}} = \frac{3}{2}$$

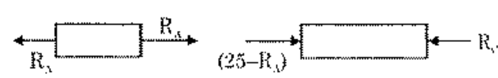
33. (c)



Here (+)ve means to the right

Here (–)ve means to the left.

The free diagram is as shown below.



Total elongation of bar between A & C = 0.

$$\Rightarrow \quad \frac{R_A \times 2L}{AE} - \frac{(25 - R_A)3L}{AE} = 0$$

$$\Rightarrow \quad 2R_A - 75 + 3R_A = 0$$

$$\Rightarrow \quad R_A = 15 \text{ kN}$$

$$R_C = 25\text{-}15 = 10 \text{ KN}$$

$$R_A = (-)15 \text{ kN [Acting to the left]}$$

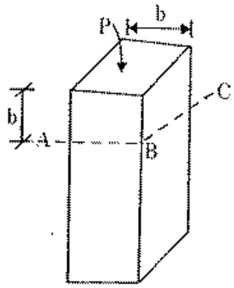
$$R_C = (-)\ 10 \text{ kN [Acting to the left]}$$

34. (d)

$$\text{Extension} = \frac{PL}{\frac{\pi d_1 d_2}{4}.E}$$

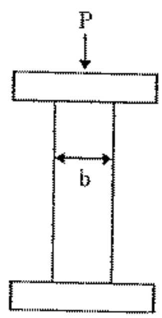
$$= \frac{12000 \times 4 \times 300}{\pi \times (30)(12) \times 2 \times 10^5} = \frac{1}{5\pi}$$

35. (d) From saint venant principle, if 'b' is the dia/width of the bar, stress in the bar will becomes uniform approx. at a distance 'b' from the point of application of force.

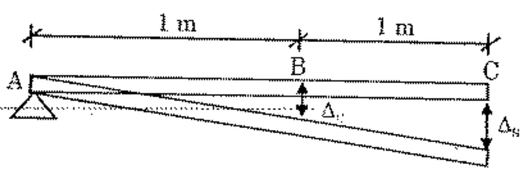


At Section ABC, stress becomes constant at P/A.

However if the load application is as shown in the fig. below, stress will become constant only from the starting.



36. (c)



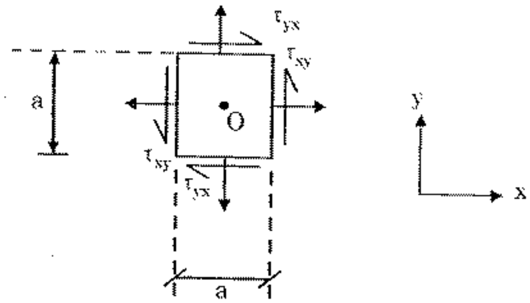
As the bar ABC is rigid, it will not bend hence the bar ABC remains straight. Thus by similar triangle.

$$\frac{\Delta_C}{1} = \frac{\Delta_S}{2}$$

$$\frac{\Delta_S}{\Delta_C} = 2$$

37. (c) Stress tensor $= \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$

When τ_{ij} = shear stress on plane 'i' in the direction of 'j'



From the moment equilibrium about 0.

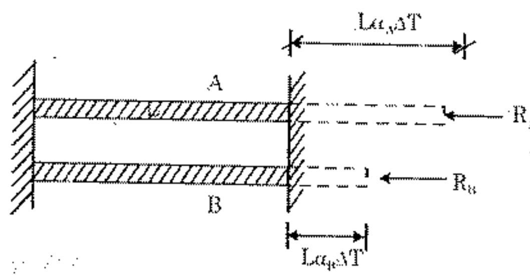
$$\tau_{xy} \times a = \tau_{yx} \times a$$

$$\Rightarrow \quad \tau_{xy} = \tau_{yx}$$

$\Rightarrow$ complimentary shear stresses are equal.

Thus the symmetry about the main diagonal is obtained from moment equilibrium.

38. (d)



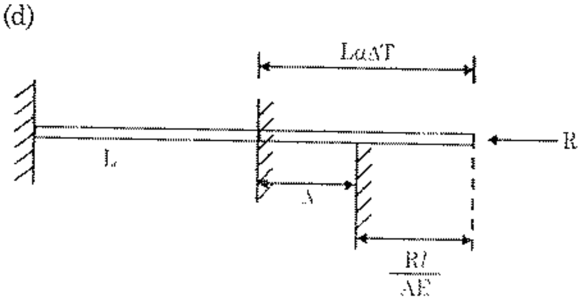
Due to temp rise, as seen in the diagram, forces generated are compressive only.

39. (b) Stress generated due to prevention of elongation is $E\alpha\Delta T$

$\Rightarrow$ Ratio of stress generated in A to B

$$\Rightarrow \frac{E_A\alpha_A\Delta T}{E_b\alpha_B\Delta T} = \frac{2E \times 2\alpha}{E\alpha} = \frac{4}{1}$$

40. (d)



$$\Rightarrow \quad (l\alpha\Delta T - \Delta) = \frac{Rl}{AE}$$

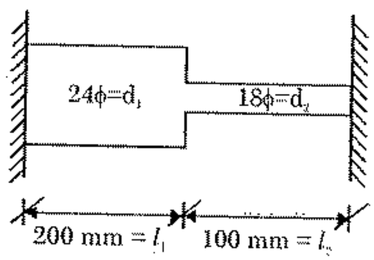
$$\Rightarrow \quad \text{Stress induced } \frac{R}{A} = \frac{E}{l}(l\alpha\Delta T - \Delta)$$

$$\Rightarrow \quad \text{Stress Induced} = E\alpha\Delta T - \frac{E\Delta}{l}$$

$$= 1\times10^{6}\times20\times10^{-6}\times100 - \frac{1\times10^{6}\times0.01}{25}$$

$$= 2000 - 400 = 1600 \text{ kg/cm}^2$$

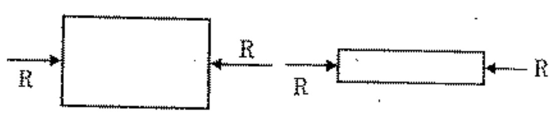
41. (d)



Total elongation due to temperature change

$$= l_1\alpha\Delta T + l_2\alpha\Delta T$$

Total contraction due to reaction load generated



$$\Rightarrow \quad \frac{Rl_1}{A_1E} + \frac{Rl_2}{A_2E} = (l_1 + l_2)\,\alpha\Delta T$$

$$\Rightarrow \quad \text{Max stress generated} = \frac{R}{A_2}$$

$$\Rightarrow \quad \frac{R}{A_2}\left[\frac{l_2}{E} + \frac{A_2 l_1}{A_1 E}\right] = (l_1 + l_2)\alpha\Delta T$$

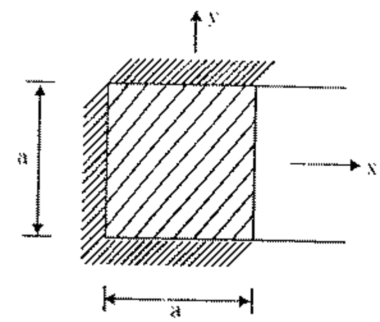
$$\Rightarrow \quad \frac{R}{A_2} = \frac{(l_1 + l_2)\alpha\Delta T}{\left(\frac{l_2}{E} + \frac{A_2}{A_1}\cdot\frac{l_1}{E}\right)}$$

$$= \frac{300\times12.5\times10^{-6}\times30\times E}{\left[100 + \frac{(18)^2}{(24)^2}\times200\right]}$$

$$= \frac{9000\times12.5\times10^{-6}\times2\times10^{5}}{\left(100 + \frac{(18)^2}{(24)^2}\times200\right)} \text{ N/mm}^2$$

$$= 105.88 \text{ N/mm}^2$$

42. (a)



Free expansion of 4th edge is along x-axis.

$$= \left[\frac{\sigma_x}{E} - \frac{\mu\sigma_y}{E} - \frac{\mu\sigma_z}{E}\right]\cdot a = \quad [\text{strain along } x - \mu\frac{\sigma_y}{E} - \frac{\mu\sigma_z}{E}]a$$

$$= \quad a\alpha\Delta T - \frac{\mu}{E}\left[-E\alpha\Delta t\right]\times a - 0$$

$$= \quad a\alpha\Delta T + a\mu\alpha\Delta T$$

$$= \quad a\alpha\Delta T(1+\mu)$$

43. (b) Elongation due to tensile stress $= \frac{\sigma}{E}L$

Elongation due to temp. change $= L\alpha\Delta T$

$$\Rightarrow \quad \frac{\sigma L}{E} = L\alpha\Delta T$$

$$\Rightarrow \quad \Delta T = \frac{\sigma}{E\alpha}$$

$$= \frac{2000}{1.5\times10^{6}\times16.66\times10^{-6}}$$

$$= \frac{2000}{1.5\times\frac{50}{3}} = 80°F$$

44. (b)

45. (b)

46. (b) Bulk modulus $K = \frac{E}{3(1-2\mu)}$

For K to be min, $\mu = 0$

$$\Rightarrow \quad K_{min} = \frac{E}{3}$$

47. (d) Elongation of the bar in longitudinal direction = Strain in longitudinal direction × length

$$= \left(\frac{\sigma_x}{E} - \frac{\mu\sigma_y}{E}\right) \times L$$

$$= \frac{440 - 0.3 \times (-200)}{2 \times 10^5} \times 400$$

$$= \frac{500 \times 400}{2 \times 10^5} = 1 \text{ mm}$$

48. (c)

$$\mu = \frac{-\text{Lateral strain}}{\text{Longitudinal strain}}$$

$$= \frac{-(-0.005)/40}{0.15/400} = \frac{10}{30} = 0.33$$

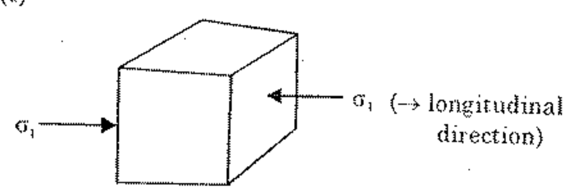
49. (d) Volumetric strain = $\varepsilon_x + \varepsilon_y + \varepsilon_z$

$$= \frac{\sigma_x + \sigma_y + \sigma_z}{E}(1 - 2\mu)$$

If $\mu = 0.5$

$\Rightarrow$ volumetric strain $\left(\frac{\Delta V}{V}\right) = 0$

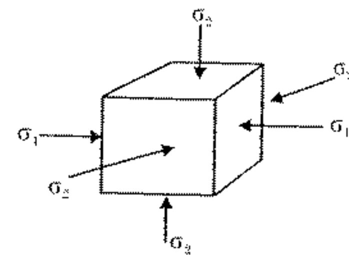
50. (c)



When only σ_1 acting, strain in lateral direction.

$$\varepsilon_a = \frac{-\mu(-\sigma_1)}{E} = \frac{\mu\sigma_1}{E}$$

When σ_1 and σ_2, σ_2 is acting as shown below.



Strain in lateral direction

$$\varepsilon_b = -\frac{\sigma_2}{E} - \mu\left(-\frac{\sigma_2}{E}\right)\mu\left(-\frac{\sigma_1}{E}\right)$$

$$\varepsilon_b = \frac{\mu\sigma_2 + \mu\sigma_1 - \sigma_2}{E}$$

Since $\varepsilon_b = \frac{1}{2}\varepsilon_a$

$$\Rightarrow \frac{\mu\sigma_1}{E} + \frac{\mu\sigma_2}{E} - \frac{\sigma_2}{E} = \frac{\mu\sigma_1}{2E}$$

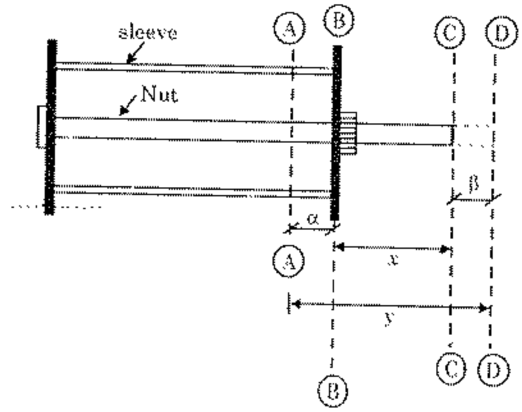
$$\mu\sigma_1 - \frac{\mu\sigma_1}{2} = \sigma_2 - \mu\sigma_2$$

$$\sigma_2 = \frac{\mu\,\sigma_1}{2(1-\mu)}$$

51. (d) Final position of bolt is at sec A-A when bolt is tightened the sleeve is compressed whereas bolt is elongated

$\Rightarrow$ Movement of nut = $(y - x) = \alpha + \beta$

= compression of sleeve + extension of bolt



2

Shear Force and Bending Moment

STRUCTURAL MEMBERS

- A beam is a structural member of sufficient length compared to lateral dimensions.
- Ties, struts, shafts and beams are all one-dimensional or line-elements, where the length is much greater than the depth or width, and have different names depending upon the main action they are designed to resist.
- Thus, ties and struts resist uniaxial tension or compression, shafts resist torque and beams resist bending moments (and shear forces). Beams may be concrete, steel or even composite beam, having any type of sections such as angles, channels, I-section, rectangle, square, hat section etc.

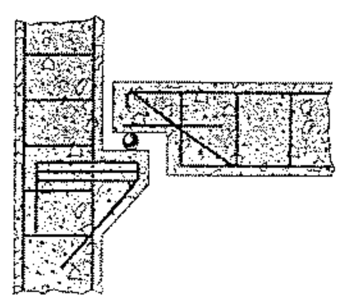
SPAN OF BEAM

- The clear horizontal distance between the supports is called the clear span of the beam.
- The horizontal distance between the centres of the end bearings is called the effective span of the beam.
- If the intensity of the bearing reaction is not uniform, the effective span is the horizontal distance between the lines of action of the end reactions.

TYPES OF SUPPORT

The supports can be classified into following categories:

(a) A simple or free support/Roller support/Rocker support
(b) Hinged or pinned support,
(c) A built-in or fixed or encastre support
(d) Slider support
(e) Link support



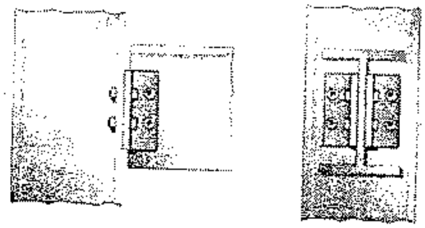
Typical "roller-supported" connection (concrete)

A Simple or Free Support/Roller Support/Rocker Support

It is a support on which beam rests freely. A roller support is the simplest and gives only one reaction, because only one deflection is restrained.

Hinged or Pinned Support

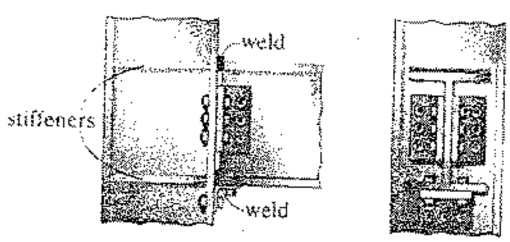
In this case, the beam is hinged or pinned to the support. A pinned or hinged support gives only two reactions, one against vertical movement and another against horizontal movements (say R_x and R_y) but offers no resistance to the angular rotation of the beam at the hinge.



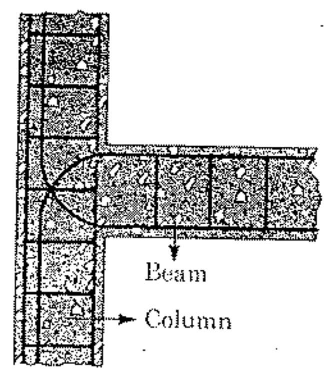
Typical "pin-supported" connection (steel construction)

A Built-in or Fixed or Encastre Support

It is a support which restrains complete movement of the beam both in position as well as direction. The support gives all the three relevant reactions (say R_x, R_y, and M_z), i.e. the reactions in x and y directions and fixing moment M_z. In other words, the fixed support offers resistance against translation in both the directions and also against the rotation.

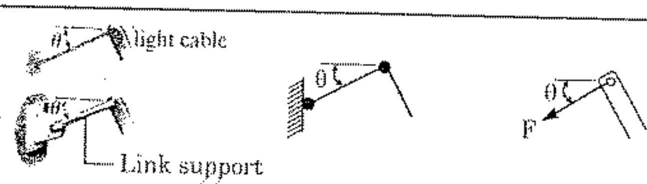


Typical "fixed-supported" connection (steel construction)

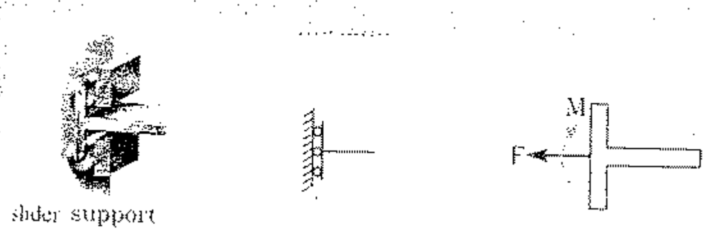


Typical "fixed-supported" connection (concrete)

Other Supports



One reaction. The reaction is a force that acts in the direction of the cable or link.



Two unknowns reactions are a force and a moment. There can not be reaction parallel to roller because the movement is free

TYPES OF BEAMS

Depending on the type and number of supports, the beams are divided into two categories:

(i) Statically determinate beam, and

(ii) Statically indeterminate beam.

Statically Determinate Beam (2D)

A beam is said to be statically determinate, when it can be analysed using three equations of static equilibrium i.e. $\Sigma F_x = 0$, $\Sigma F_y = 0$ and $\Sigma M_z = 0$, where

ΣF_x = algebraic sum of horizontal forces

ΣF_y = algebraic sum of vertical forces

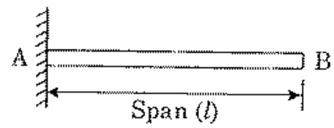
ΣM_z = algebraic sum of moments of all the forces at a point about z-axis.

Examples of statically determinate beams are as follows:

(i) Cantilevers, (ii) Simply supported beams, and (iii) Overhanging beams.

Cantilever Beams

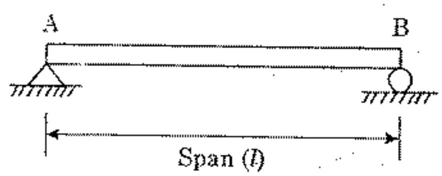
A beam fixed at one end and free at the other end is known as cantilever.



Cantilever beams

Simply Supported Beam

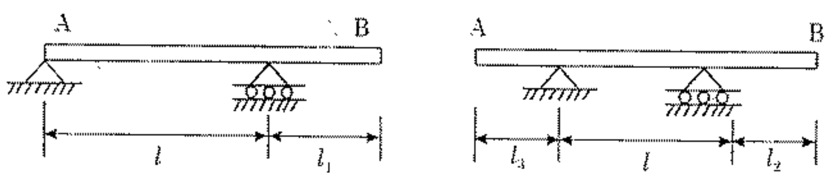
A beam supported or resting freely on the walls or columns at its both ends is known as simply supported beam. The support may be hinged or roller support.



Simply supported beam

Overhanging Beam

A beam having its end portion (or portions) extended in the form of a cantilever beyond the support is known as overhanging beam. A beam may be overhanging on one side or on both sides.



Overhanging beam

l_1, l_2, l_3 are the lengths of overhanging portions.

Statically Indeterminate Beam

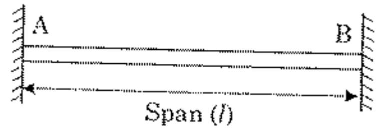
When the number of unknown reactions or stress components exceed the number of static equilibrium equations available, (i.e. $\Sigma F_x = 0$, $\Sigma F_y = 0$ and $\Sigma M_z = 0$) the beam is said to be statically indeterminate. This means, that the three equilibrium equations are not adequate to analyse the beam. In this case additional equations of compatibility are required to analyse the beam.

Examples are as follows:

(i) Fixed beams, (ii) Propped cantilevers, and (iii) Continuous beams.

Fixed Beam

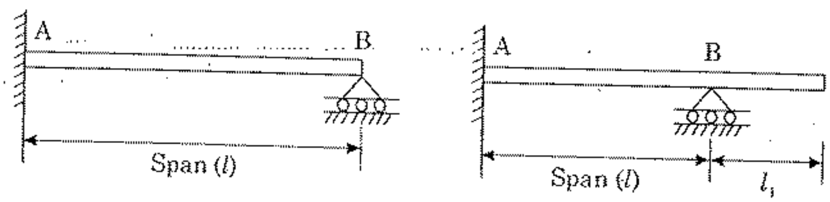
A beam rigidly fixed at its both ends like beam-column rigid/monolithic construction or built-in walls is known as rigidly fixed beam or a built-in beam.



Fixed beam

Propped Cantilever

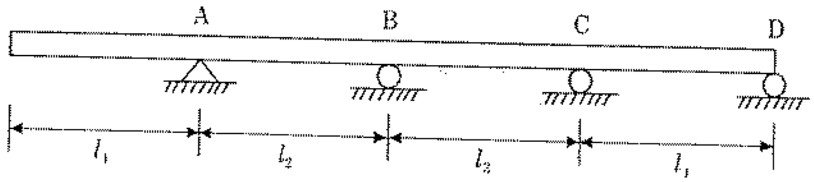
If a cantilever beam is supported by a simple support at the free end or in between, is called propped cantilever. It may or may not be having overhanging portion.



Propped cantilever

Continuous Beam

A beam which is provided with more than two supports is called a continuous beam. It may be noted that a continuous beam may or may not be an overhanging beam.



Continuous beam

l_1 = length of the overhanging portion, and

l_2 = span AB; l_3 = span BC; l_4 = span CD

TYPES OF LOADINGS

A beam may be subjected to the following types of loads.

(i) Concentrated or point load

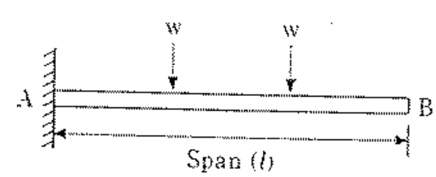
(ii) Uniformly distributed load

(iii) Uniformly varying load

(iv) Couples.

Concentrated or Point Load

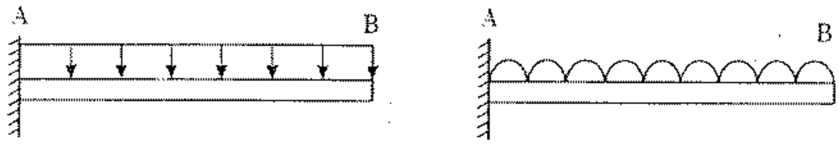
A load acting at a point on a beam is known as concentrated or a point load. In practice, it is not possible to apply a load at a point, i.e. at a mathematical point. as it will be distributed over a small area. But this very small area, as compared to the length of the beam, is negligible.



Concentrated or point load

Uniformly Distributed Load

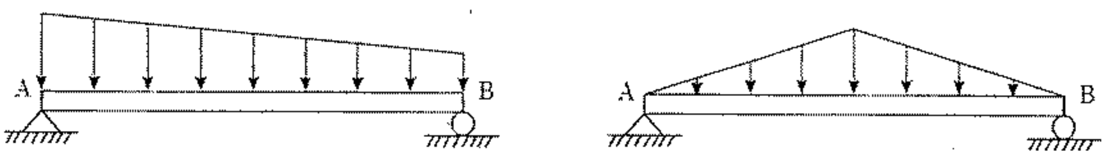
A load which is spread over a beam in such a manner that each unit length is loaded to the same extent, is known as uniformly distributed load (briefly written as u.d.l.).



Uniformly distributed load

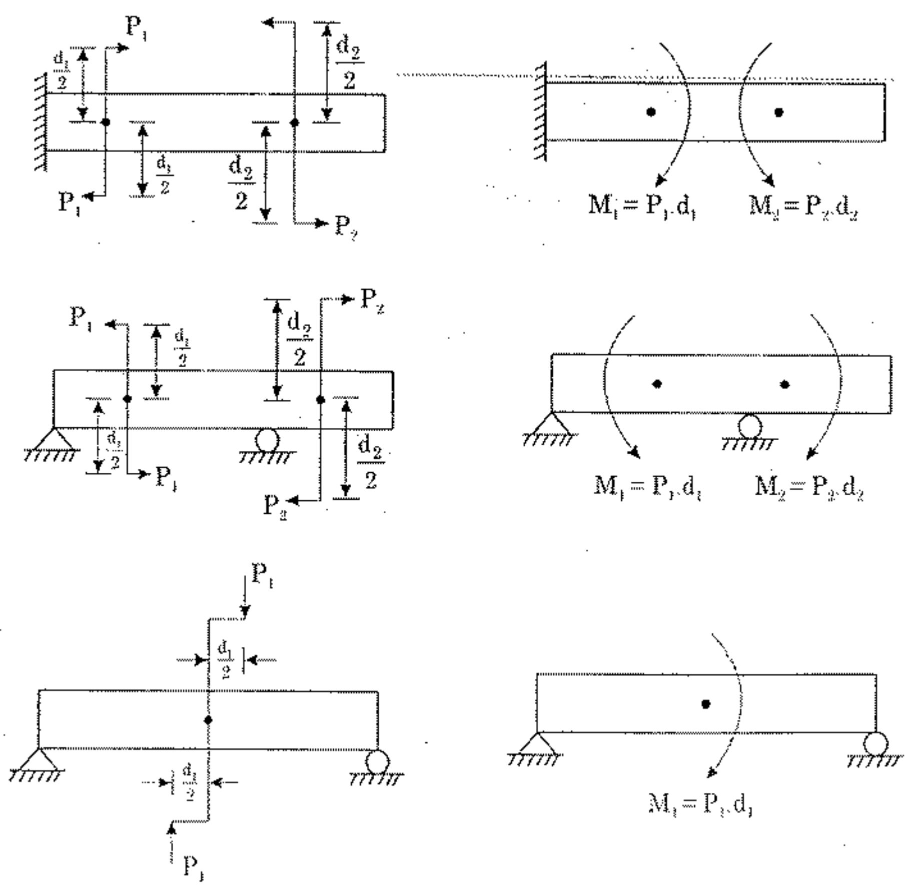
Uniformly Varying Load

A load which is spread over a beam in such a manner that its extent varies uniformly on each unit length is known as uniformly varying load.



Uniformly varying load

Beams Subjected to Couples

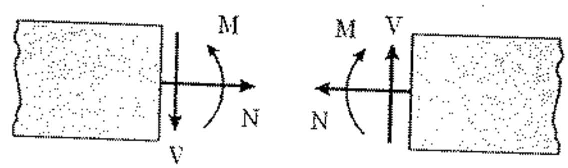


Beams subjected to couples

A beam may carry anyone of the above load systems or a combination of two or more systems at a time. Loads may be static or dynamic. Static loads are those which are applied gradually and do not change their magnitude, direction or point of application with time. Dynamic loads are those which vary in time with speed.

Significance of Bending Moment and Shear Force Diagrams

- At any point in a loaded member of structure when a section is cut perpendicular to the axis of the member, we get *internal loading.*
- For Coplaner condition, i.e. when loads are in single plane, bending moment, shear force and axial thrust are the internal loading.

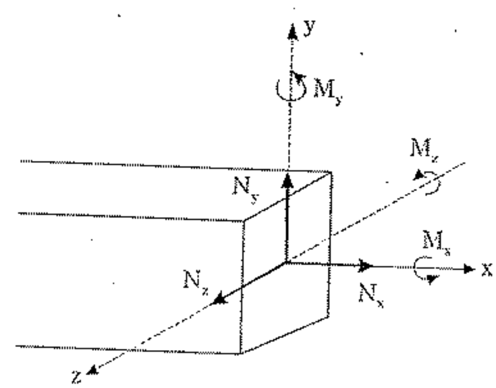


Internal loadings

M = *Bending moment* at a section
V = *Shear force* at the section
N = *Axial thrust* at the section

Note: For *out of plane loading, internal forces are*

N_x	N_y	N_z	M_x	M_y	M_z
↓	↓	↓	↓	↓	↓
Axial thrust	Shear	Shear	Torsion	Bending	Bending

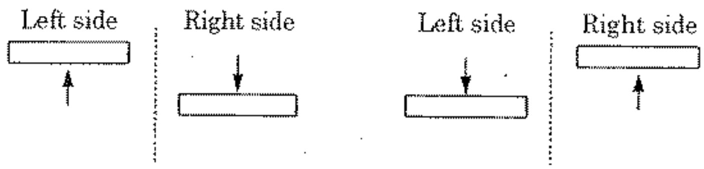


If internal loadings are known, *stresses* at the section can be calculated, (like bending stress = $\frac{M \cdot y}{I}$, M = bending moment). Thus, to check the safety of a chosen design section, knowledge of internal loadings at various sections is essential. Hence bending moment and shear force at all sections is found out and their variation is plotted along the length of beam. The diagrams so obtained are called bending moment diagram and shear force diagram.

DEFINITION OF SHEAR FORCE AT A SECTION

Shear force at a section is the resultant of all transverse forces to the right or left of the section.

> *SF at the section is (+) ve if resultant of all transverse forces to the right of the section is downward or resultant of transverse forces to the left of section is upwards.*

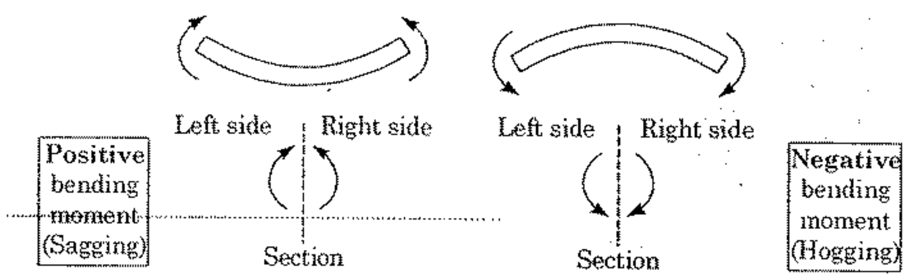


Positive shear force   Negative shear force

DEFINITION OF BENDING MOMENT

Bending moment at a section is the resultant moment at the section due to all the transverse forces either to the left or right of the section.

At sections, where the bending moment is such that it tends to bend the beam at that point to a curvature having concavity at the top as shown in figure is taken as positive, and where the bending moment is such that it tends to bend the beam at that point to a curvature having convexity at the top as shown in figure is taken as negative.



Sagging movement   *Hogging movement*

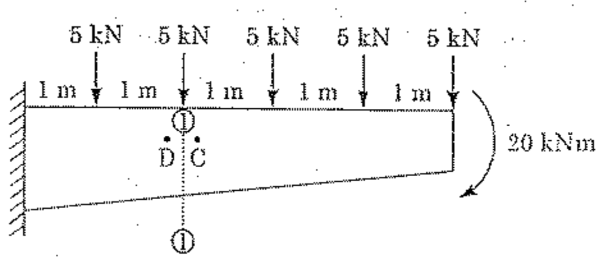
> ***Note:*** *There is a difference between bending moment at a section and moment at a point. Bending moment is as discribed above whereas moment at a point is the summation of moment due to all loading on the beam produced at that point. For equilibrium this summation is always zero.*

DEFINITION OF AXIAL THRUST

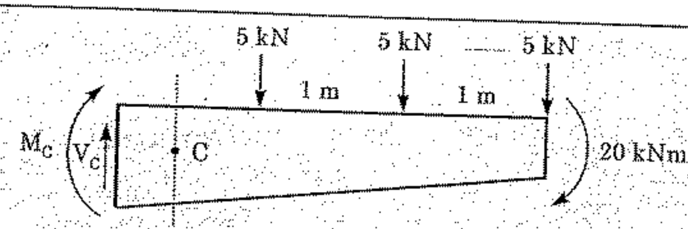
Axial thrust is the force acting along the longitudinal axis of the members. Axial thrust is (+) ve if it tries to elongate the member.

Example 1

Find bending moment and shear force at C and D.



Sol: C and D are points just to the right and left of section (1)-(1)



$\boxed{V_C = 15 \text{ kN}}$ ⇒ (+) ve)

Summation of all forces to the right of section at C is downward. The shear force V_C to the left of the section is upwards.

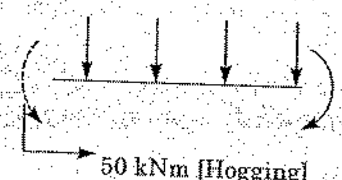
⇒ Shear force is (+) ve

Taking clockwise moment of all forces about C, $\Sigma M_{\text{about C}} = 0$

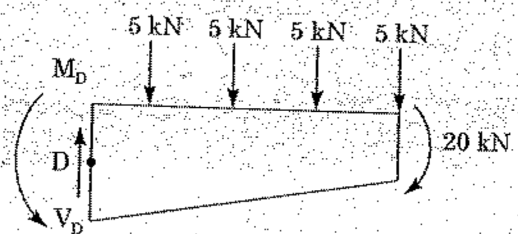
$$\Rightarrow (5 \times 3) + (5 \times 2) + (5 \times 1) + M_C + 20 = 0$$

$$\Rightarrow \boxed{M_C = -50 \text{ kNm}}$$

(–) ve sign of M_C implies that the direction of M_C is opposite to that assumed in the figure above. Hence



Similarly,



$$\boxed{V_D = 20 \text{ kN}}$$

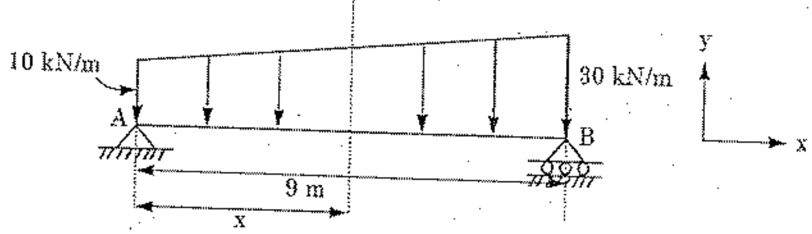
$$\Rightarrow \boxed{M_D = -50 \text{ kNm}} \text{ [Hogging]}$$

Note:

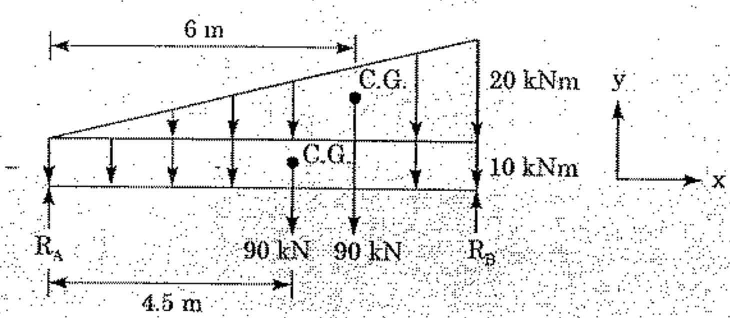
- *Shear force is different on either side of concentrated load.*
- *BM remains the same on either side of concentrated load.*

Example 2

Find BM and SF at a distance 'x' from end 'A'.



Sol: *Step 1: Calculate reactions*



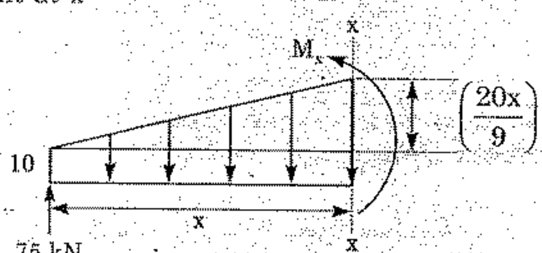
$\Sigma F_y = 0, \quad \Rightarrow \quad R_A + R_B = 10 \times 9 + \frac{1}{2} \times 9 \times 20 = 180 \text{ kN}$

$\Sigma M_A = 0, \quad \Rightarrow \quad R_B \times 9 - 90 \times 6 - 90 \times 4.5 = 0$

$\Rightarrow \quad R_B = 60 + 45 = 105 \text{ kN}$

$R_A = 75 \text{ kN}$

M_x = Sagging moment at x



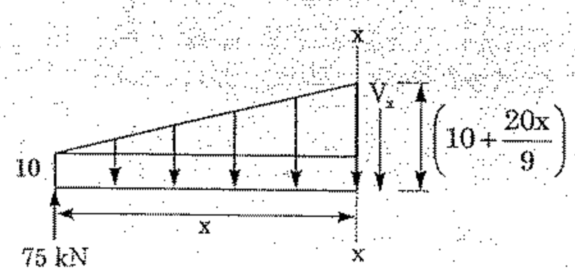
Sagging moment from left at section x-x

$$= 75x - (10x)\left(\frac{x}{2}\right) - \left(\frac{20x}{9} \times \frac{1}{2} \times x\right) \times \frac{x}{3}$$

Note: *Upward force to the right or left of a section produces sagging moment at the section.*

$\boxed{M_x = 75x - 5x^2 - \frac{10x^3}{27}}$ when this relation is plotted we get BMD

SF at x:



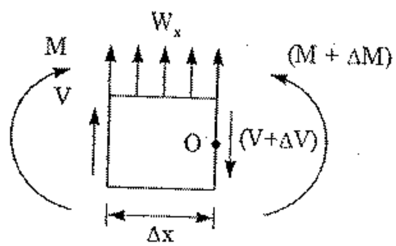
Total upward force to the left of section

$$V_x = 75 - 10x - \frac{1}{2} \times \frac{20x}{9} \times x$$

$\boxed{V_x = 75 - 10x - \frac{10x^2}{9}}$ when this relation is plotted we get SFD

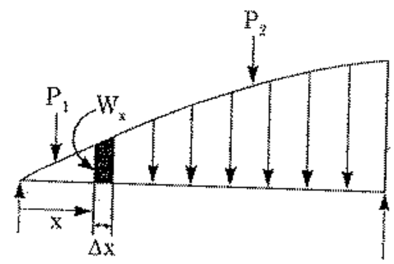
> ***Note:*** *We will always proceed from left to right i.e. towards increasing value of x, for plotting. BMD and SFD.*

RELATION BETWEEN BM, SF AND LOADING



(i)

Sign convention: upward loading (+) ve



(ii)

For equilibrium $\Sigma F = 0$,

$$V + W_x \cdot \Delta x - (V + \Delta V) = 0$$

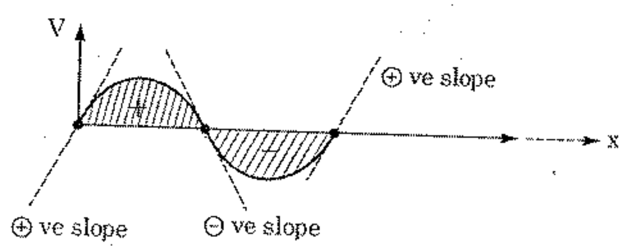
$$\Rightarrow \quad \frac{\Delta V}{\Delta x} = W_x$$

for very small value of x

$$\Rightarrow \quad \boxed{\frac{dV}{dx} = W_x} \qquad \text{(i)}$$

⇒ Slope of shear force diagram = Intensity of distributed load

Slope of SFD



SFD

If slope of SFD is positive, this implies that load intensity at that point is (+) ve i.e. upwards and if slope of SFD is (–) ve, this implies that load intensity at that point is (–) i.e. downwards.

For equilibrium, $\Sigma M_0 = 0$

$$M + V\Delta x + W_x \cdot \Delta x \cdot \frac{\Delta x}{2} - (M + \Delta M) = 0$$

$$\Delta M = V\Delta x + W_x \frac{(\Delta x)^2}{2}$$

$$\lim_{\Delta x \to 0} \frac{\Delta M}{\Delta x} = \lim_{\Delta x \to 0}\left(V + W_x \frac{(\Delta x)}{2}\right)$$

$$\Rightarrow \quad \boxed{\frac{dM}{dx} = V} \qquad \text{(ii)}$$

Slope of bending moment diagram at any section = Shear force at that section

The above results (i) and (ii) can also be expressed as:

$$\Delta V = \int W_x dx \quad \text{......... (iii)}$$

$\Rightarrow$ Change in shear, $(V_{final} - V_{initial})$ = Area under distributed loading diagram between those two points.

Similarly,

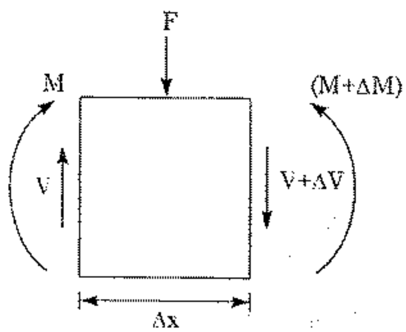
$$\Delta M = \int V dx \quad \text{......... (iv)}$$

Change in BM diagram between two section = $(M_{final} - M_{initial})$ = Area under SF diagram between those two sections (i)–(ii).

The above equations can be grouped into two sets: Set A→{(i), (ii) and (iii)}; Set B→{(ii), (iii) and (iv)}

Set A doesnot hold true in case of point load and Set B does not hold true in case of concentrated moments. Hence we will consider the effects of point load and concentrated moments separately.

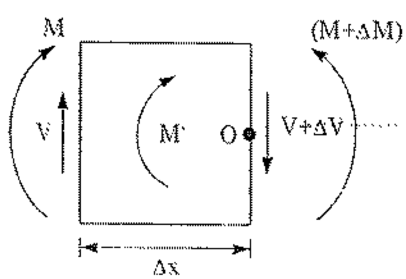
EFFECT OF CONCENTRATED LOAD AND CONCENTRATED MOMENT



$$\Sigma F = 0, \Rightarrow \Delta V + F = 0$$

$$\Rightarrow \Delta V = -F$$

- When F acts downward at a section, SF drops by amount F at that section
- When F acts upward at a section, S.F. jumps up by amount 'F' at that section



$\Sigma M_0 = 0 \Rightarrow M + V\Delta x + M' - M - \Delta M = 0$ {M' = concentrated moment of section}

$$\Rightarrow \boxed{\Delta M = M' + V\Delta x}$$

For very small Δx i.e. when $\Delta x \to 0$

$$\boxed{\Delta M = M'}$$

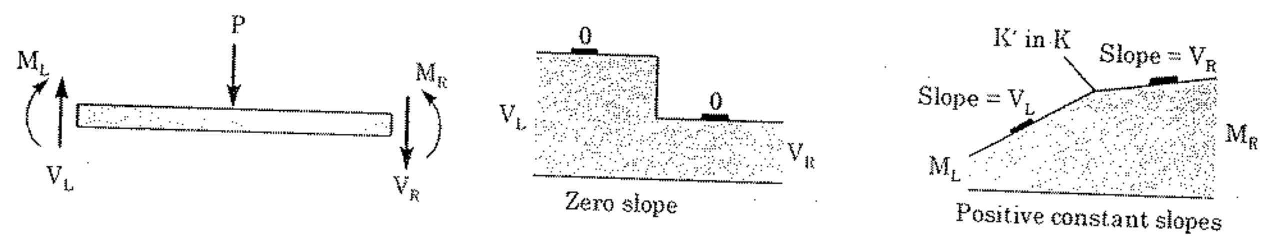
$$\boxed{\int dM = M' + \int V dx}$$

$\boxed{M_2 - M_1 = M' + \int V dx}$ i.e. ΔM = concentrated moment + area under SFD

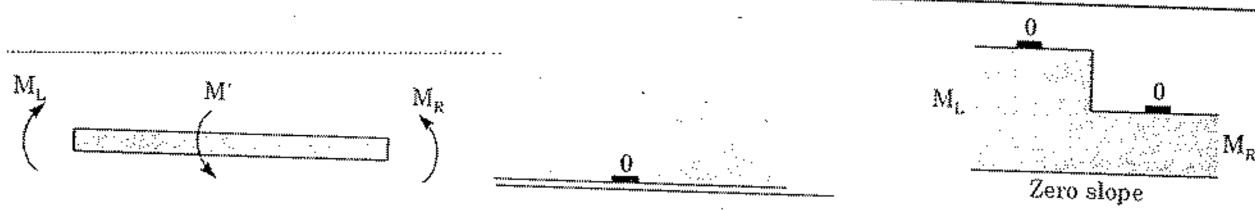
- Due to clockwise external couple M′, BMD Jumps up by M′
- Due to counterclockwise external couple M′, BMD drops down by M′

The conclusion derived above can be used to plot BMD and SFD as discussed below:

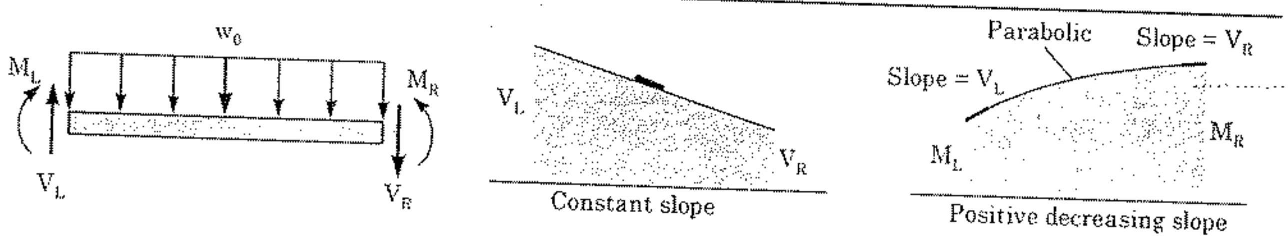
Loading	**Shear diagram,** $\frac{dV}{dx} = w$	**Moment diagram,** $\frac{dM}{dx} = V$



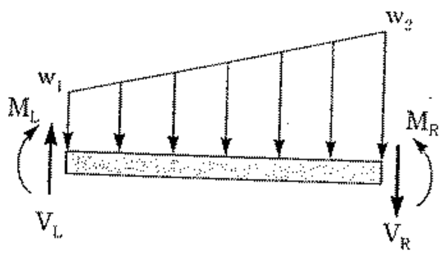
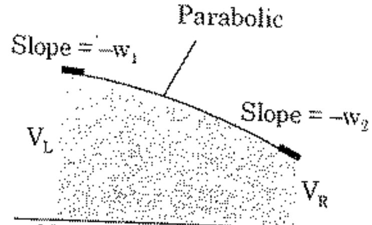
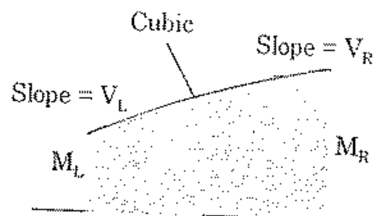
- SF on left side is (+) V_L and that on right side is (+) V_R ⇒ BMD slope on left side is (+) V_L and on right side is (+) ve V_R
- Point load causes sudden change in SFD and kink in BMD
- For point load in any portion of beam, SFD = zero slope curve and BMD = linear curve



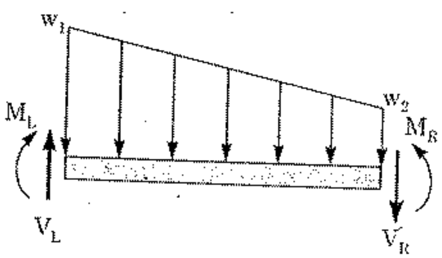
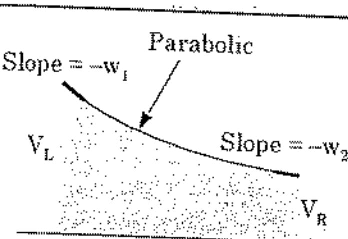
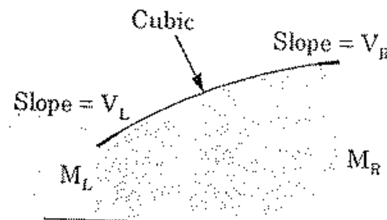
- SF = 0 ⇒ BM slope = 0
- Concentrated anticlockwise moment causes drop in BMD.
- Pure bending on beam causes zero slope BMD.



- Loading is (–) ve and constant ⇒ SF slope is (–) and constant
- SF is (+) ve decreasing ⇒ Bending moment slope is (+) ve decreasing
- If load intensity is udl ⇒ SFD is linear → BMD is parabolic
- Slope of BMD at any section is equal to SFD ordinate at that section
- Slope of SFD at any section is equal to load intensity at that section
- If load intensity is n-degree curve, SFD will be (n + 1) degree curve and BMD will be (n + 2) degree curve.

Loading	Shear diagram, $\frac{dV}{dx} = w$	Moment diagram, $\frac{dM}{dx} = V$
		
	Negative increasing slope	Positive decreasing slope

- Loading is (–)ve and increasing ⇒ SFD slope is (–)ve and increasing
- SF is (+) ve and decreasing ⇒ BMD slope is (+) ve and decreasing
- If load intensity is uvl (uniformly varied load) SFD is parabolic and BMD is cubic

		
	Negative decreasing slope	Positive decreasing slope

SF is (+) ve and decreasing ⇒ BMD slope is (+) ve and decreasing

Loading is (–) ve and decreasing ⇒ SFD slope is (–) ve and decreasing.

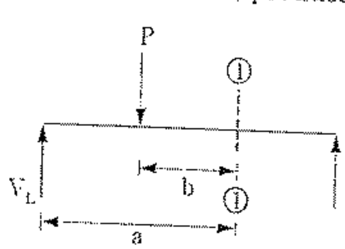
ADDITIONAL POINT

1. When V = 0 i.e. $\frac{dM}{dx} = 0$, B.M. is max/min. Also when shear force changes sign, BM is max at that section.
2. **Point of Contraflexure:** Point where BMD changes sign [(BMD = 0) at this section].
   Point of Inflection: Point where deflected shape changes curvature [BMD = 0 at this section].

Curvature of deflected shape corresponding to (+) ve BM

Curvature of deflected shape corresponding to (–) ve BM

3. Upward acting forces to the right or left of a section, produces (+) ve BM at the section

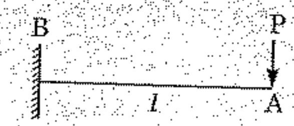


$$\boxed{BM_{(1)-(1)} = V_L a - P \times b}$$

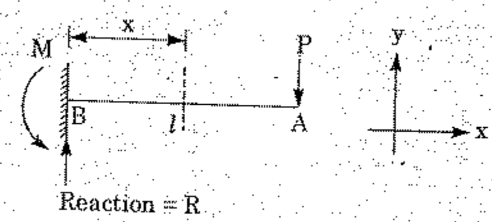
We do not need to bother about the sign of BM at section (1)-(1) if moment due to upward load is taken as (+) ve and that due to downward load is taken as (–) ve. The net value will have the sign as per our specification.

Example 3

Draw bending moment diagram and shear force diagram for the following beams.



Sol:



Step 1: Calculate reaction

$\Sigma F_y = 0 \Rightarrow R - P = 0$

$\Sigma M_B \circlearrowright = 0 \Rightarrow Pl - M = 0 \Rightarrow M = Pl$ (Hogging)

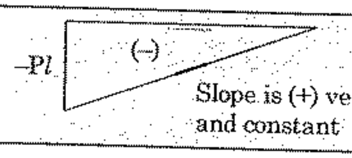
Step 2: Analysis of shear force

- At B, loading is upward [point load]
  $\Rightarrow$ SF jumps up by, $R = P$
- B to A, load intensity $= 0 \Rightarrow \dfrac{dV}{dx} = w = 0$
  $\Rightarrow$ Slope of SFD = 0
- At A, there is a downward point load (P)
  $\Rightarrow$ SF jumps down by P
  $\Rightarrow$ SFD is P [(+) ve] P

Note: *SF plotted above base line (horizontal) is (+) ve.*

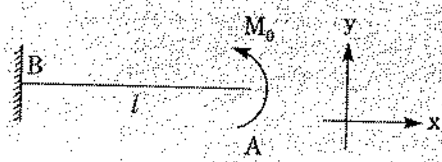
Step 3: Analysis of bending moment

- BM at B = $-Pl$ (as per our sig convention, Hogging = (–) ve)
- B to A, shear force is (+) ve and constant
  $\Rightarrow$ Slope of BMD is (+) ve and constant.
- BM at A = M_A, BM at B = $M_B = -Pl$
  $\Rightarrow \Delta M = M_A - M_B$ = Area under SFD = Pl
  where M_A = final BM, M_B = initial BM
  $\Rightarrow M_A - (-Pl) = Pl$
  $\Rightarrow M_A = 0$
  $\Rightarrow$ BMD is as shown below



Example 4

Draw the SFD and BMD diagram.



Sol:

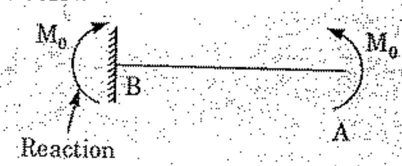
Step 1: Calculate reaction

$$\Sigma F_y = 0 \quad \Rightarrow \quad R = 0$$

$$\Sigma M_B \circlearrowleft = 0 \quad \Rightarrow \quad -M_0 - M = 0$$

$$\Rightarrow \quad M = -M_0$$

Hence the net reaction is as shown below



Step 2: Analysis of SFD

As there is no loading force anywhere

$\Rightarrow$ SFD = 0

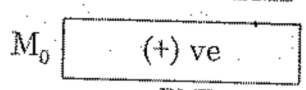
Step 3: Analysis of BMD

At B: $M = M_0$ [(+) ve means Sagging)]

B to A: $\dfrac{dM}{dx} = V = 0 \Rightarrow$ BM is constant

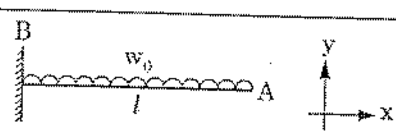
At A: Anticlockwise concentrated moment $\Rightarrow$ BMD drops by M_0

BMD is as shown below $\Rightarrow$



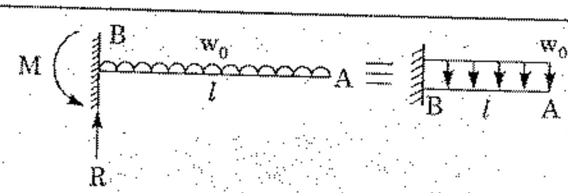
BMD

Example 5



(Cantilever loaded with udl)

Step 1: Find reactions:



$\Sigma F_y = 0 \quad \Rightarrow \quad R - w_0 l = 0$

$\Rightarrow \quad R = w_0 l$

$\Sigma M_B = 0 \quad \Rightarrow \quad w_0 \times l \times \frac{l}{2} - M = 0$

$$M = \frac{w_0 l^2}{2}$$

Step 2: Analysis of SFD

- At B, reaction is upward

  $\Rightarrow$ SFD jumps up by $w_0 l$ $\Rightarrow$ [SF at A = $V_A = w_0 l$]

- B to A, $\frac{dV}{dx} = -w_0$ [(–) ve because load is downward]

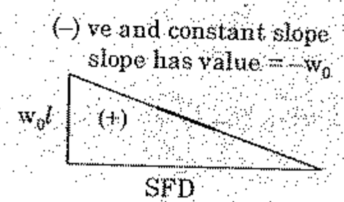
  $\Rightarrow$ Slope of SFD is (–) ve and constant.

  $\Delta V = V_A - V_B$ = area under loading diagram

  $\Rightarrow V_A - V_B = -w_0 \times l$

  $\Rightarrow V_A - w_0 l = -w_0 l$

  $\Rightarrow V_A = 0$



Step 3: Analysis of BMD

At B:

$BM = (-)\frac{w_0 l^2}{2}$ [(–) ve because Hogging]

B to A:

$\frac{dM}{dx} = V$ = shear force

i.e. slope of BMD is (+) ve and decreasing

Slope of BMD at (x = 0), = $w_0 l$ = (SF) at B.

At A:

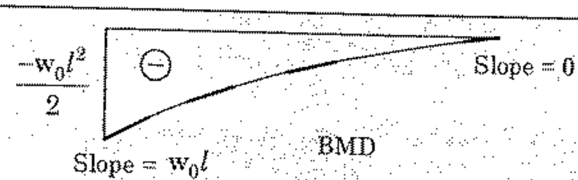
ΔM = area under SFD between A to B

$M_A - M_B = \frac{1}{2} \times (w_0 l) \times l$

$M_A - \left(-\frac{w_0 l^2}{2}\right) = \frac{w_0 l^2}{2}$

$\Rightarrow \quad M_A = 0$

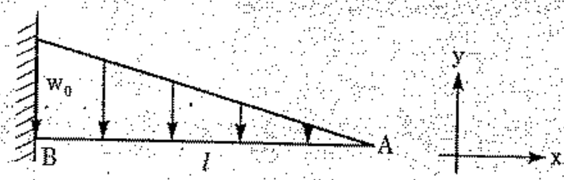
$(\text{Slope of BMD})_A = (SF)_A = 0$



Note: *Slope of BMD is (+) ve and decreasing because SF is (+) ve and decreasing.*

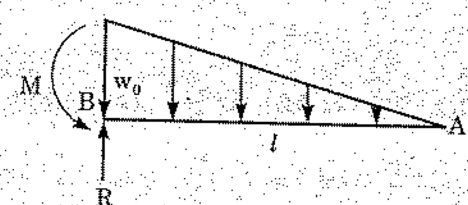
Example 6

Draw shear force and BMD for the beam shown below.



Sol:

Step 1: Find reactions



$\Sigma F_y = 0 \quad \Rightarrow \quad R - \frac{1}{2} \times w_0 \times l = 0$

$\Rightarrow \quad R = \frac{w_0 l}{2}$

$\Sigma M_B \circlearrowright = 0 \Rightarrow \quad \frac{w_0 l}{2} \times \frac{l}{3} - M = 0$

$\Rightarrow \quad M = \frac{w_0 l^2}{6}$ [(+) ve means same direction as assumed].

Step 2: Analysis of SFD

At B: SF is (+) ve [because left side upward is (+) ve shear force]

$\Rightarrow$ SFD jumps up by $\frac{w_0 l}{2}$

B to A:

$$\frac{dV}{dx} = -w_0\left[1 - \frac{x}{L}\right] = \text{loading}$$

$\Rightarrow$ • Loading is (–) ve and decreasing

$\Rightarrow$ • Slope of SFD is (–) ve and decreasing

• At B, slope of SFD = $-w_0$ [load intensity]

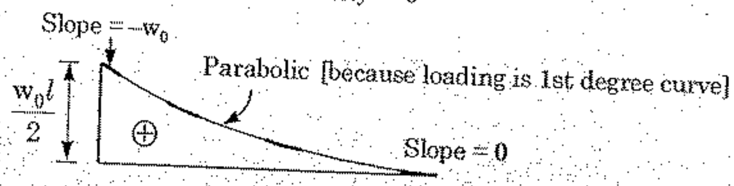
At A:

ΔV = Area under loading diagram

$\Rightarrow \quad V_A - V_B = \left[-w_0 \times \frac{1}{2} \times l\right]$

$\Rightarrow V_A = 0$

At A, slope of SFD = load intensity = 0



SFD

Note: *Slope of SFD is (–) ve decreasing becausing loading is (–) ve and decreasing.*

Step 3: Analysis of BMD

At B:

$$BM = -\frac{w_0 l^2}{2} \quad [(-)\text{ve means Hogging}]$$

B to A:

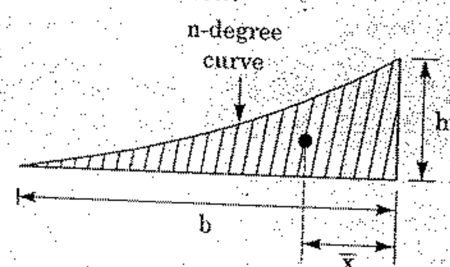
$$\frac{dM}{dx} = V \text{ shear force}$$

SF is (+) ve and decreasing

$\Rightarrow$ Slope of BMD is (+) ve and decreasing

$\Delta M = M_A - M_B$ = area under SFD

Note: *For n-degree curve like the figure shown below.*



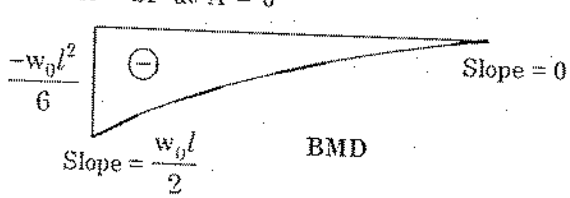
$$\text{Area} = \frac{bh}{n+1}, \qquad \bar{x} = \frac{b}{n+2}$$

$$\text{At A, } \Delta M = M_A - \left(-\frac{w_0 l^2}{6}\right) = \frac{1}{3} \times \frac{w_0 l}{2} \times l$$

$$\Rightarrow M_A = 0$$

Slope of BMD at B = SF at B = $\frac{w_0 l}{2}$

Slope of BMD at A = SF at A = 0



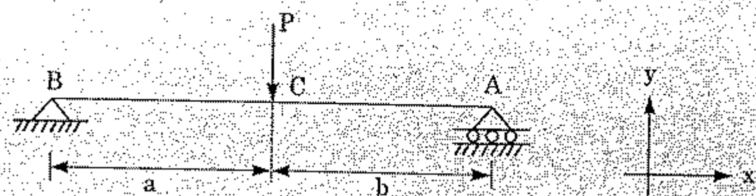
BMD

Note: *Slope is (+) ve and decreasing. Hence slope of BMD is (+) ve and decreasing.*

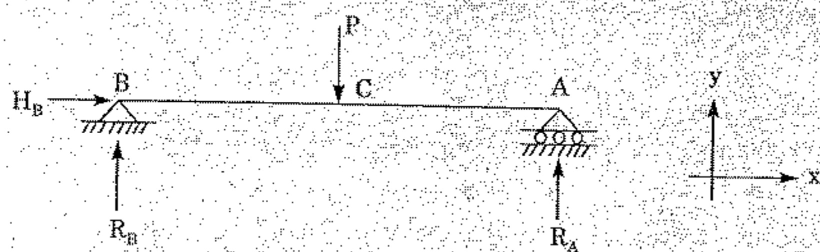
SIMPLY SUPPORTED BEAMS

Example 7

Draw SFD and BMD for the beam shown below.



Step 1: Find reaction



$\Sigma F_x = 0 \quad \Rightarrow \quad H_B = 0$ ---------- (i)

$\Sigma F_y = 0 \quad \Rightarrow \quad R_A + R_B - P = 0$ ---------- (ii)

$\Sigma M_B = 0 \quad \Rightarrow \quad R_A(a+b) - P(a) = 0$ ---------- (iii)

$$\Rightarrow \quad \boxed{R_A = \frac{Pa}{a+b}} = \frac{P \times \text{far distance for A}}{\text{Span}}$$

$$\Rightarrow \quad \boxed{R_B = \frac{Pb}{a+b}} = \frac{P \times \text{far distance for B}}{\text{Span}}$$

Remember this formula for direct computation of reactions

Step 2: Analysis of shear force

At B: Shear force is (+) ve equal to $\frac{Pb}{a+b}$

Shear force jumps up by $\frac{Pb}{a+b}$

B to C: No loading $\Rightarrow \frac{dV}{dx} = 0$

$\Rightarrow$ SF is constant

At C: Point loading (P) in ↓ direction

$\Rightarrow$ SF jumps down by P

Net SF at C just to the left of C = $\frac{Pb}{a+b}$

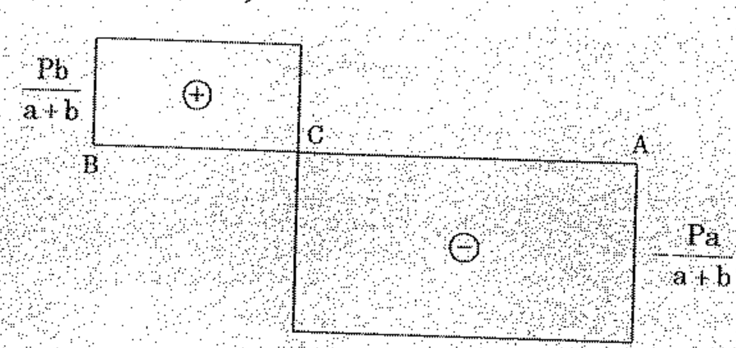
Net SF at C just to the right of C = $\frac{Pb}{a+b} - P = -\frac{Pa}{a+b}$

C to A: *No loading intensity*

$\Rightarrow$ SF is constant.

At A: Upward point load $\left(\frac{Pa}{a+b}\right)$

$\Rightarrow$ SF jumps up by $\left(\frac{Pa}{a+b}\right)$



SFD

Step 3: Analysis of bending moment

At B: BM = 0, Slope of BMD = $\frac{dM}{dx} = V = \frac{Pb}{a+b}$

B to C: $\frac{dM}{dx} = V$ = shear force = $\frac{Pb}{a+b}$ = constant

$\Rightarrow$ BMD slope is (+) ve and constant.

At C: SF changes abruptly

$\Rightarrow$ $\frac{dM}{dx} = V$ changes abruptly

$\Rightarrow$ Slope of BMD changes abruptly

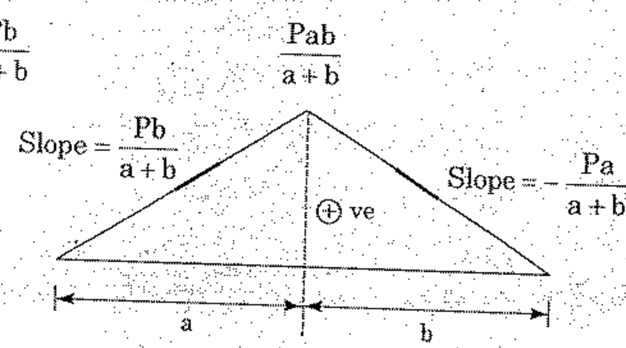
$\Rightarrow$ There is kink at C in BMD

Slope of BMD just to the left of C = $\frac{Pb}{a+b}$

Slope just to the right of C = $-\frac{Pa}{a+b}$

$\Delta M = M_C - M_B$ = Area under SFD

$M_C - 0 = \frac{Pb}{a+b} \times a = \frac{Pab}{a+b}$



BMD

C to A: $\frac{dM}{dx} = V = -\frac{Pa}{a+b}$

$\Rightarrow$ Slope is constant and (–) ve equal to $-\frac{Pa}{a+b}$

ΔM = Area under SFD between C and A

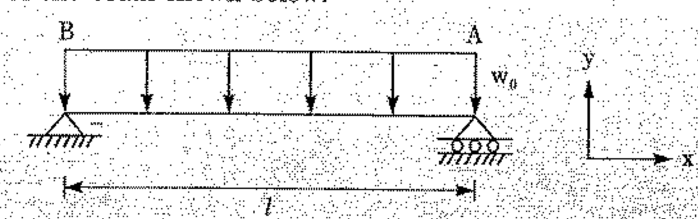
$M_A - M_C = -\frac{Pa}{a+b} \times b = -\frac{Pab}{a+b}$

$M_A - \frac{Pab}{a+b} = -\frac{Pab}{a+b}$

$\Rightarrow \quad M_A = 0$

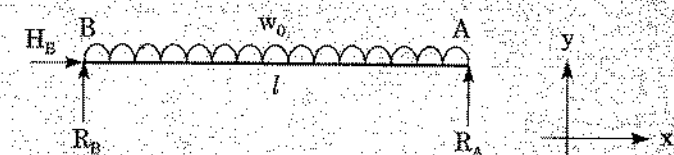
Example 8

Draw SFD and BMD for the beam shown below.



Sol:

Step 1: Find reactions



- As there is no horizontal force except H_B in the system, for equilibrium, $H_B = 0$.
- Due to symmetry, $R_A = R_B = \frac{w_0 l}{2}$

Step 2: Analysis of shear force

At B, shear force $= \frac{w_0 l}{2}$ (+ ve)

B to A, $\frac{dV}{dx}$ = load intensity $= -w_0$ = slope of SFD

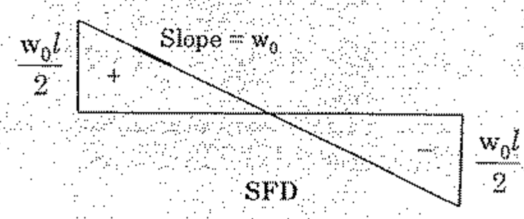
$\Rightarrow$ Slope of SFD is (–) ve and constant.

At A: $\Delta V = V_A - V_B$ = area under loading diagram

$\Rightarrow \quad V_A - \frac{w_0 l}{2} = -w_0 l$

$\Rightarrow \quad V_A = -\frac{w_0 l}{2}$

Shear force at mid span $= \frac{wl}{2} - \frac{wl}{2} = 0$



Step 3: Bending moment diagram analysis

At B: BM = 0

B to mid span:

$\Rightarrow \quad \frac{dM}{dx} = V$ is (+) ve and decreasing

$\frac{dM}{dx} = 0$ at mid span $\Rightarrow$ max BM occurs at mid span

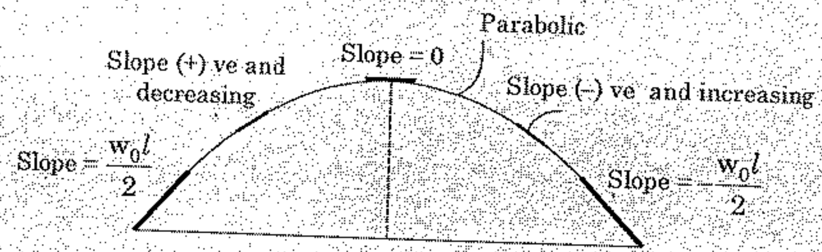
$\frac{dM}{dx}$ = (–) ve and increasing from mid span to A

$M_{mid\ span} - M_B$ = Area under SFD

$= \frac{1}{2} \times \frac{w_0 l}{2} \times \frac{l}{2}$

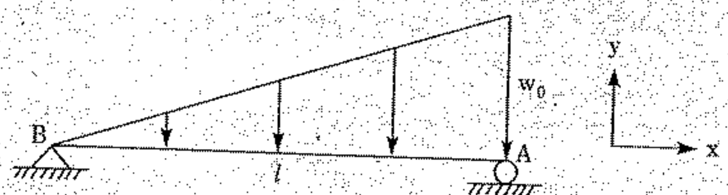
$M_{mid\ span} - 0 = \frac{w_0 l^2}{8}$

$$M_{mid\,span} = \frac{w_0 l^2}{8}$$

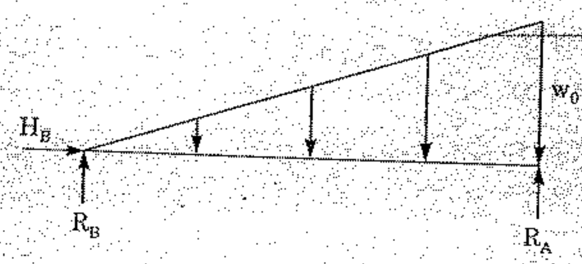


Example 9

Find BMD and SFD of the beam shown below.



Sol:



Step 1: Find reactions

$$\Sigma F_x = 0 \Rightarrow H_B = 0$$

$$\Sigma F_y = 0 \Rightarrow R_A + R_B - \frac{w_0 l}{2} = 0$$

$$\Sigma M_B = 0 \Rightarrow R_A \times l - \frac{w_0 l}{2} \times \frac{2l}{3} = 0$$

$$\Rightarrow R_A = \frac{w_0 l}{3}$$

$$\Rightarrow R_B = \frac{w_0 l}{6}$$

Step 2: Analysis of shear force

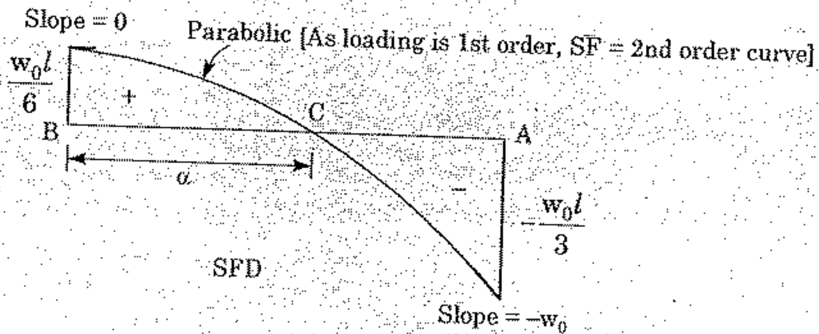
At B: SF is $\frac{w_0 l}{6}$ [(+) ve]

B to A: $\frac{dV}{dx}$ = load intensity is (–) ve and increasing

$\left.\frac{dV}{dx}\right|_B = 0$ because load intensity is zero at B

$\left.\dfrac{dV}{dx}\right|_A = -w_0$, because load intensity is $-w_0$ at A

Shear force at A = $-\dfrac{w_0 l}{3}$ [Right side upward SF = –ve]



Let the shear force be zero at α from end B i.e. at C

$$\Rightarrow \quad \Delta V = 0 - \frac{w_0 l}{6} = \text{area under loading diagram}$$

$$0 - \frac{w_0 l}{6} = -\frac{1}{2}\frac{w_0 \alpha}{l} \times \alpha$$

$$\Rightarrow \quad \alpha = \frac{l}{\sqrt{3}}$$

Step 3: Analysis of BM diagram

At B: BM = 0

From B to C:

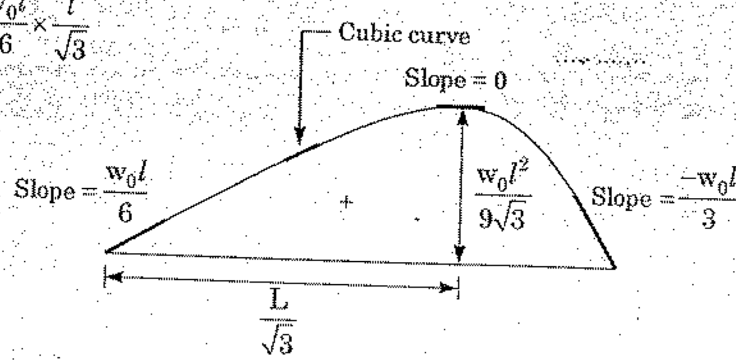
Shear force is (+) ve upto α-distance from B.

$\Rightarrow$ Slope of BMD is (+) ve upto that point.

- SF is (+) ve upto C and decreasing $\Rightarrow$ BM slope is (+) ve and decreasing upto C
- $M_C - M_B$ = Area under SFD = $\dfrac{2}{3} \times \dfrac{w_0 l}{6} \times \dfrac{l}{\sqrt{3}}$

$$\Rightarrow \quad M_C - 0 = \frac{w_0 l^2}{9\sqrt{3}}$$

$$\Rightarrow \quad \boxed{M_C = \frac{w_0 l^2}{9\sqrt{3}}}$$



- ***From C to A:*** SF is (–) and increasing

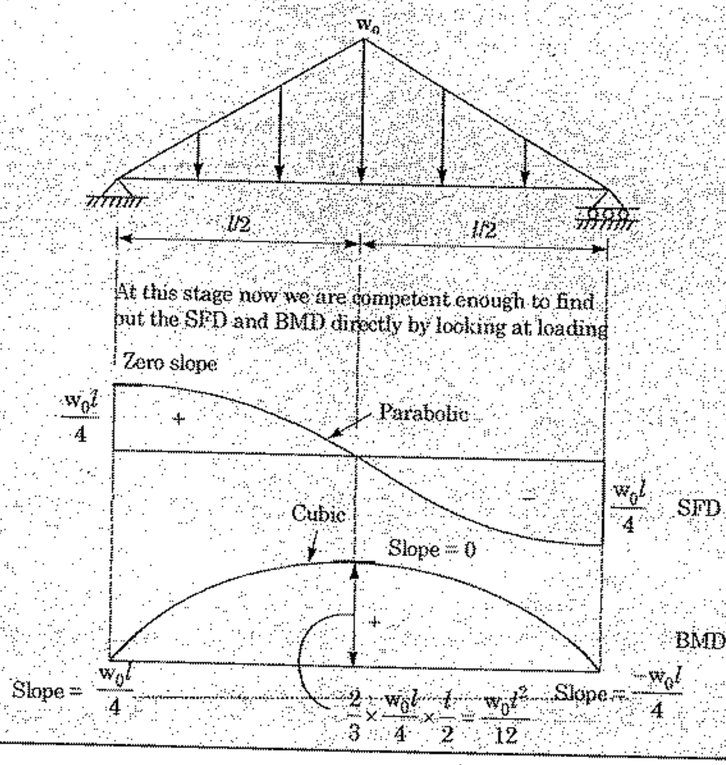
$\Rightarrow$ Slope of BMD is (–) ve and increasing

- BM at A = 0

Example 10

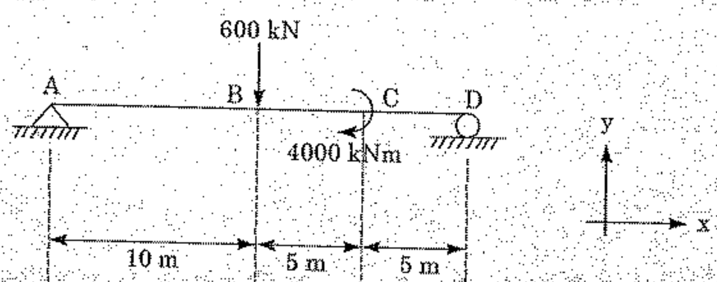
Draw BMD and SFD for the beam shown below.

Sol:



Example 11

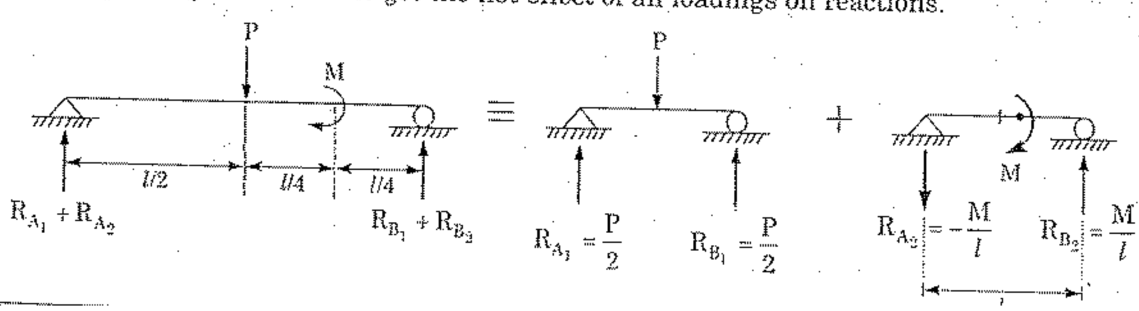
Draw SF and BM diagram for the following beam



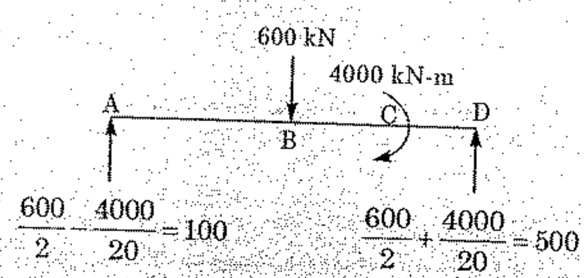
Sol: To find out reactions, we can follow the usual approach of equation of static equilibrium

i.e. $\Sigma F_y = 0$; $\Sigma M = 0$

However, we can also follow the principle of superposition in which effect of each loading or reaction is considered separately and added to get the net effect of all loadings on reactions.

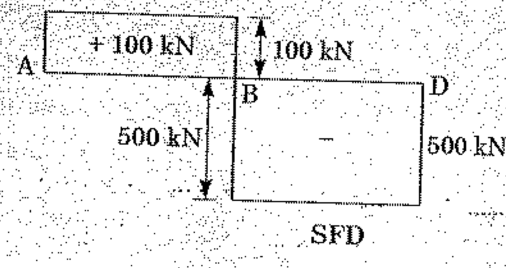


Hence



Shear Force Diagram

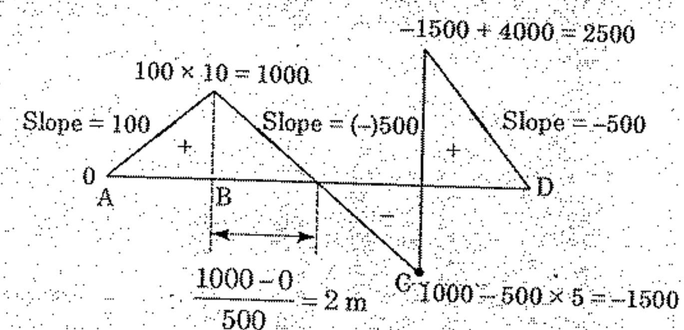
- Rise up due to 100 kN at A
- Horizontal in AB
- Drop by 600 kN at B
- Horizontal in BD



Note: *Concentrated moment in span doesnot after the SFD. It only affects the SFD ordinate at supports.*

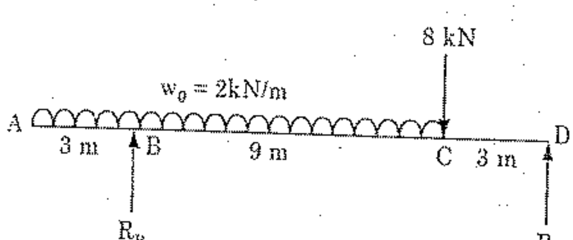
Bending Moment Diagram

- BM = 0 at A and D.
- Jump up of BMD due to clockwise moment at C. Jump is equal to magnitude of concentrated moment.
- (+) slope of 100 kN in AB.
- (–) slope of 500 in BD.



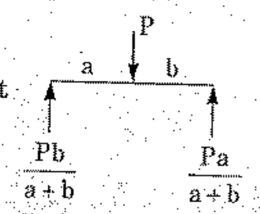
Example 12

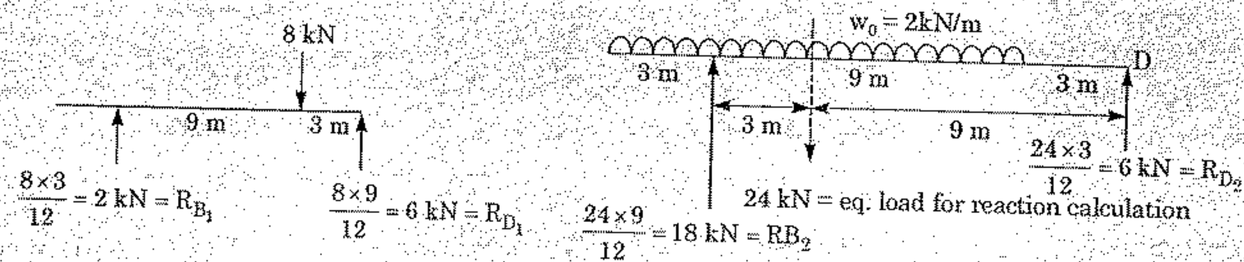
Draw BMD and SFD for the beam shown below.



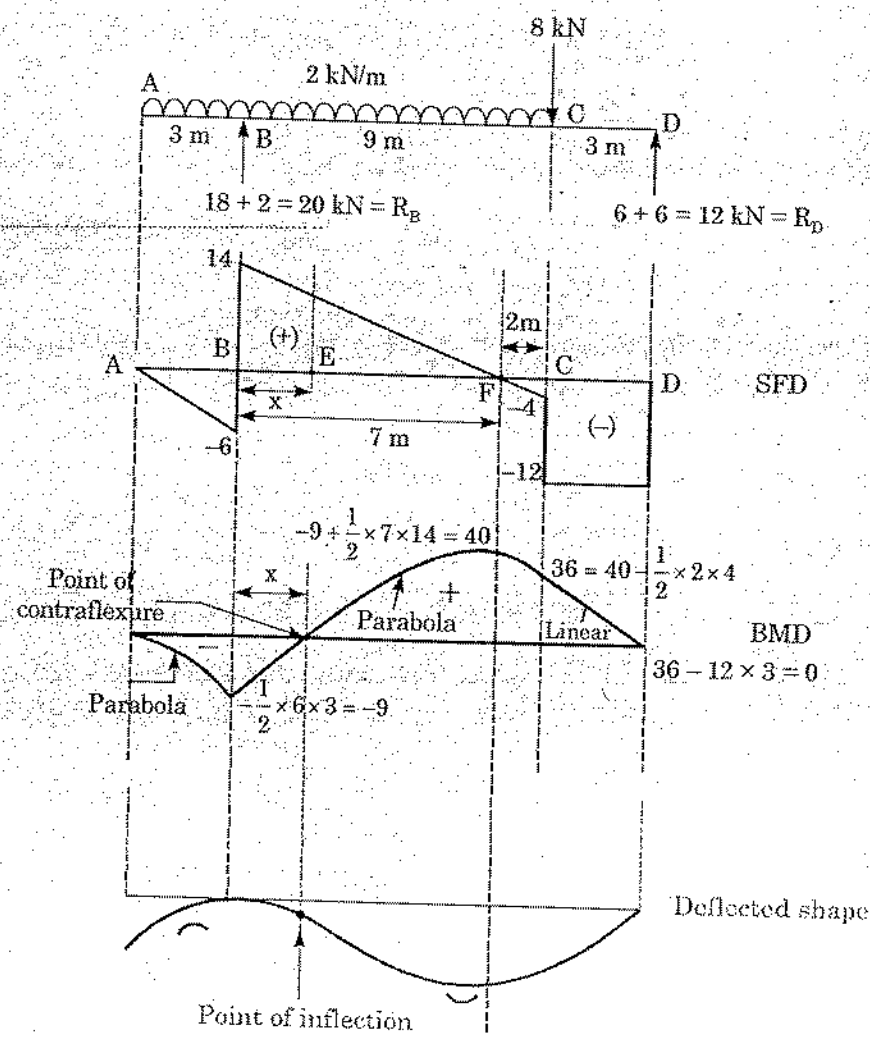
Sol:

Step 1: Find reactions using the standard result





Final reactions are as shown below.



Justification

SFD

- **At A:** SF = 0 [No load to the left of A]
- ***A to B:*** SF is (–) ve, slope is (–) 2 [equal to load intensity]

  ⇒ Value of SF at B (from left) = –2 × 3 = –6 kNm
- Curve will be linear because loading is constant.
- ***At B*** sudden jump of 20 kN

  ⇒ Value reaches –6 + 20 = 14 kNm
- ***B to C*** slope is (–) ve and constant [load intensity being (–) ve and constant.]

  Slope = –2

  ⇒ Value of SF at C from left

  = 14 – 2 × 9 = –4 kNm
- SF becomes zero at $\frac{14-0}{2}$ = 7 m from B.
- Curve is linear because loading is constant.
- ***At C:*** there is sudden drop by amount 8 kN.

  ⇒ SF at C from right = –4 – 8 = –12 kN
- ***C to D:*** SF = constant [No load intensity]
- ***At D:*** sudden jump up due to point load of 12 kN.

BMD

- BM at A = 0 [no moment to the left of A].
- ***AB:*** slope of BMD is (–) ve and increasing. [Because SF is (–) ve and increasing].
- Curve is parabolic because SF is linear.
- ***BM at B:*** $M_B - M_A$ = Area under SFD between A and B

$$M_B - 0 = \frac{1}{2} \times 6 \times 3 = -9$$

⇒ $M_B = -9$ kNm

B to F: Slope of BMD is (+) ve and decreasing [because SF is (+) ve and decreasing].

- Shear force is zero at F.
- ⇒ BM is max at F

$$M_F - M_B = \frac{1}{2} \times 14 \times 7 = 49$$

$$M_F - (-9) = 49$$

- M_F = 40 kNm
- Curve is parabolic because SF is linear.

F to C: Slope of BMD is (–) and increasing [parabolic curve]

$$M_C - M_F = -4 \times \frac{1}{2} \times 2$$

$M_C - 40 = -4$

$\Rightarrow M_C = 36$ kNm

C to D: BMD is slope is constant [load intensity = 0]

Slope = shear force = –12

At D: BM = 0

Point of contraflexure can be located by equating BM at point of contraflexure to be zero. Let x be the distance of point of contraflexue from 'B'

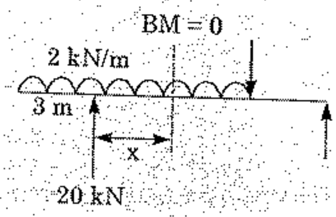
⇒ Area under SFD in x-distance must be equal to '9'. So that

$M_F = 0$

Note: $M_F - M_B$ = area under SFD between B and F

$\Rightarrow M_F - (-9)$ = area under SFD between B and F

$0 + 9$ = Area under SFD



$$\Rightarrow \frac{14+14-2x}{2} \times x = 9$$

$$\Rightarrow (14 - x)x = 9$$

$$\Rightarrow x^2 - 14x + 9 = 0$$

$$\Rightarrow x = \frac{14-\sqrt{(14)^2 - 4\times 9}}{2} = 0.675 \text{ m}$$

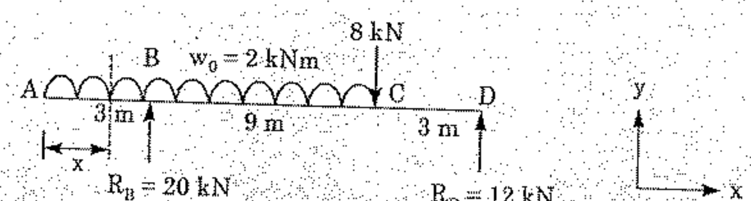
This analysis can also be carried out as

$$M_x = 20x - \frac{(3+x)^2 \times 2}{2} = 0$$

$$\Rightarrow 20x - x^2 - 6x - 9 = 0$$

$$x^2 - 14x + 9 = 0$$

$$\Rightarrow x = 0.675 \text{ m}$$



Analytical Approach

Reactions are found out in the same way as we have done earlier.

AB: [x measured from A]

SF = V = –2x [linear] ---------- (A)

$BM = M = -\frac{2x^2}{2}$ [parabolic] ---------- (B)

BC: [x measured from A] ---------- (C)

SF = V = 20 – 2x ---------- (D)

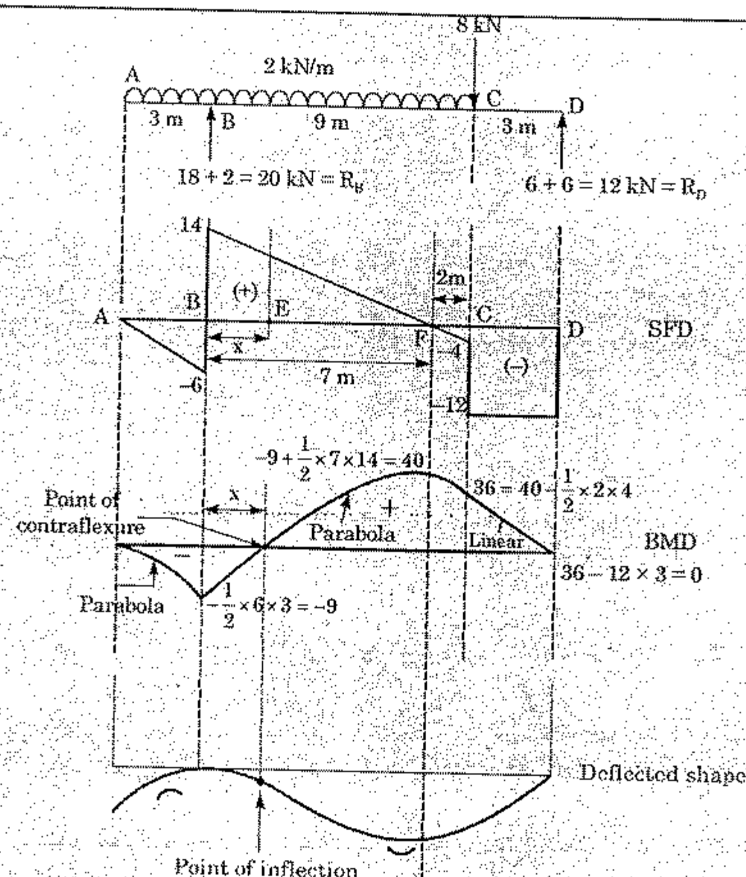
$BM = M = 20[x - 3] - \frac{2x^2}{2}$ ---------- (D)

CD: [x measured from A]

SF = V = 20 – 8 – 2 × 12 = –12 kN ---------- (E)

BM = M = 12 (15 – x) [BM taken from right] ---------- (F)

The equations A–F, can than be plotted to obtain BMD and SFD



AB: $M = -x^2$ (parabolic)

$\frac{dM}{dx} = -2x \Rightarrow$ (–) ve increasing slope

at B, $M = -(3)^2 = -9$ kNm

BC: $M = 20(x-3) - x^2$ [parabolic]

$\frac{dM}{dx} = 20 - 2x \Rightarrow$ Slope is (+) ve and decreasing upto x = 10 m there after it is (–) ve and increasing

at x = 10 m, M = 40 kNm

at x = 12 m, M = 36 kNm

CD: $M = 2(15 - x) \Rightarrow$ at x = 15, M = 0, linear curve

Point of centraflexure can be obtained by taking $M_F = 0$

$\Rightarrow \quad 20(x-3) - x^2 = 0$

$20x - 60 - x^2 = 0$

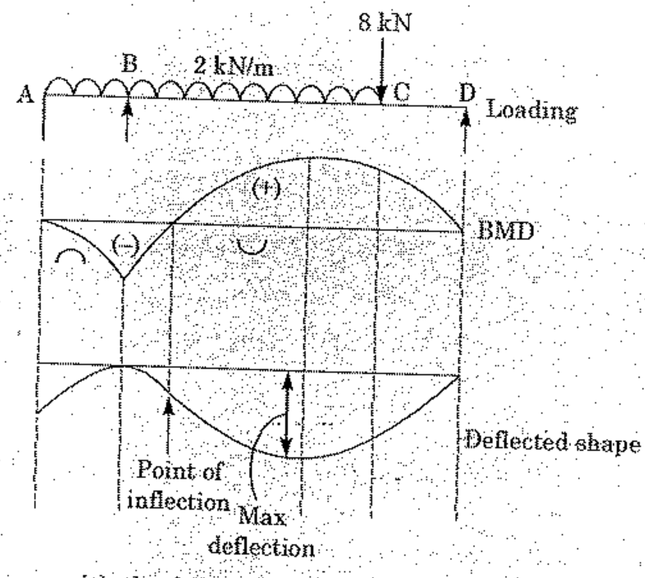
$x^2 - 20x + 60 = 0$

$$x = \frac{20 \pm \sqrt{400-240}}{2} = \frac{20 \pm \sqrt{160}}{2}$$

$$x = \frac{20 - 12.65}{2} = 3.675 \text{ m}$$

i.e. distance from B = 0.675 m

Deflected Shape

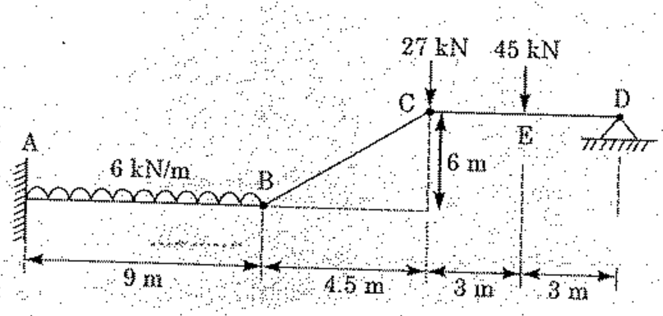


Deflected shape can be drawn with the following understanding.

1. Deflection at B and D = 0.
2. (–) ve BM produces ⌒ curvature.
3. (+) ve BM produces ◡ curvature.
4. Point of inflection (point of zero BM).
5. Max deflection need not be at the point of max bending moment.

Example 13

Draw BMD and SFD.



Sol:

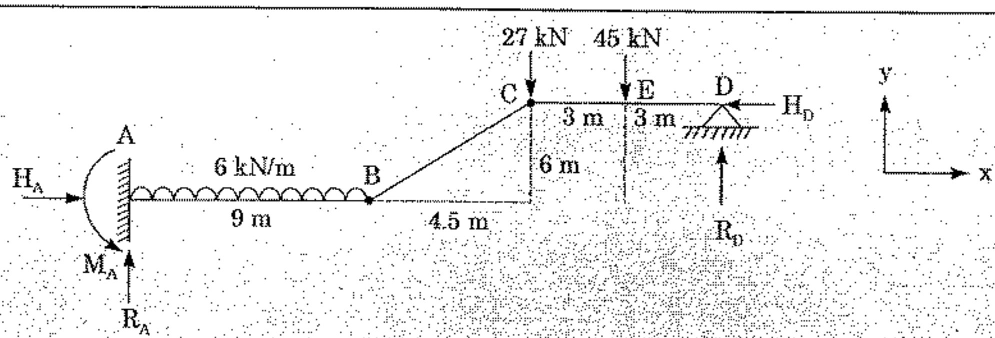
Note: *BC is a link [A link carries only axial force. Provided that link portion BC does not have a force on it].*

⇒ SF and BM on BC = 0.

Reaction Calculation:

[If a structure has supports at two different levels then even if loading is only vertical, horizontal reactions may develop.]

This beam is a determinate beam because no. of unknown rections are 5 [R_A, H_A, M_A, R_D, H_D] and no. of conditions available are 5[$\Sigma F_x = 0$, $\Sigma F_y = 0$, $\Sigma M = 0$, $M_B = 0$, $M_C = 0$]. Hence reactions can be calculated without using equation of compatibility.



$\Sigma F_x = 0 \quad \Rightarrow \quad H_A = H_D$ ---------- (i)

$\Sigma F_y = 0 \quad \Rightarrow \quad R_A + R_D - 45 - 6 \times 9 - 27 = 0$ ---------- (ii)

$\Sigma M_A) = 0 \Rightarrow \quad -H_D \times 6 - R_D \times (19.5) + (45 \times 16.5) + (27 \times 13.5) + (6 \times 9 \times 4.5) - M_A = 0$ (iii)

$M_C = 0 \quad \Rightarrow \quad R_D \times 6 - 45 \times 3 = 0$

$R_D = 22.5$ kNm ---------- (iv)

$M_B = 0 \quad \Rightarrow \quad (R_D \times 10.5) - (45 \times 7.5) - (27 \times 4.5) + (H_D \times 6) = 0$

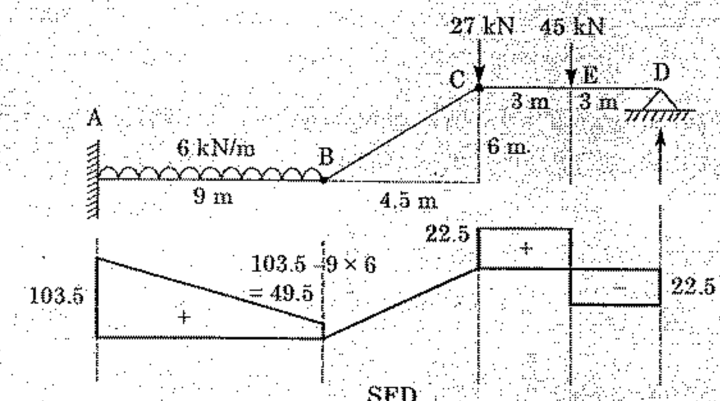
$\Rightarrow \quad H_D = 37.125$ ---------- (v)

$\Rightarrow$ From (ii), $R_A + 22.5 - 45 - 27 - (9 \times 6) = 0$

$\Rightarrow \quad R_A = 103.5$

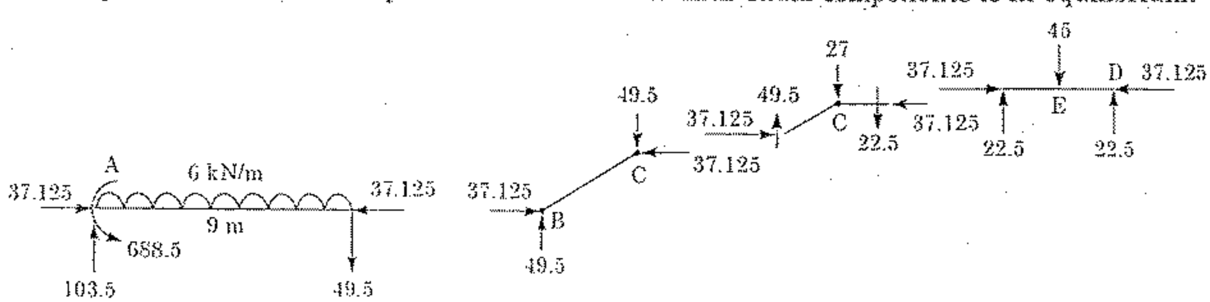
From (i), $H_D = H_A = 37.125$ kN

From (iii), $M_A = 688.5$ kNm



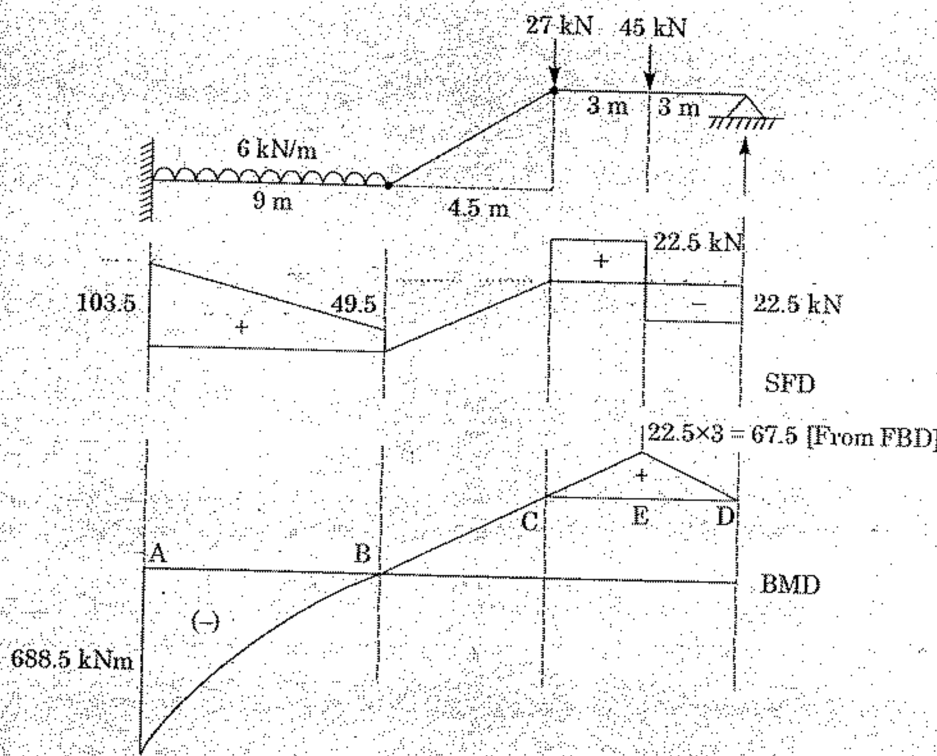
Note that at point 'C' in beam portion CD, shear force will not be 27 kN. This is because if we see the reaction at D, it is 22.5 kN upward. Total load on CD is 45 kN downwards; This implies that reaction at C in CD will be 22.5 kN upward.

To avoid such details, discussed above, it is better to draw free body diagram which clearly shows forces in the various components of beam [free body diagram is drawn by breaking a structure in components and writing the forces in the components such that each individual components is in equilibrium.



Note that whenever there is a load at the joint, joint free body diagram should also be shown. If however, the joint does not carry any force, the free body diagram of joint need not be drawn. In this case only member free body diagram is sufficient.

Now looking at the free body diagram we can plot the SFD and BMD of individual component, and join them to get the SFD and BMD of complete structure.



BM ordinates can also be found out using SFD, like

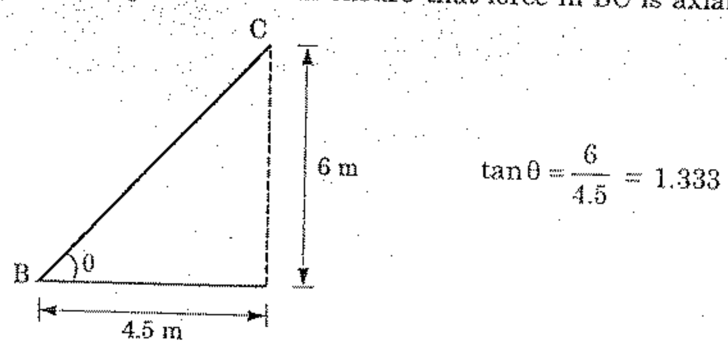
$$M_D - M_E = \text{Area under SFD} = -22.5 \times 3$$

$$\Rightarrow \; 0 - M_E = -67.5$$

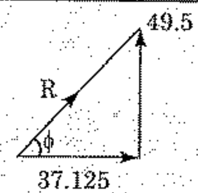
$$\Rightarrow \; M_E = 67.5 \text{ kNm}$$

Shear force in BC will be zero because it is a link. However it can also be verified from the free body diagram.

If resultant of 49.5 kN and 37.125 kN is along BC. This will ensure that force in BC is axial.



Geometrical triangle



Force Triangle

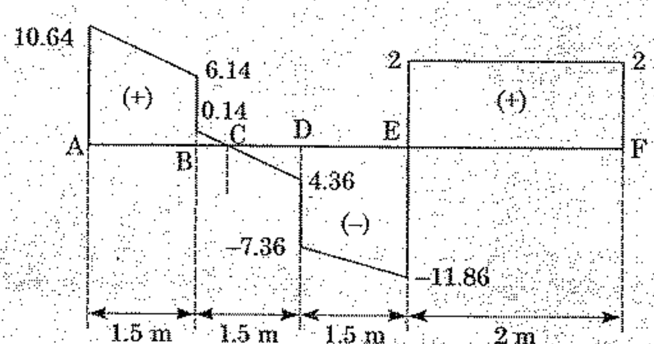
$\tan\phi = \frac{49.5}{37.125} = 1.333$

$\Rightarrow \theta = \phi$

As $\theta = \phi \Rightarrow$ **force is axial.**

Example 14

The shear force diagram in a statically determinate beam is shown in figure below. Determine the loading diagram and draw bending moment diagram.



Sol:

At A: SF = 10.64

$\Rightarrow$ There is an upward load of 10.64 kN at A.

AB: Slope of SFD $= \frac{6.14 - 10.64}{1.5} = -3/\text{m}$

$\Rightarrow \frac{dV}{dx} = w = -3/\text{m}$

$\Rightarrow$ There is a udl of 3/m downward.

B: At B there is a sudden drop is SFD.

$\Rightarrow$ There is a point load at B.

Change in SF = 0.14 – 6.14 = –6.

$\Rightarrow$ There is a downward point load of 6.

BD: $\frac{dV}{dx} = \frac{[-(4.36) - 0.14]}{1.5} = -3/\text{m}$

$\Rightarrow$ downward udl of 3/m

D: At D there is sudden drop in SFD

$\Rightarrow$ There is point load at D.

Change in SF = –7.36 – (–4.36) = –3

$\Rightarrow$ Point load is downwards of magnitude 3.

DE:

$\frac{dV}{dx} = \frac{-11.86 - (-7.36)}{1.5} = -3/\text{m}$

⇒ There is downward udl of 3/m.

E: At E, SF changes abruptly.

⇒ There is a point load at E.

Change in SF = 2 – (–11.86) = 13.86

⇒ There is an upward point load of 13.86.

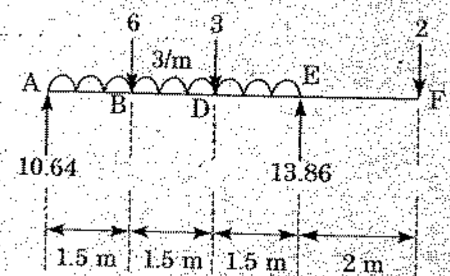
EF: SF is constant.

⇒ No loading on EF.

F: SF drops suddenly by 2.

⇒ There is downward point load at F equal to 2.

Based on above analysis the probable loading diagram is as given below.



Loading diagram drawn from SFD always satisfy force equilibrium. But it may not satisfy moment equilibrium. (i.e. ΣM may not be equal to zero).

Hence we will check for moment equilibrium

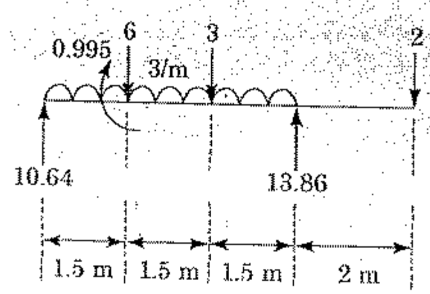
$$\Sigma M_A = (2 \times 6.5) - (13.86 \times 4.5) + (3 \times 3) + (6 \times 1.5) + (3 \times 4.5 \times 2.25)$$

$$= -0.995 \neq 0$$

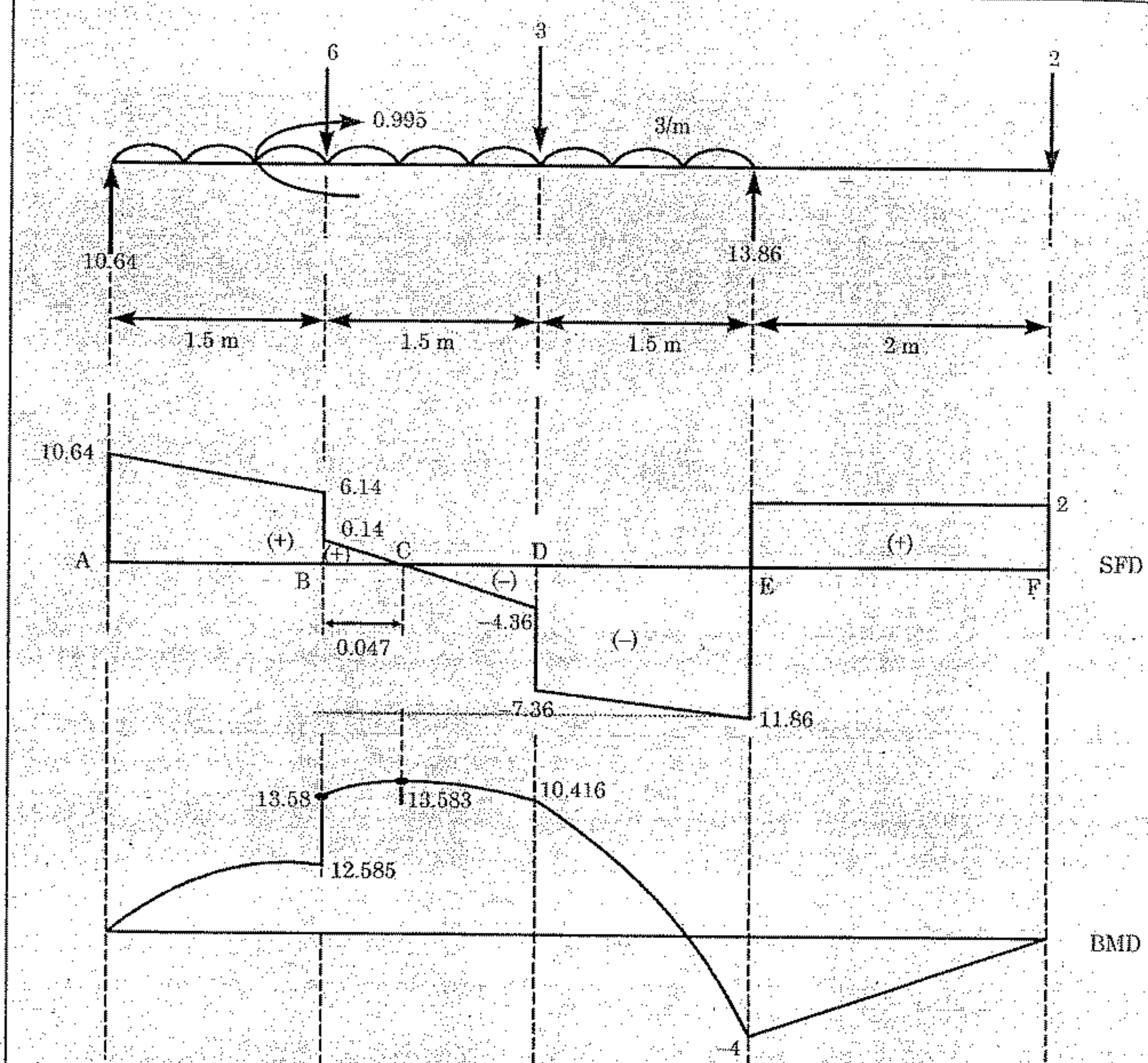
To make it zero, then should be a clockwise couple some where in the beam having magnitude 0.995.

Note that provision of a concentrated clockwise moment of 0.995 any where in the beam will satisfy the required equilibrium. Hence it can be placed any where.

Let us apply a concentrated clockwise moment of 0.995 at B. Hence final loading diagram is



Note that this beam is statically determinate.



BMD has been drawn from the usual analysis as discussed earlier.

Note the following calculations:

$$\frac{0.14}{3} = .047$$

$$\frac{10.64 + 6.14}{2} \times 1.5 = 12.585$$

$$12.585 + .995 = 13.58$$

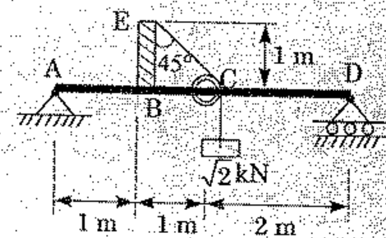
$$13.58 + \frac{1}{2} \times .14 \times .047 = 13.583$$

$$13.583 - \frac{4.36}{2} \times (1.5 - 0.047) = 10.416$$

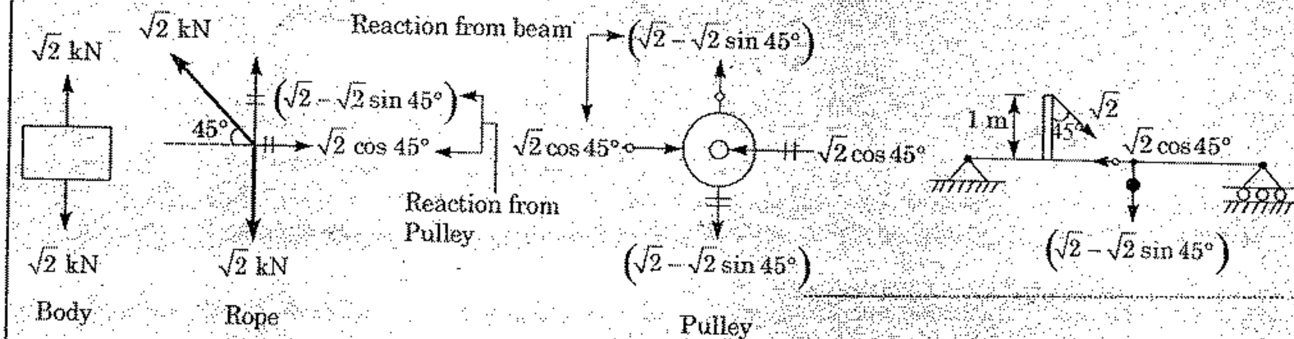
$$10.416 - \frac{7.36 + 11.86}{2} \times 1.5 = -4$$

Example 15

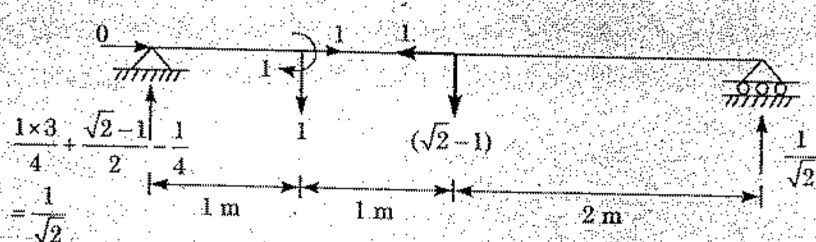
A beam ABCD is hinged at 'A' and simply supported at D. A vertical strut BE, 1 m long is fixed at B, 1m from A and a frictionless pulley is attached to the beam at 'C', 2 m from A. A flexible string carries a load of $\sqrt{2}$ kN and passes over the pulley and is attached to the strut at 'E' as shown in the figure. Draw BMD, SF and axial force diagram of beam.



Sol: The free body diagram of the above the structure is

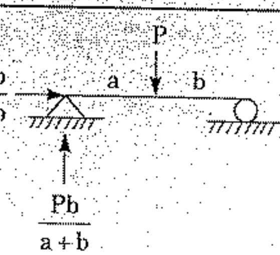


Hence net force on beam is



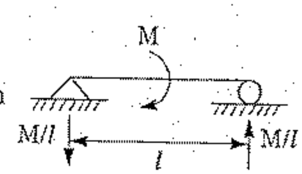
***Note:** We have taken the effect of individual loads on reactions and by principle of superposition added them to obtain the net reactions.*

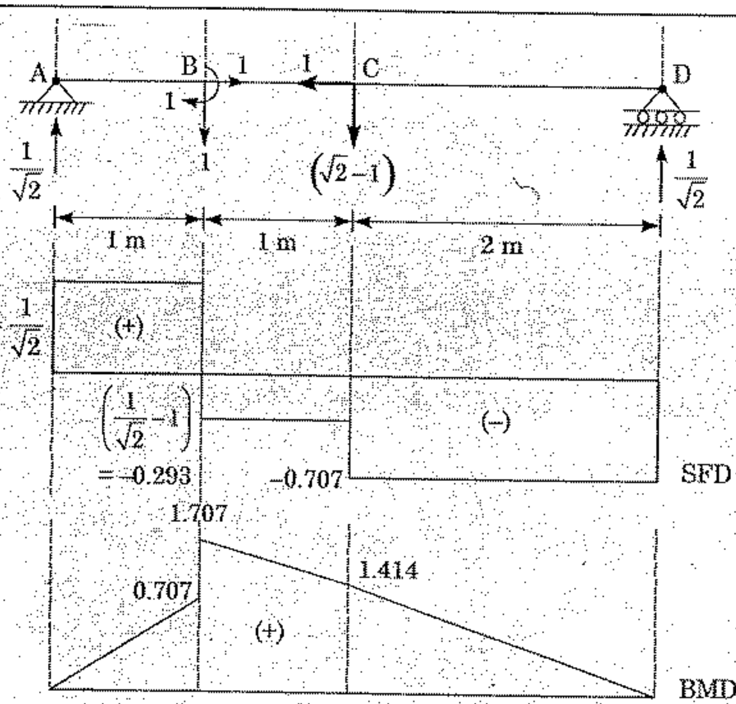
$1 \times \frac{3}{4}$ has been obtained from $\frac{P \times b}{a+b}$



$\frac{\sqrt{2}-1}{2}$ has been obained from symmetry of loading

$-\frac{1}{4}$ has been obtained from





Justification

SFD

- At A ⟶ Rise by $\frac{1}{\sqrt{2}}$ kN due to upward reaction.
- A to B ⟶ constant [No load intensity].
- At B ⟶ Sudden drop by 1 kN [due to point load].
- B to C ⟶ constant [No load intensity].
- At C ⟶ drop by $(\sqrt{2}-1)$ [due to point load].
- C to D ⟶ constant (no load intensity).
- At A ⟶ Sudden rise by $\frac{1}{\sqrt{2}}$ [due to upward reaction].

BMD

- At A ⟶ BM = 0
- A to B ⟶ $\frac{dM}{dx}$ = constant = 0.707 [because SF = 0.707 constant].

A to B: From left $M_B - 0$ = Area under SFD between A to B.

$M_B = 0.707 \times 1 = 0.707$ kNm

Due to concentrated clockwise moment BM jumps up by 1 kNm.

⇒ BM at B to the right of B = 1 + 0.707 = 1.707 kNm

B to C: $\frac{dM}{dx} = -0.293$ = SF in BC

$M_C - M_B = -0.293 \times 1 = -0.293$

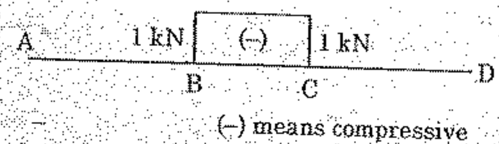
$M_C - 1.707 = -0.293$

$M_C = 1.414$ kNm

C to D: $\frac{dM}{dx} = -0.707$

BM at D = 0

Axial Force Diagram

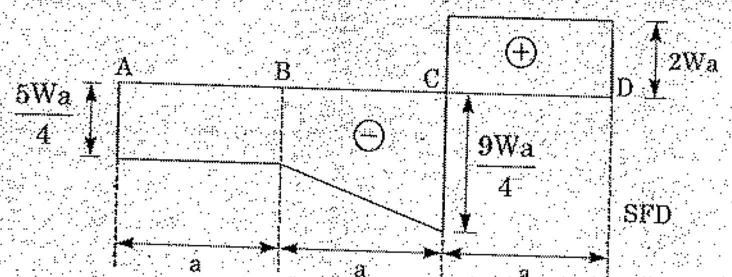


Axial force (compressive) only exists in BC.

⇒ The axial force diagram is as shown above.

Example 16

SFD is as shown below. Find loading and BMD.



Sol:

Analysis of Loading Diagram

- ***At A:*** SF is $-\frac{5Wa}{4}$

  ⇒ Downward loading at A equal to $\frac{5Wa}{4}$.
- ***A to B:*** SF = constant.

  ⇒ No load intensity in AB.
- ***B to C:*** $\frac{dV}{dx} = -\frac{\frac{-9Wa}{4} - \left(-\frac{5Wa}{4}\right)}{a} = -W$

  ⇒ Downward udl of W.

At C: Sudden jump up of SFD by

$2Wa - \left(-\frac{9Wa}{4}\right) = \frac{17Wa}{4}$

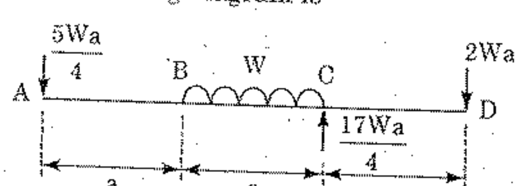
C to D: SFD = constant

⇒ No loading in CD

At D: SFD falls down by 2Wa

⇒ Point load downward by 2Wa at D.

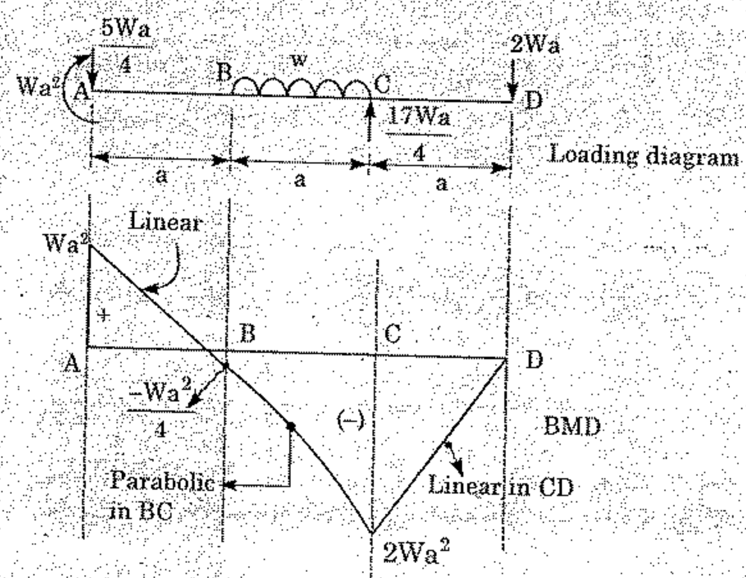
Based as above analysis, probable loading diagram is



Loading derived from SFD always satisfies the force equilibrium but moment equilibrium may not be satisfied.

$$\Rightarrow \Sigma M_A \circlearrowleft = (2Wa \times 3a) - \left(\frac{17\,Wa}{4} \times 2a\right) + \left(Wa \times \frac{3a}{2}\right) = -Wa^2 \neq 0$$

To make $\Sigma M_A \circlearrowleft = 0$ there must be a concentrated moment some where in AD of clockwise nature and of magnitude Wa^2. Let the concentrated moment be applied at A. Hence the possible loading diagram and the corresponding BMD are



Justification

BMD

At A: BM is sagging (+) = Wa^2

A to B: $\frac{dM}{dx} = SF = -\frac{5Wa}{4}$

$$\Rightarrow M_B = Wa^2 - \frac{5Wa}{4} \times a = -\frac{Wa^2}{4}$$

B to C: $\frac{dM}{dx}$ = (–)ve and increasing

$M_C - M_B$ = Area under SFD between C and B.

$$M_C - \left(-\frac{Wa^2}{4}\right) = \frac{-\frac{9Wa}{4} - \frac{5Wa}{4}}{2} \times a$$

$$M_C + \frac{Wa^2}{4} = -\frac{14Wa^2}{8}$$

$$M_C = -\frac{16Wa^2}{8} = -2Wa^2$$

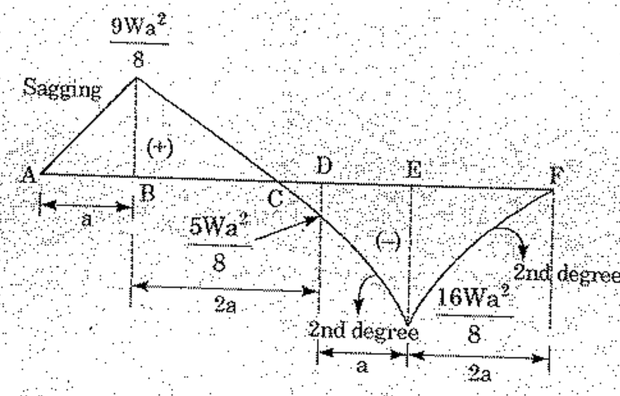
$$M_C = -2Wa^2$$

C to D: $\frac{dM}{dx} = 2Wa = SF = constant$

At D: $BM = 0$

Example 17

Draw the shear force, load and deflected shape diagrams corresponding to the BMD given in the figure below. Specify the values at all change of load position and at all points of maximum moments.



Sol: ***Slope:***

$\frac{dM}{dx} = V \Rightarrow$ Slope of BMD = Shear force

1. At 'A' slope of BMD is (+) ve and constant upto 'B'

   $\Rightarrow$ Shear force is (+) ve and constant upto B

   Shear force = slope of BMD = $\frac{9Wa^2}{8a} = \frac{9Wa}{8}$

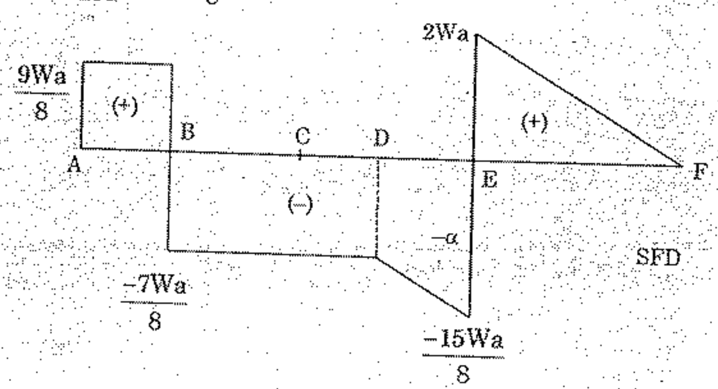
2. At B sudden change in slope of BMD occurs.

   $\Rightarrow$ SF changes abruptly $\Rightarrow$ point load

   New slope is (–) ve and constant

   $\Rightarrow$ SF is (–) ve and constant up to D

$$SF = \frac{\frac{-5Wa^2}{8} - \frac{9Wa^2}{8}}{2a} = \frac{-14Wa^2}{16a} = \frac{-7Wa}{8} = \text{Slope of BMD}$$



From D to E

- BMD is parabolic $\Rightarrow$ SF is linear
- Slope of BMD is (–) ve and increasing

  $\Rightarrow$ SF is (–) ve and increasing

Let the shear force at E just before E = $-\alpha$

$$M_E - M_D = -\frac{\left(\frac{7wa}{8} + \alpha\right)}{2} \times a$$

$$-\frac{16wa^2}{8}+\frac{5wa^2}{8}=\frac{-7wa^2}{16}-\frac{\alpha a}{2}$$

$$\frac{-11wa^2}{8}+\frac{7wa^2}{16}=-\frac{\alpha a}{2}$$

$$\frac{15wa^2}{16}=\frac{\alpha a}{2}$$

$$\Rightarrow \alpha=\frac{15wa}{8}$$

$$\frac{dV}{dx} \text{ in DE} = \frac{\frac{-15Wa}{8}-\left(-\frac{7Wa}{8}\right)}{a} = -W \Rightarrow \text{downward udl of W in DE}$$

From E to F

- BMD is parabolic $\Rightarrow$ SF is linear $\Rightarrow$ loading is udl
- Slope of BMD is (+) ve and decreasing
  $\Rightarrow$ SF is (+) ve and decreasing
- Abrupt change in BMD slope of E
  $\Rightarrow$ SF changes abruptly $\Rightarrow$ point load at E

Let the slope of BMD at F = 0
$\Rightarrow$ SF = 0 at F $\Rightarrow$ No point load at F

In EF loading is udl = w_0 (say) [Because BMD is parabolic]

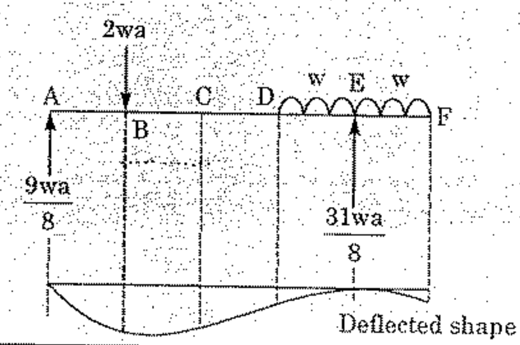
$$\Rightarrow \frac{w_0(2a)^2}{2}=\frac{16w^2}{8} = \text{BM at E}$$

$$4w_0a^2 = 4wa^2$$

$$\Rightarrow w_0 = w$$

$$\Rightarrow \text{(+) ve SF at E} = w(2a) = 2wa$$

Loading Diagram



Note:

1. Looking at BMD it is clear that there is no abrupt change in BMD any where.
   $\Rightarrow$ There is no concentrated moment in the beam.
2. (+) ve BM gives ◡ curvature.
   (–) ve BM gives ◠ curvature.
   At support location there is no deflection. Assuming support at A and E, deflected curve will be as shown in figure above.

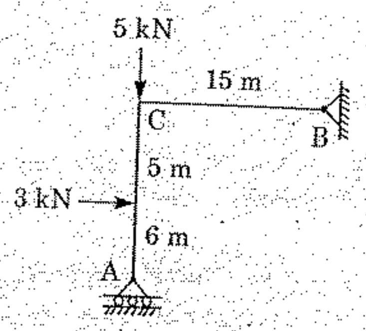
SHEAR AND MOMENT DIAGRAM FOR FRAMES

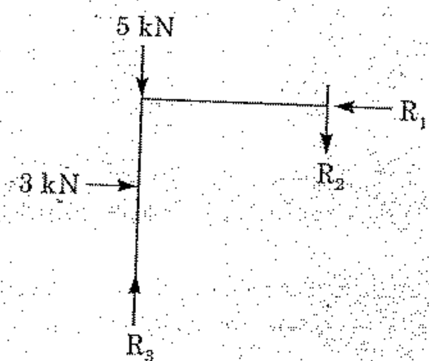
- For frames, after analysis, free body diagram of each component is drawn showing axial force, shear force and moment at ends of each component.
- BM is drawn positive on the compression face. (This convention suits the method we have been following before).

Note: Forces are shown parallel and $\perp^r$ to the individual components at ends.

Example 18

Draw SFD and BMD.





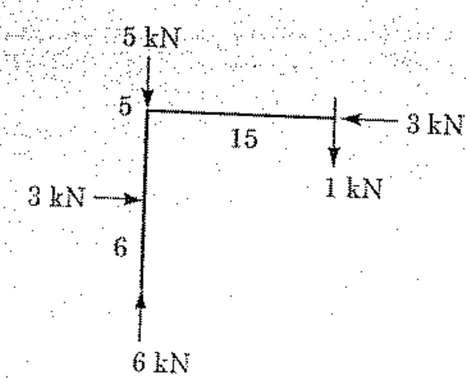
Sol:

$\Rightarrow$ $R_1 = 3$ kN (i)

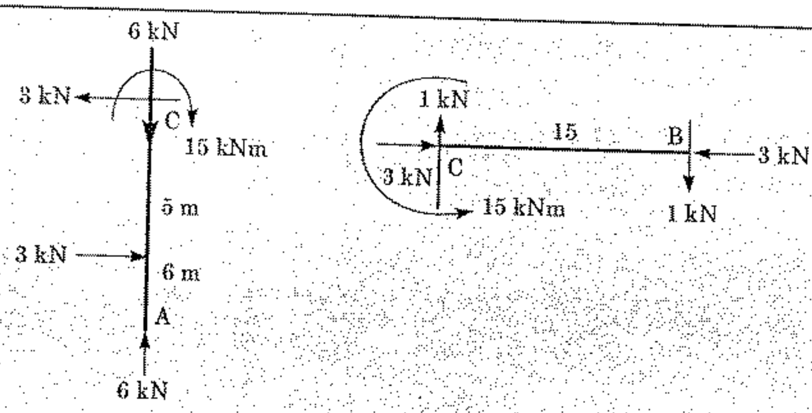
$R_3 - R_2 = 5$ (ii)

$(R_3 - 5) \times 15 - 3 \times 5 = 0$

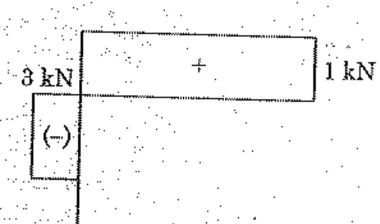
$R_3 = 6$ kN, $R_2 = 1$ kN, $R_1 = 3$ kN



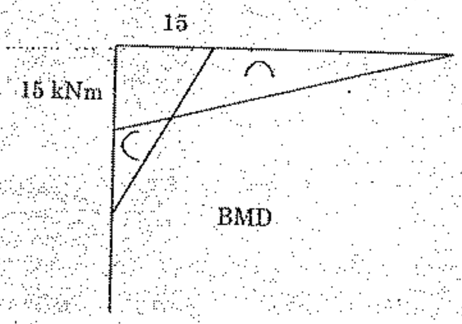
Free body diagram for above loading is as shown below.



Free body diagram



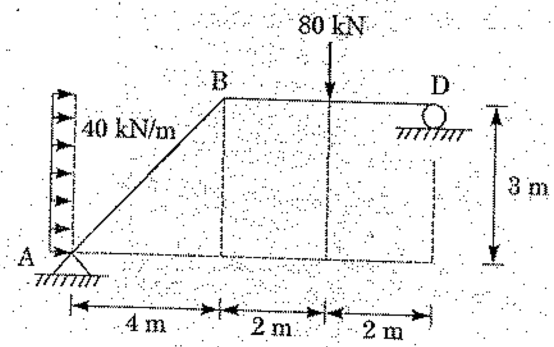
SFD



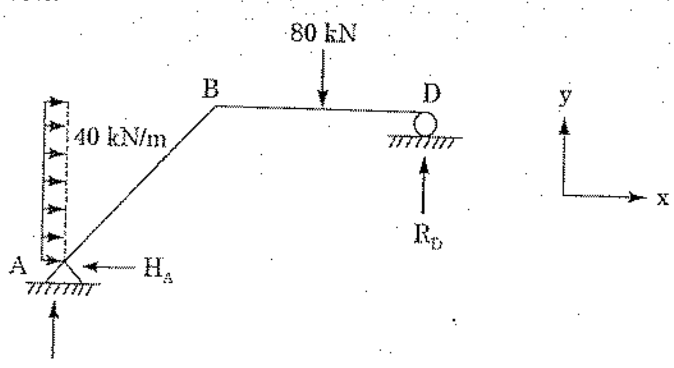
BM has been shown to produce curvature as shown like ⌒ in the figure.

Example 19

Draw the SF and BM diagram for the loading shown below.



Sol: ***Reaction calculation:***



$\Sigma F_x = 0$

$\Rightarrow 40 \times 3 - H_A = 0$

$\Rightarrow H_A = 120$ kN ---------- (i)

$\Sigma F_y = 0$

$\Rightarrow R_A + R_D - 80 = 0$ ---------- (ii)

$\Sigma M_A = 0$

$\Rightarrow (R_D \times 8) - (80 \times 6) - (40 \times 3) \times 1.5 = 0$ ---------- (iii)

$\Rightarrow R_D = 82.5$ kN , $R_A = -2.5$ kN

For the above reaction, the free body diagram is as shown below.

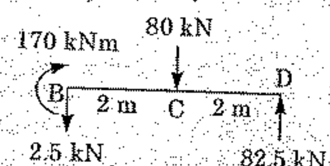


Fig. A

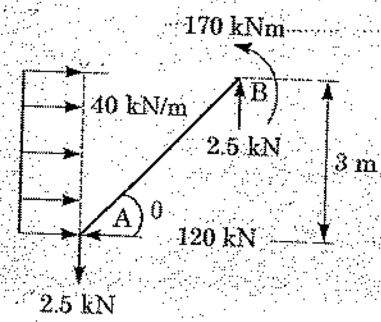


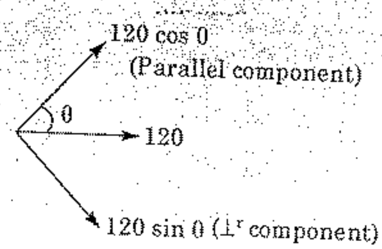
Fig. B

$\tan\theta = \frac{3}{4}$

$\Rightarrow \sin\theta = \frac{3}{5}$

$\cos\theta = \frac{4}{5}$

Shear force is produced due to forces perpendicular to the element.
Hence forces on AB has to be taken perpendicular to AB. Total horizontal force due to 40 kN/m udl = $40 \times 3 = 120$ kN. Component of this 120 kN in direction $\perp^r$ and $\parallel$ to AB is



Perpenciular component

$$= 120 \times \frac{3}{5} = 72 \text{ kN}$$

$$\text{udl } \perp^r \text{ to AB} = \frac{72}{\sqrt{3^2+4^2}} = 14.4 \text{ kN/m}$$

Hence for member AB free body diagram can be shown as below.

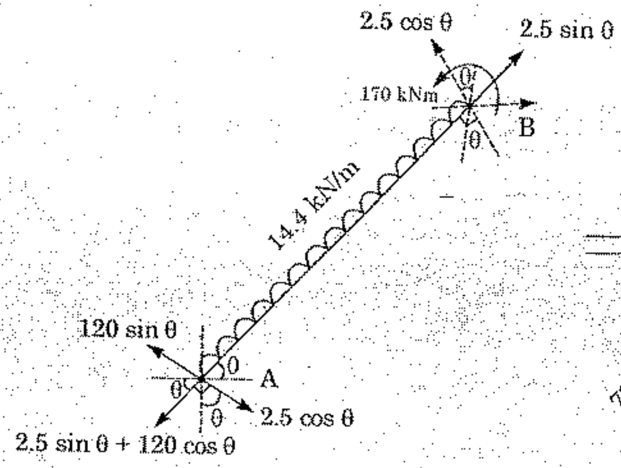


Fig. C

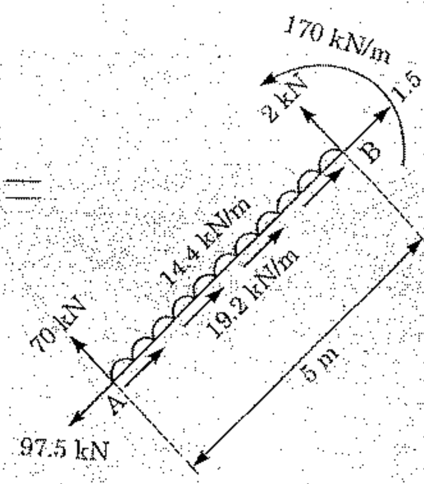
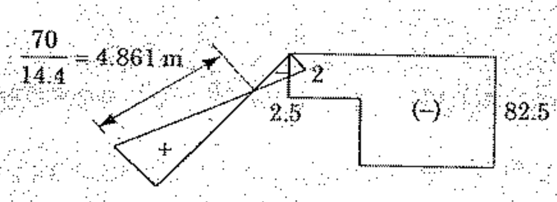
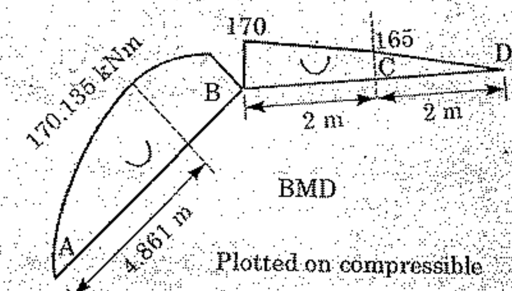


Fig. D

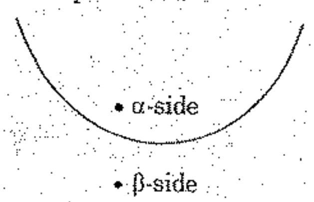
Required free body diagram for SF and BM calculation is given by Fig. A and Fig. D.



SFD



Instead of giving (+) or (–) sign in this case to the bending moment we give the symbol ◡ or ◠ just to show in which direction compresion will be produced

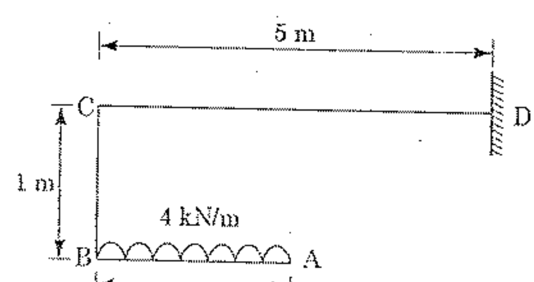


α side → compression

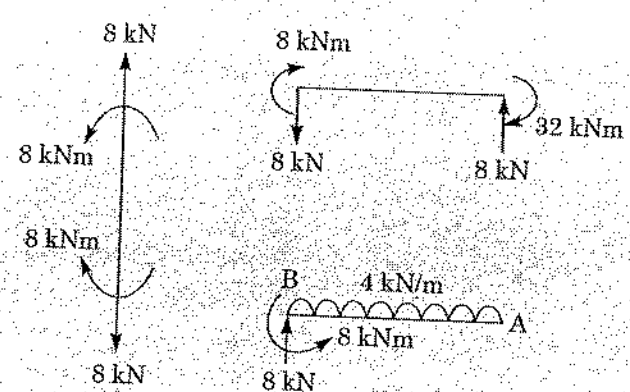
β side → tension

Example 20

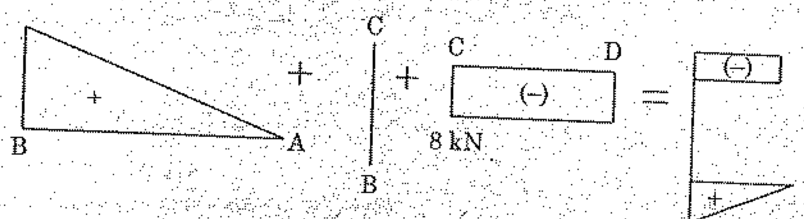
Draw BMD and SFD for the loading condition shown below.

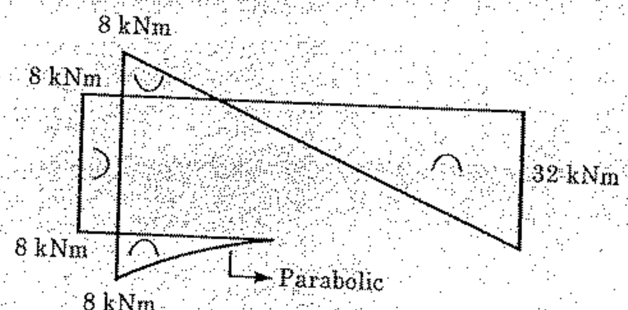


Free body diagram for the above case is as under:



Hence SFD is





BMD

OBJECTIVE QUESTIONS

1. Match **List-I** (Type and position of load on cantilever) with **List-II** (Shape of moment diagram for cantilever) and select the correct answer using the codes given below the lists:

List-I

A. Carrying linearly varying load from zero at its free end and maximum at the fixed end

B. Subjected to uniformly distributed load

C. Carrying concentrated load at its free end

D. Whose free end is subjected to bending moment

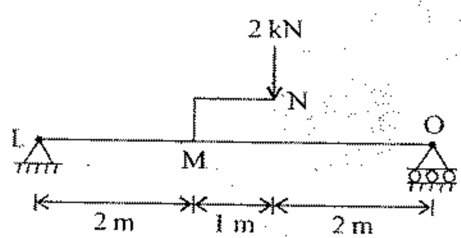
List-II

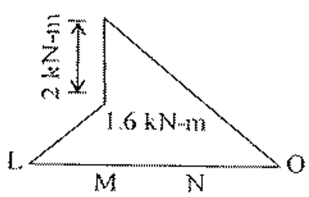
1. Parabola
2. Rectangle
3. Cubic parabola
4. Triangle

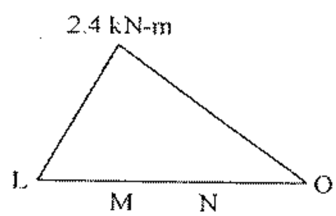
Codes:

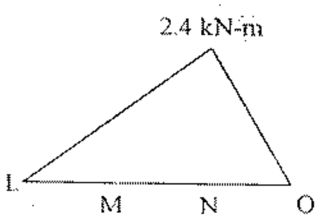
	A	B	C	D
(a)	3	2	1	4
(b)	4	3	2	1
(c)	3	1	4	2
(d)	2	4	1	3

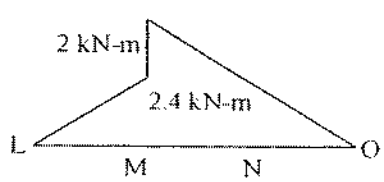
2. The bending moment diagram of the beam shown figure is ______



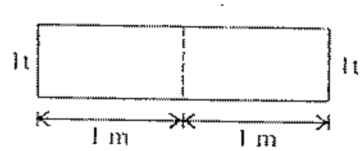
(a) 

(b) 

(c) 

(d) 

3. The SFD and BMD for a beam are shown in the given figures I and II. The corresponding loading diagram would be



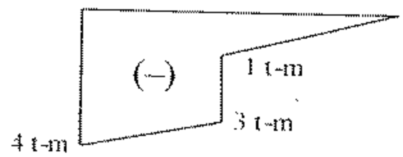
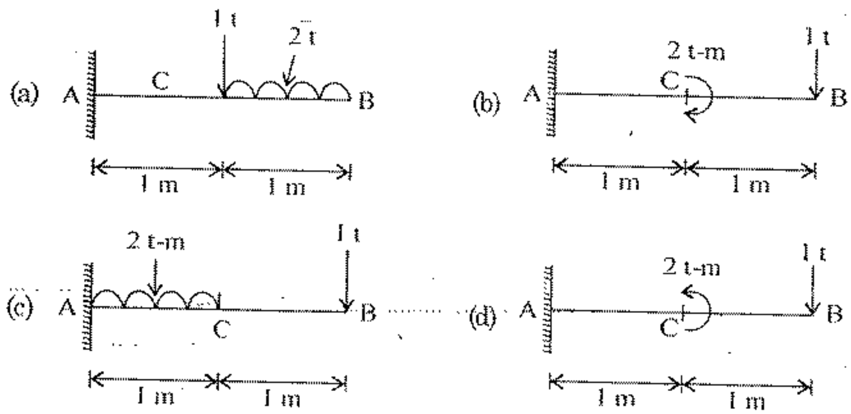


Figure-II



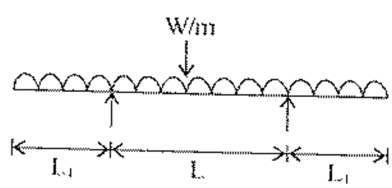
4. Which one of the following statements is correct?
   (a) Shear force is the first derivative of bending moment.
   (b) Shear force is the first derivative of intensity of load.
   (c) Load intensity on a beam is the first derivative of bending moment.
   (d) Bending moment is the first derivative of shear force.

5. **Assertion (A):** The maximum bending moment occurs where the shear force is either zero or changes sign.

   Reason (R): If the shear force diagram line between the two points is horizontal, the BM diagram line is inclined. But if the SF diagram is inclined, the BM diagram is a parabola of second degree.

   Of these statements
   (a) both A and R are true and R is the correct explanation of A
   (b) both A and R are true but R is not a correct explanation of A
   (c) A is true but R is false
   (d) A is false but R is true

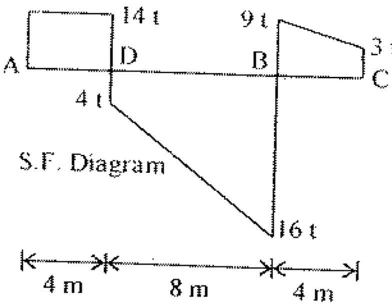
6. For the beam shown in the given figure, the maximum positive bending moment is equal to negative bending moment. The value of L_1 is



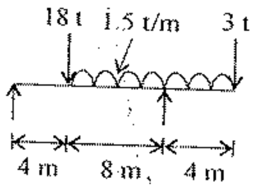
(a) $\frac{L}{\sqrt{2}}$ (b) $\frac{L}{\sqrt{3}}$

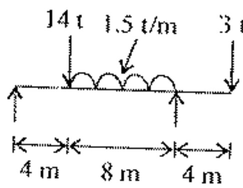
(c) $\frac{L}{2}$ (d) $\frac{L}{2\sqrt{2}}$

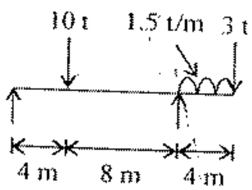
7. Consider the shear force diagram shown in given figure

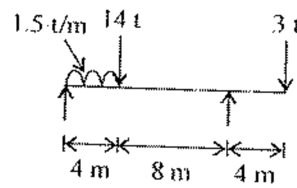


The loaded beam will be

(a)


(b)


(c)


(d)


8. **Assertion (A)**: Bending moment may be defined as the algebraic sum of the moments of all forces on either side of the section.

Reason (R): The rate of change of bending moment is equal to shear force at the section.

Of these statements

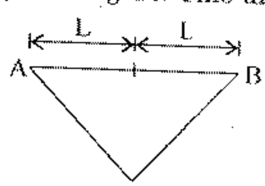
(a) both A and R are true and R is the correct explanation of A

(b) both A and R are true but R is not a correct explanation of A

(c) A is true but R is false

(d) A is false but R is true

9. For a cantilever AB with clamped end A and subjected to concentrated loads, the shape of the bending moment diagram is shown in the adjacent figure. This diagram is



(a) an absurdity

(b) possible only if the free end B has an additional couple

(c) possible only if the left half of the beam has twice the moment of inertia as compared to that of the right half

(d) none of the above

10. Consider the following statements:

A simply-supported beam is subjected to a couple somewhere in the span. It would produce

1. a rectangular SF diagram
2. parabolic BM diagrams
3. both +ve and –ve BMs which are maximum at the point of application of the couple.

Which of these statements are correct?

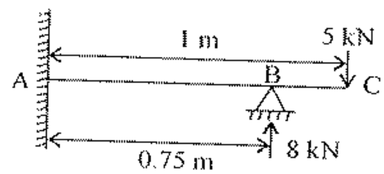
(a) 1, 2 and 3 (b) 1 and 2

(c) 2 and 3 (d) 1 and 3

11. A beam of length 10 m carries a UDL of 20 kN/m over its entire length and rests on two simple supports. In order that the maximum BM produced in the beam is the least possible, the supports must be placed from the ends at a distance of

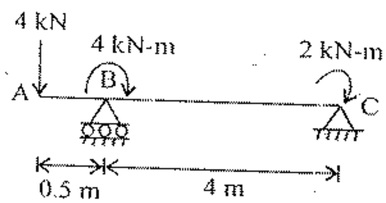
(a) 5.86 m (b) 4.14 m
(c) 2.93 m (d) 2.07 m

12. The beam ABC shown in the given figure is horizontal. The distance to the point of contraflexure from the fixed end A is



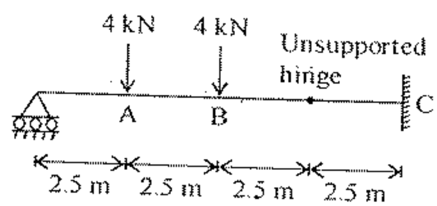
(a) 0.333 m (b) 0.666 m
(c) 0.25 m (d) 0.75 m

13. The beam shown in the given figure has a design bending moment value of



(a) 8 kN-m (b) 6 kN-m
(c) 4 kN-m (d) 2 kN-m

14. The bending moments at point A, B and C of the beam shown in the given figure will be



(a) 10 kN-m, 10 kN-m and 10 kN-m
(b) 10 kN-m, 10 kN-m and – 10 kN-m
(c) 20 kN-m, 10 kN-m and – 10 kN-m
(d) 10 kN-m, – 10 kN-m and 20 kN-m

15. The bending moment diagram of the beam shown in the figure-I is

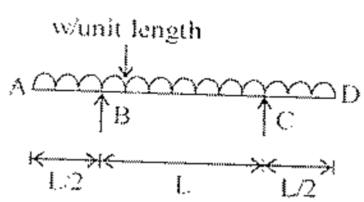
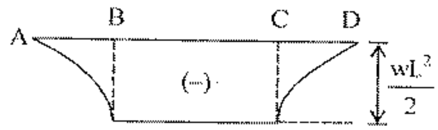
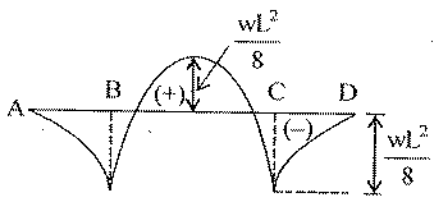
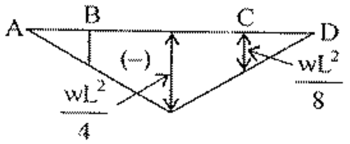
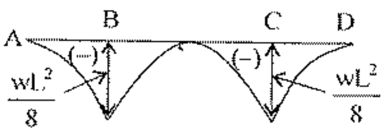


Figure-I

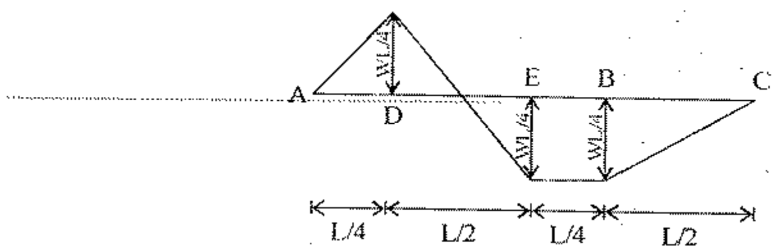
(a) 

(b) 

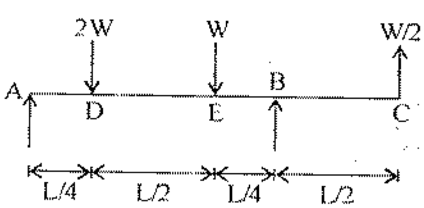
(c) 

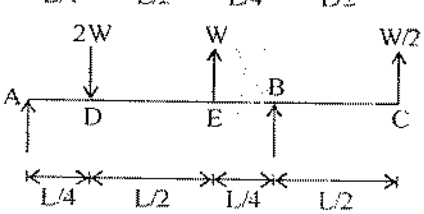
(d) 

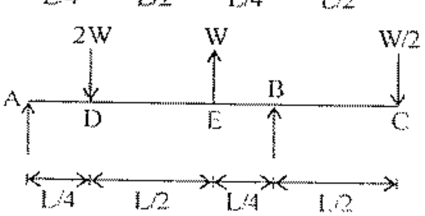
16. A beam ABC has simply supported span AB and overhanging span BC. The bending moment diagram for the beam is given in the following figure:

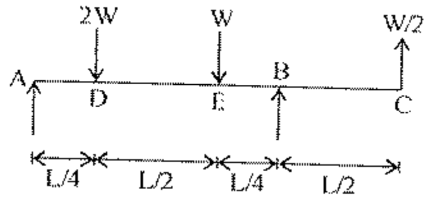


The loading for the beam would correspond to

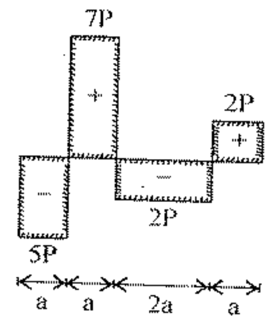
(a) 

(b) 

(c) 

(d) 

17. The shear force diagram of a beam is shown in the figure.



The absolute maximum bending moment in the beam is

(a) (2P × a) (b) (5P × a)

(c) (4P × a) (d) (7P × a)

18. In which one of the following, the point of contraflexure will **NOT** occur?

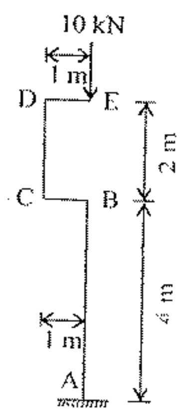
(a) A two span continuous beam of equal spans, simply supported and loaded by UDL over both spans

(b) A simply supported beam loaded by UDL

(c) A fixed beam loaded by UDL

(d) A propped cantilever loaded by UDL

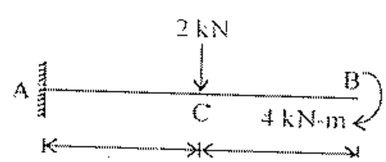
19. What is the bending moment at A for the bent column shown in the figure given below?



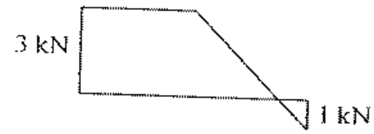
(a) 40 kN-m (b) 20 kN-m

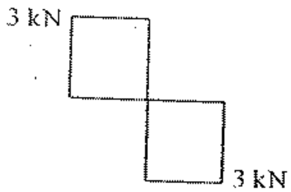
(c) 10 kN-m (d) Zero

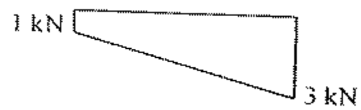
20. A cantilever beam AB carries loading as shown in the figure below. Which one of the following is the SFD for the beam?



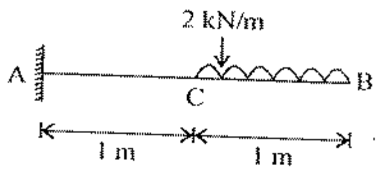
(a) 

(b) 

(c) 

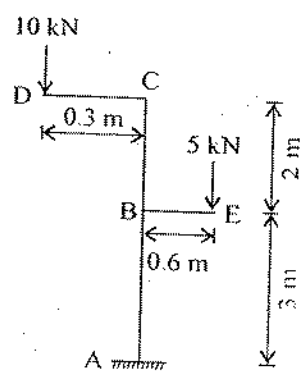
(d) 

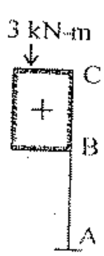
21. A cantilever AB is loaded as shown in the figure given below. What is the shape of the bending moment diagram for portion AC?

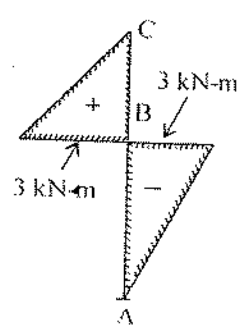


(a) Parabolic
(b) Linearly varying with maximum value of bending moment at C
(c) Linear with constant bending moment value from C to A
(d) Linearly varying with maximum value at A

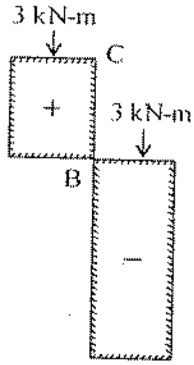
22. Which one of the following is the bending moment diagram for the vertical cantilever beam loaded as shown in the figure below?



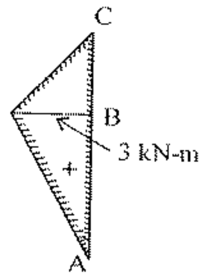
(a) 

(b) 

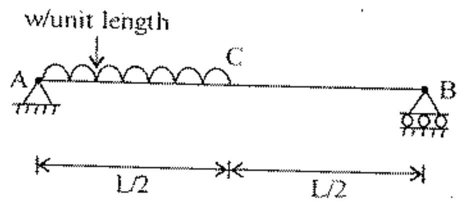
(c)



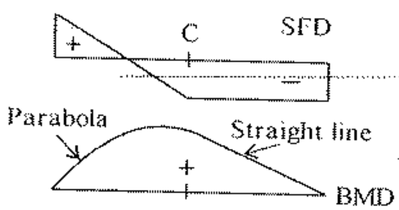
(d)



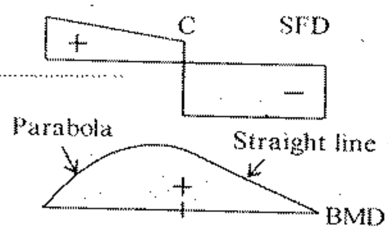
23. A simply supported beam AB has span L as shown in the figure below, Point C is the midspan of the beam. It is subjected to UDL w/unit length, in the portion A to C. Which of the following are the SFD and BMD for the beam?



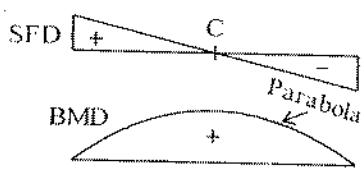
(a)



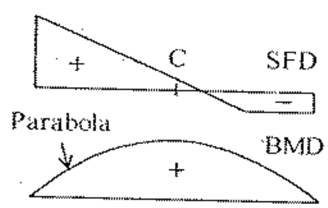
(b)



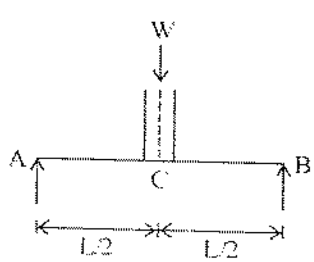
(c)



(d)



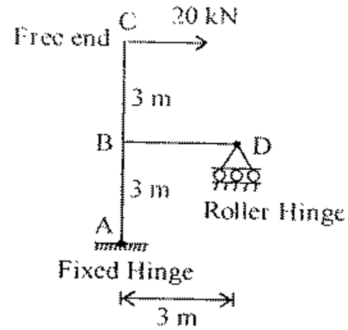
24. For the simply supported beam in the figure below, C is the centre of the span. C is also the point through which the resultant of the column load W passes. The column rests on the beam over a small length δl, symmetrically on either side of C. What is the shearing force at C?



(a) W/2 (b) W/4

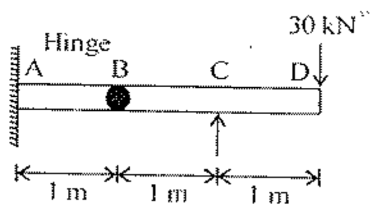
(c) W (d) zero

25. What is the reaction at the support D of the rigid-jointed structure shown below?

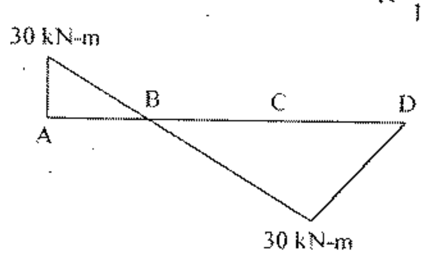


(a) 10 kN (b) 20 kN
(c) 30 kN (d) 40 kN

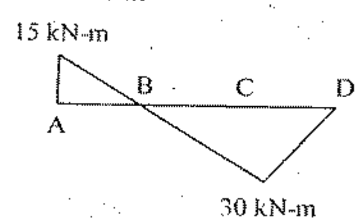
26. Which one of the following is the correct bending moment diagram for a propped cantilever beam shown in figure below?



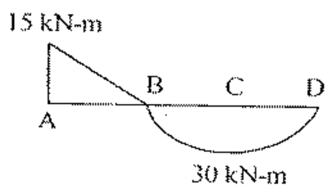
(a)



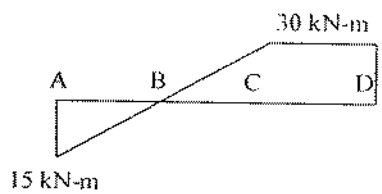
(b)



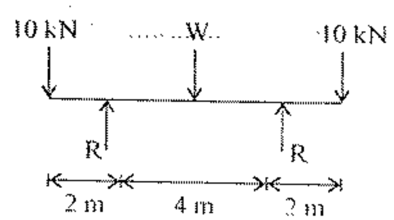
(c)



(d)



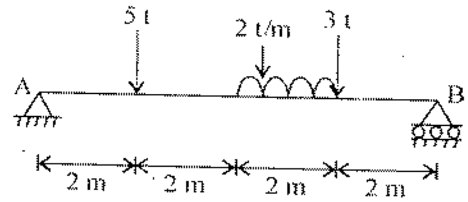
27. A beam of uniform flexural rigidity supports a set of loads as shown in figure below.



What is the value of W if the magnitudes of bending moment at midspan and at support of the beam are numerically equal ?

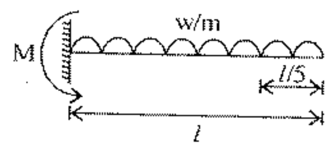
(a) 20 kN (b) 40 kN
(c) 60 kN (d) 80 kN

28. The ratio of reaction R_A and R_B of the simply supported beam in the given figure is



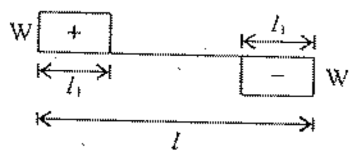
(a) 1/2 (b) 2/3
(c) 3/2 (d) 1

29. In the given figure the maximum bending moment at the fixed end of the cantilever caused by the UDL is M. The bending moment at a section $l/5$ from the free end is



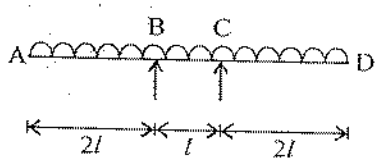
(a) 4% of M (b) 5% of M
(c) 10% of M (d) 20% of M

30. The shear force diagram for a simple supported beam of span l is given in the figure. The maximum bending moment is



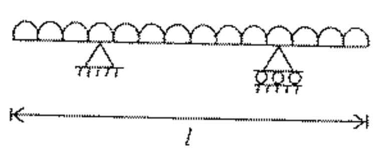
(a) $\frac{Wl}{2}$ (b) $W\left(\frac{l}{2} - l_1\right)$
(c) Wl_1 (d) $W(l - 2l_1)$

31. ABCD is a beam of length $5l$ which is supported at B and C (having supported length BC = l) and having two equal overhangs AB and CD of length $2l$ each. It carries a uniformly distributed load of intensity 'w' per unit length throughout the beam as shown in the given figure.



The points of contra-flexure will occur
(a) at B and C (b) at the mid-point of BC
(c) nowhere in the beam (d) at the mid-points of AB and CD

32. A simply supported beam with equal overhang on both sides is loaded as shown in the figure. If the bending moment at mid-span is zero, then the percentage overhang on each side will be



(a) 33.3 (b) 25
(c) 20 (d) 15

33. Match **List-I** (Beam/Column and loading) with **List-II** (Number of points of contra-flexture) and select the correct answer using the codes given below the lists:

List-I

A. Propped cantilever beam under midpoint loading
B. Fixed beam under uniformly distributed load
C. Fixed beam subjected to a moment at mid point
D. Simply supported column subjected to eccentric load at an intermediate point

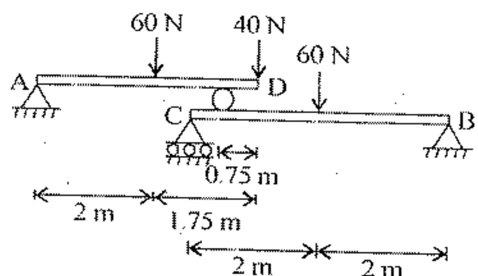
List-II

1. Two
2. Three
3. One

Codes:

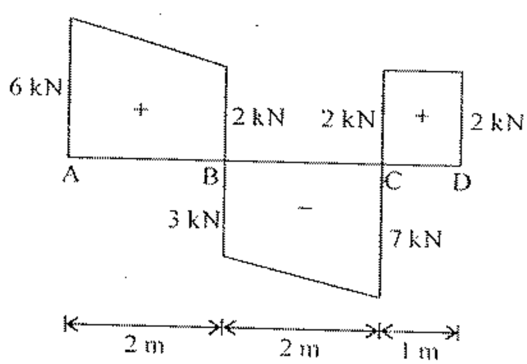
	A	B	C	D
(a)	3	1	2	3
(b)	3	3	2	1
(c)	2	3	3	1
(d)	2	1	3	3

34. In the set of beams shown below, the reactive force at support A is



(a) 30 N (b) 10 N
(c) 20 N (d) 40 N

35. Consider the following statements in regard to the shear force diagram for an overhanging beam supported at A and C:



1. The beam is carrying a uniformly distributed load of 2 kN/m throughout.
2. The beam is carrying a uniformly distributed load of 2 kN/m over the supported span AC and concentrated load of 2 kN at the free end D.
3. The beam is carrying a uniformly distributed load of 2 kN/m over the supported span AC, and concen-trated load of 5 kN at the centre of supported span AC and also a point load of 2 kN at the free end D.
4. The point of contra-flexure occurs between the supported region AC and nearer to support C.

Which of these statements is/are correct?

(a) 1, 2, 3 and 4 (b) 4 only

(c) 2 and 3 (d) 3 and 4

36. A cantilever beam of span l carries a uniformly varying load of zero intensity at the free end and 'w' per metre length at the fixed end. What does the integration of the ordinate of the load diagram between the limits of free and fixed ends of the beam give?

(a) Bending moment at the fixed end.

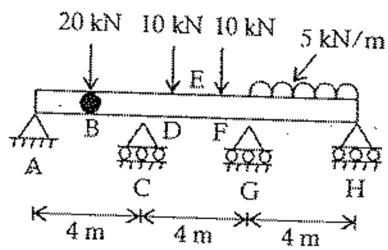
(b) Shear force at the fixed end

(c) Bending moment at the free end

(d) Shear force at the free end.

Common Data for Question 37 and 38:

A three-span continuous beam has an internal hinge at B. Section B is at the mid-span of AC. Section E is at the mid-span of CG. The 20 kN load is applied at section B whereas 10 kN loads are applied at sections D and F as shown in the figure. Span GH is subjected to uniformly distributed load of magnitude 5 kN/m. For the loading shown, shear force immediate to the right of section E is 9.84 kN upwards and the hogging moment at section E is 10.31 kN-m.



AB = BC = 2 m

CD = DE = EF = FG = 1 m

GH = 4 m

37. The magnitude of the shear force immediate to the left and immediate to the right of section B are, respectively

(a) 0 and 20 kN

(b) 10 kN and 10 kN

(c) 20 kN and 0

(d) 9.84 kN and 10.16 kN

38. The vertical reaction at support H is

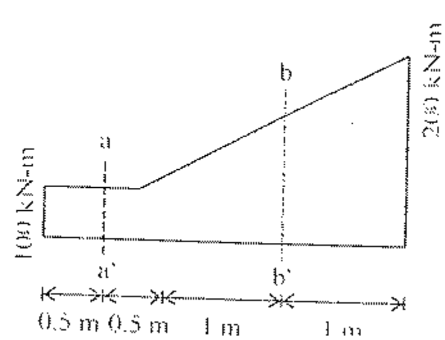
(a) 15 kN upward

(b) 9.84 kN upward

(c) 15 kN downward

(d) 9.84 kN downward

39. The bending moment diagram for a beam is given below:



The shear force at sections aa′ and bb′ respectively are of the magnitude

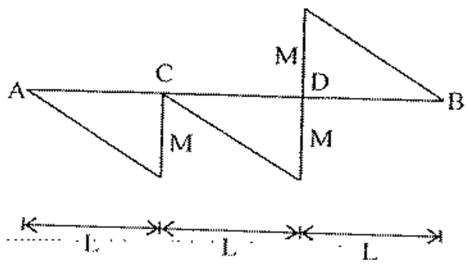
(a) 100 kN, 150 kN

(b) zero, 100 kN

(c) zero, 50 kN

(d) 100 kN, 100 kN

40. A simply supported beam AB has the bending moment diagram as shown in the following figure:



The beam is possibly under the action of following loads

(a) Couples of M at C and 2M at D

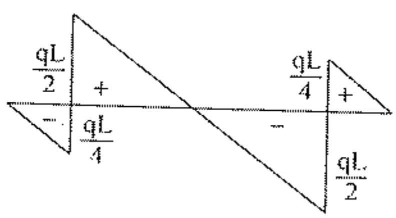
(b) Couples of 2M at C and M at D

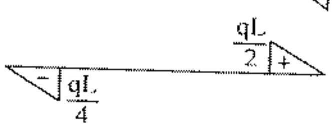
(c) Concentrated loads of M/L at C and 2M/L at D

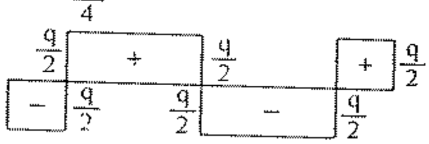
(d) Concentrated load of M/L at C and couple of 2M at D

41. Match **List-I** (Shear Force Diagrams) beams with **List-II** (Diagrams of beams with supports and loading) and select the correct answer by using the codes given below the lists:

List-I

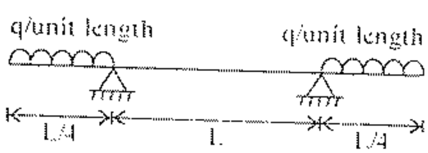
A.


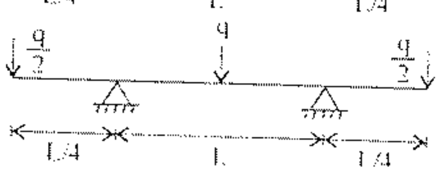
B.


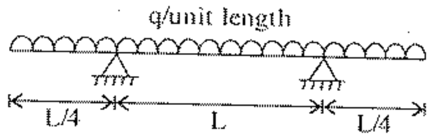
C.


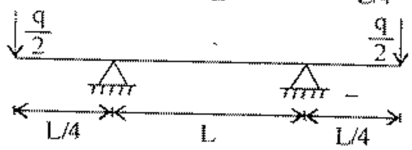
D.


List-II

1.


2.


3. 

4. 

Codes:

	A	B	C	D
(a)	3	1	2	4
(b)	3	4	2	1
(c)	2	1	4	3
(d)	2	4	3	1

ANSWERS

1. (c)
2. (a)
3. (b)
4. (a)
5. (a)
6. (d)
7. (a)
8. (b)
9. (d)
10. (d)
11. (d)
12. (a)
13. (d)
14. (b)
15. (d)
16. (c)
17. (b)
18. (b)
19. (d)
20. (a)
21. (d)
22. (a)
23. (a)
24. (d)
25. (d)
26. (a)
27. (b)
28. (d)
29. (a)
30. (c)
31. (c)
32. (b)
33. (a)
34. (b)
35. (d)
36. (b)
37. (a)
38. (b)
39. (c)
40. (a)
41. (a)

SOLUTION...

1. (c) We know that $\frac{dM}{dx} = V$, $\frac{dV}{dx} = w$

$$\Rightarrow \quad \frac{d^2M}{dx^2} = w$$

- If load intensity variation is linear, BMD will be cubic. If load intensity is constant, BMD will be parabolic. For point loads, BMD is linear. For constant BM, BMD is constant (Rectangular).

  Based on the above

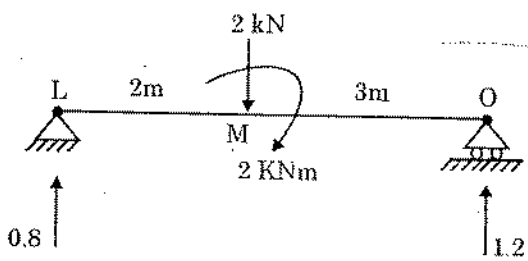
  Udl → Parabolic

  UVl → cubic

  Point load → linear

  Constant BM → Rectangular

2. (a) The equivalent loading is

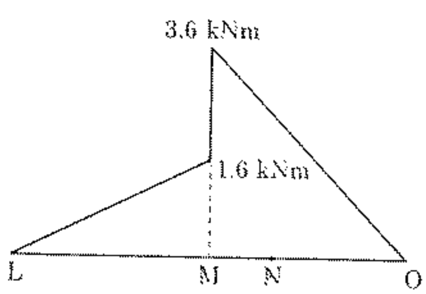


$$\left(\frac{2\times 3}{5}-\frac{2}{5}\right)=0.8$$

BM at M from left = 0.8 × 2 = 1.6 kN-m

Due to clockwise concentrated moment there will be a sudden jump in BM by 2 kN-m

Hence the final BMD is



3. (b) From the options, it is clear that the SFD and BMD

As SFD jumps up by 1 t at left support, remains constant and falls by 1 t at right end, it means there is a concentrated point load acting at the free end.

From the SFD it is clear that there is no load intensity on span because

$$\frac{dV}{dx} = 0 = w$$

but there can be a concentrated moment

There is a sudden change in BM by 2t-m at the mid span thus there is a concentrated moment of 2 kN-m acting in the mid span.

By looking at the options, it is clear that only (b) will satisfy the above conditions. (d) cannot be the answer because concentrated moment here is reducing the net moment.

4. (a) $\frac{dM}{dx} = V \Rightarrow$ Shear force is the 1st derivative of bending moment.

5. (b) The reason follows from $\frac{dM}{dx} = V$

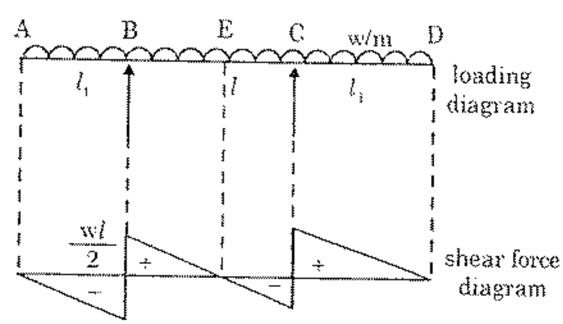
Hence magnitude of BM will be max. when

$$\frac{dM}{dx} = 0 = \text{shear force}$$

Hence, when shear force is zero, BM is max.

Thus assertion follows from reasoning

6. (d)



max. (+) Bending moment will occur at the mid span of central span(E) [(+) means sagging] and max (–)ve BM will occur at the supports B & C

⇒ (Max (+)ve BM – BM at free end)

$$= \int V dx = \text{area under SFD between A to E}$$

$$\Rightarrow [\text{Max} -\text{ve BM} - 0] = \frac{1}{2}\times\frac{wl}{2}\times\frac{l}{2} - \frac{1}{2}\times wl_1 \times l_1$$

$$= \frac{wl^2}{8} - \frac{wl_1^2}{2}$$

⇒ {Max (–ve)BM – BM at free end}

= Area under SFD between A & B.

$$\Rightarrow |(\text{Max} - \text{ve BM} - 0)| = \frac{1}{2}\times wl_1 x l_1 = \frac{wl_1^2}{2}$$

For max (+ve) BM = max (–ve) BM

$$\Rightarrow \frac{wl^2}{8} - \frac{wl_1^2}{2} = \frac{wl_1^2}{2}$$

$$\Rightarrow l_1^2 = \frac{l^2}{8}$$

$$l_1 = \frac{l}{2\sqrt{2}}$$

7. (a) At A, Sudden jump in SFD of 14 t, thus there is an upward point load of 14 t at A.

A to D, Slope of SFD is zero, hence load intensity = 0

At D, sudden drop in SFD by 18 t, hence point load of 18 t acting downwards at D.

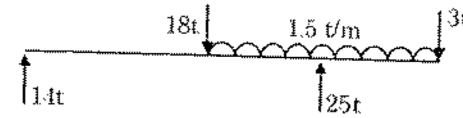
D to B, slope of SFD is (–ve) and constant, hence downward udl from D to B of magnitude $\frac{16-4}{8} = 1.5\,t/m$

At B, sudden jump in SFD by 25t, hence upward loading of 25t.

B to C, slope of SFD is (–ve) and constant, hence downward udl of magnitude $\frac{9-3}{4} = 1.5\,t/m$.

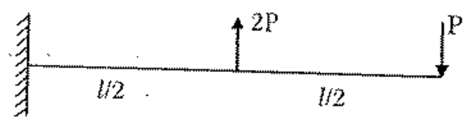
At C, sudden drop in SFD by 3t hence downward loading of 3t.

Thus loading diagram is

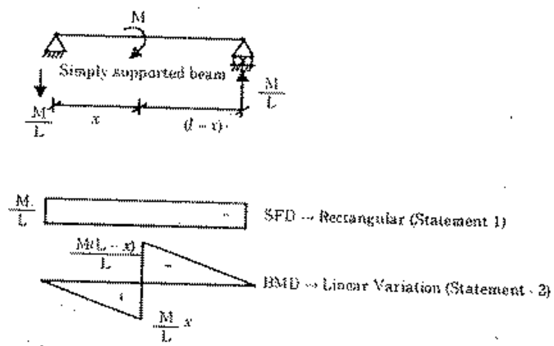


8. (b) The two statements are correct the assertion is the definition of BM but reasoning is the property of BM. Hence property must follow from definition but not the other way round. Thus Assertion does not follow from reasoning.

9. (d)



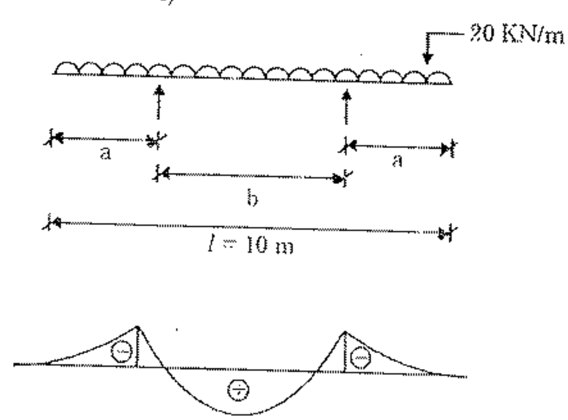
10. (d)



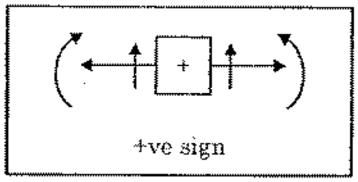
Approach - 2 : Statement-2 is wrong as BMD is one degree higher than SFD.

11. (d)

Most suitable position for location of supports in case of erection of beams (UDL = Self wt)



Conversion Followed



$$\frac{wa^2}{2} = w\frac{L}{2}\left(\frac{L}{2}-a\right)-\frac{w}{2}\left(\frac{L}{2}\right)^2$$

$$a^2 = L\left(\frac{L}{2}-a\right)-\frac{L^2}{4}$$

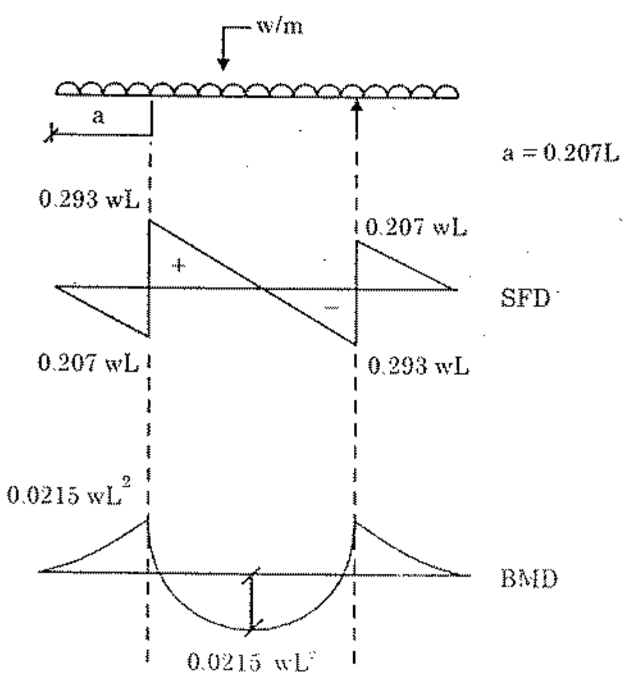
$$a^2 = \frac{L^2}{4}-aL$$

$$\boxed{a = 0.207\,L}$$

$$\Rightarrow \quad a = 0.207 \times 10m = 2.07 \text{ m}$$

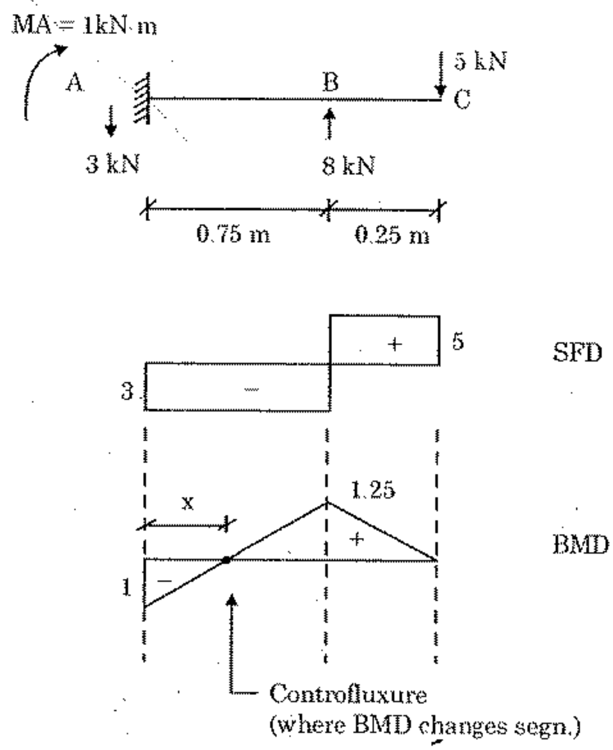
$$M_{max} = \frac{wa^2}{2} = 0.0215\ WL^2$$

Standard Result



Observe the profile of SFD & BMD.

12. (a)

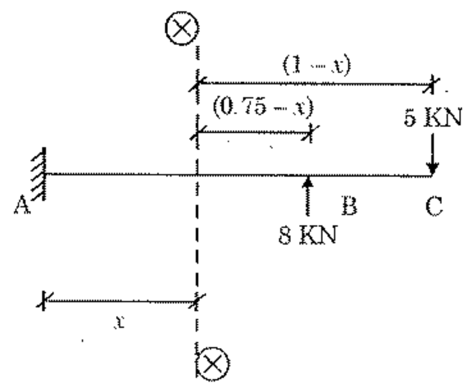


From properly of triangle,

$$\frac{x}{1} = \frac{0.75-x}{1.25}$$

$$\Rightarrow \quad x = \frac{0.75}{2.25} = 0.333\text{m}$$

Approach - 2



By visual inspection, contraflexure occurs in AB portion at sec x − x

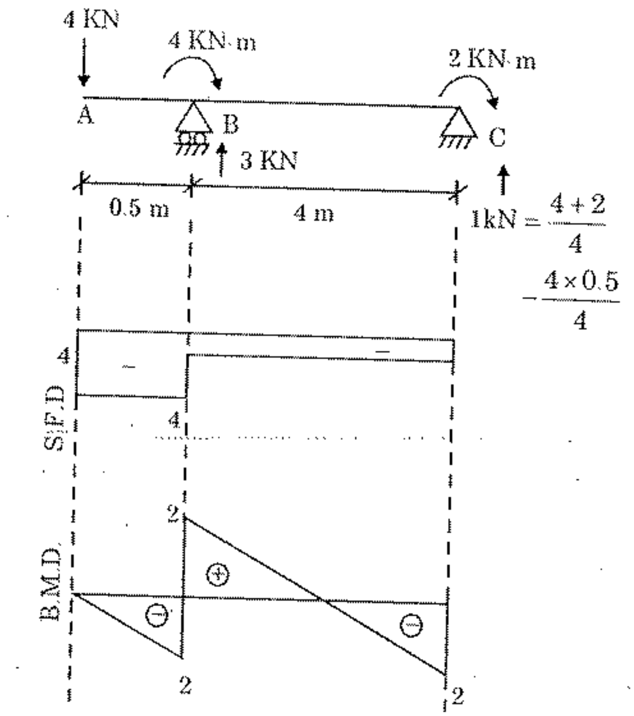
∴ Consider sec x − x.

$$M_x = 5(1-x)-8(0.75-x) = 0$$

$$5(1-x) = 8(0.75-x)$$

$$x = 0.33 \text{ m}$$

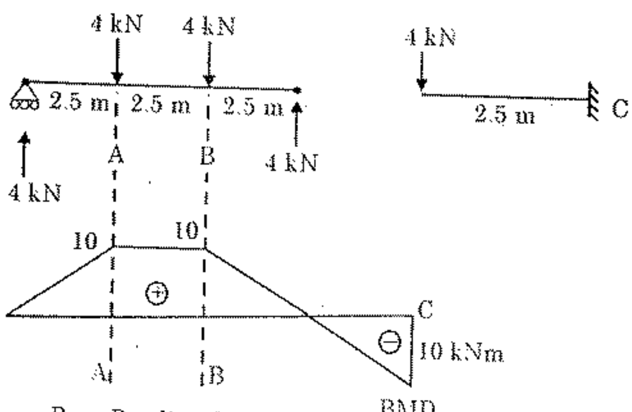
13. (d)



Design BM = Max moment (any where) = 2 kN·m

⇒ Observe BM profile

14. **(b)** Always seperate structure at internal hinges



Pure Bending Case

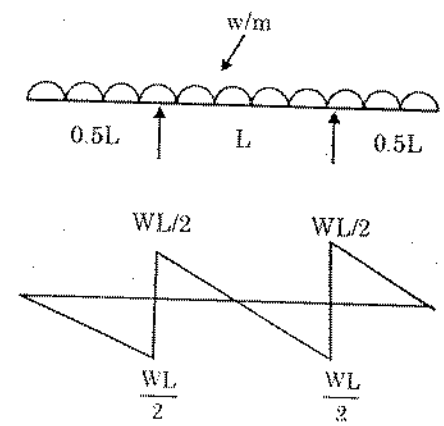
$$M_A = +10$$
$$M_C = -10$$
$$M_B = +10$$

15. (d) Hints :

Loading = UDL

SFD = Linear

BMD = Parabolic ⇒ $\frac{\text{options}}{a \& c}$ eliminated



B.M at centre

$$M = \frac{1}{2}\times\frac{WL}{2}\times\frac{L}{2}-\frac{1}{2}\times\frac{WL}{2}\times\frac{1}{2}=0$$

16. (c) Slope of BMD = SF

AD → slope of BMD is (+)ve

⇒ SF is (+)ve & constant = $\frac{WL}{4\left(\frac{L}{4}\right)} = W$

DE → slope of BMD is (–)ve and constant

⇒ SF is (–)ve and constant

$$= \frac{\left(-\frac{WL}{4}-\frac{WL}{4}\right)}{L/2} = -W$$

At D → Sudden change in slope of BMD i.e. a kink

⇒ Sudden change in SF from W in AD to (–) W in DE

EB → slope of BMD = 0 ⇒ SF = 0

at E → sudden change in slope of BMD

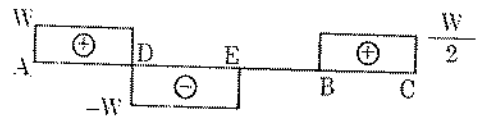
⇒ sudden change in SF from –W to zero.

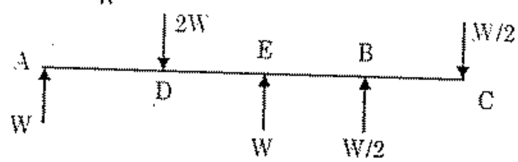
BC → slope of BMD = $\frac{0-\left(-\frac{WL}{4}\right)}{L/2} = \frac{W}{2}$

⇒ SF is $\frac{W}{2}$

At B → There is sudden change in slope from 0 to $\frac{W}{2}$ ⇒ sudden change in SF from 0 to $\frac{W}{2}$

SFD

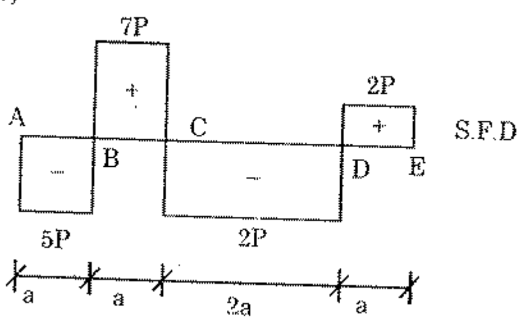


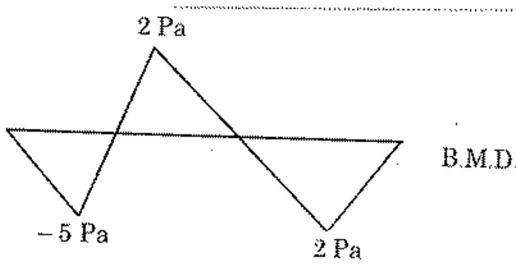


Loading Diagram

Note: $\frac{dV}{dx} = w$

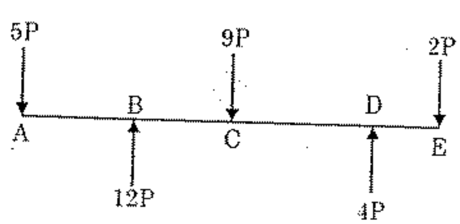
17. (b)





Approach - 2

The loading diagram can be like



To check if there is moment equilibrium take moment about A

$M_A = -2P \times 5a + 4P \times 4a - 9P \times 2a + 12P \times a = 0$

$\Rightarrow$ Hence the loading is in equilibrium.

Max moment can only be at B, C or D.

$$|M_B| = 5Pa$$

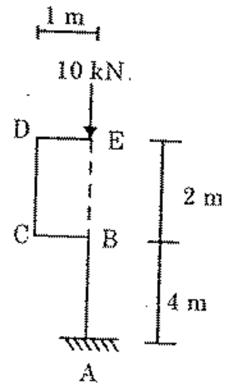
$$|M_C| = 2Pa$$

$$|M_D| = 2Pa$$

$$\Rightarrow \quad |M_{max}| = 5Pa$$

19. (d) Here BM @ A = 0

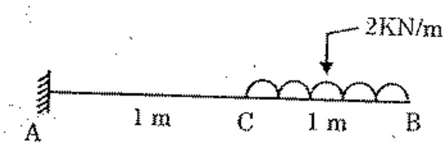
[Since loading passes through A]



20. (a) Basic Problem

Hint : No loading on CB $\Rightarrow$ SFD in CD = 0

21. (d)

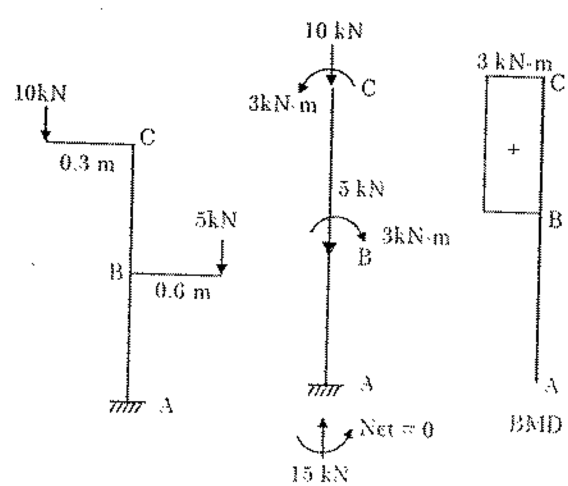


BMD over AC

(direct) No loading $\Rightarrow$ BM linearly varying

In cantilever $\rightarrow$ Max BM occurs at support for unidirectional loading

22. (a)



23. (a) $\frac{dV}{dx} = W = \text{constant in AC}$

[SF is linear in AC]

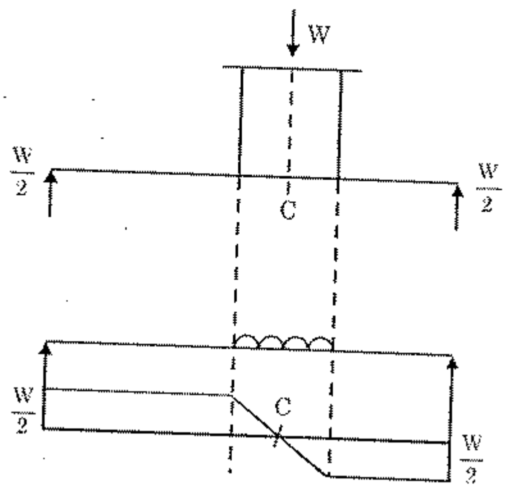
$\frac{dV}{dx} = W = 0 \text{ in CB}$

[V = constant in CB]

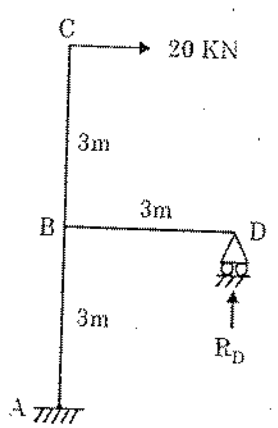
There is no point load in span

[SFD will not change abruptly]

24. (d)



25. (d)



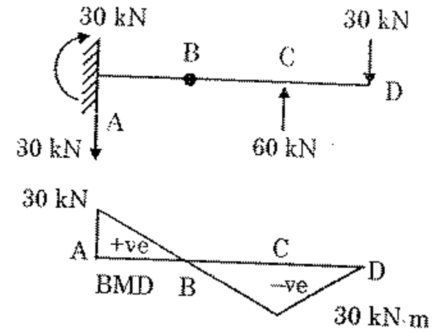
about A, Hinge ⇒ BM = 0

$R_D \times 3 = 20 \times 6$

$R_D = 40 \text{ kN}$

26. (a) Observe Loading

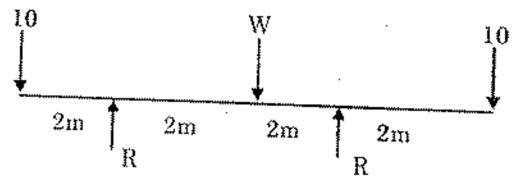
BMD should be linearly varying ⇒ c & d are eliminated



$M_B = 0 \Rightarrow R_C = 60 \text{ kN}$

$\Rightarrow \quad R_B = 30 \text{ kN} \Rightarrow M_A = 30 \text{ kNm}$

27. (b)



$|BM_{\text{below W}}| = R \times 2 - 10 \times 4$

$|BM_{\text{at support}}| = 20$

$\Rightarrow R \times 2 - 40 = 20$

$R = 30$

$W = 40 \text{ kN}$

28. (d) Hints :

1. ΣM_A and $\Sigma M_B = 0\}$ Static equilibrium equations
2. $\Sigma F_y = 0$

$(5 \times 2) + (4 \times 5) + 3 \times 6 = R_B \times 8$ [From $\Sigma M_A = 0$]

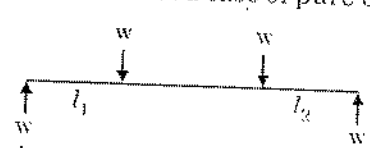
$R_B = \frac{48}{8} = 6 \text{kN}$

$5 + 3 + 4 = R_A + R_B \; [R_A : R_B] = [1 : 1]$

$R_A = 6 \text{ kN}$

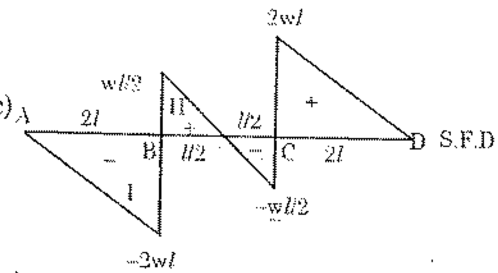
29. (a) $M = \frac{Wl^2}{2}$; $M' = \frac{Wl^2}{2 \times 5^2} = \frac{1}{125}(M) = 4\% \,(M)$

30. (c) Standard case of pure bending



Max B.M = Wl_1

31. (c)



Approaches

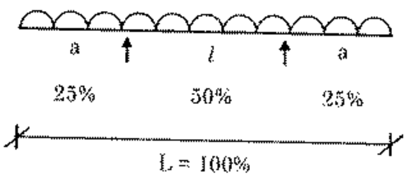
Point of contraflexure will appear only in BC at point of contraflexure BM = 0

Point of contraflexure will appear only when area II ≥ area I

Which is not there in this case

⇒ No point of contraflexure

32. (b)

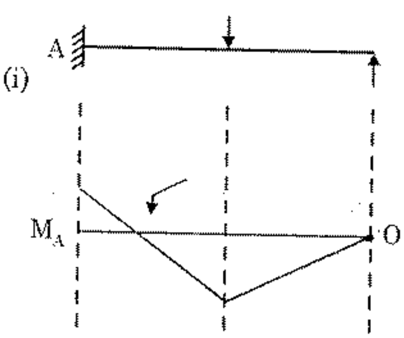


$$\frac{Wl^2}{8} - \frac{Wa^2}{2} = 0$$

$$l = 2a$$

(∴ % of over hang on each side = 25%)

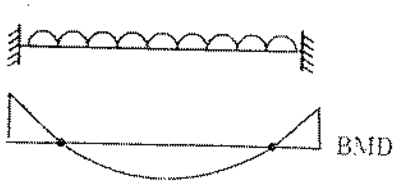
33. (a) (i)



BMD

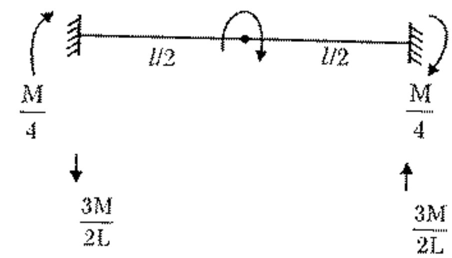
one points of contraflexure

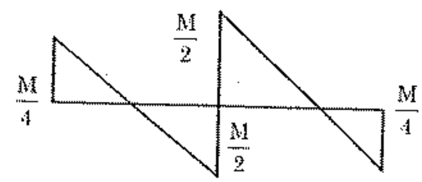
(ii)



2 points of contraflexure

(iii)

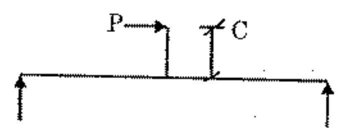


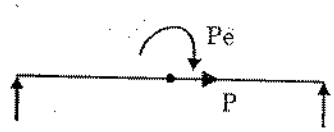


BMD

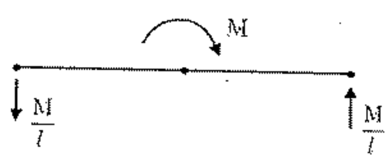
3 points of contraflexure

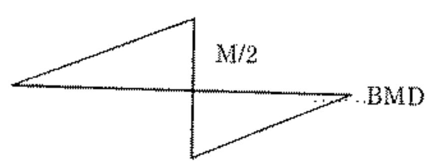
(iv)





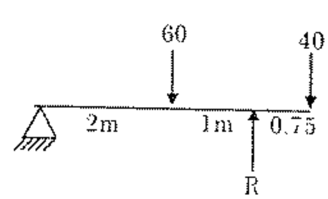
like a SSB with moment M.





One points of contraflexure

34. (b)



$$\Sigma M_A = 0$$

$$60 \times 2 + 40 \times 3.75 = R_{internal} \times 3$$

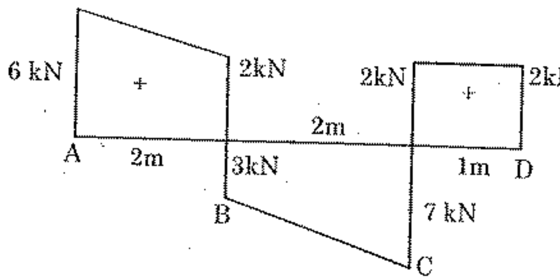
$$R = 90 \text{ kN}$$

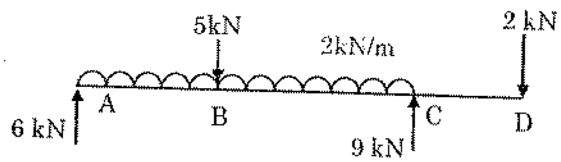
36.

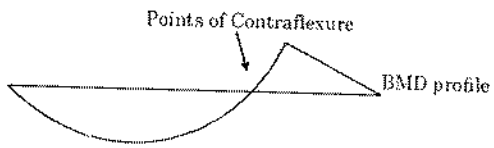
$$R_A + R_i = 60 + 40 \; ; \quad \Sigma F_y = 0$$

$$R_A = 10 \text{ kN}$$

35. (d)







SFD

Following are derived by observing SFD

1. Cantilever CD because support only at A and C
2. support @ A, C
3. Point Load @ B, 5 kN
4. UDL over ABC (linearly varying)

Check Moment equilibrium

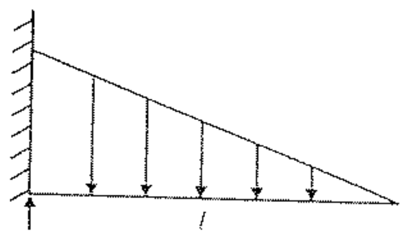
$$\Sigma M_A = -2 \times 5 + 9 \times 4 - 2 \times 4 \times 2 - 5 \times 2 = 0$$

⇒ Loading is correct (as above)

$$\text{UDL} = \frac{6-2}{2} = 2 \text{ kN/m}$$

→ Since continuous beam, point of contraflexure occurs in ABC portion.

36. (b)



$$\frac{dV}{dx} = w_x$$

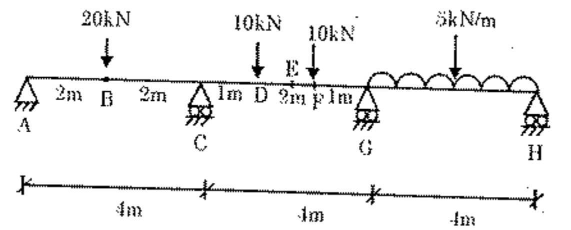
$$\Rightarrow \quad V = \int w_x \, dx$$

$$\Rightarrow V_{fixed} - V_{free} = \int_0^l w_x \, dx$$

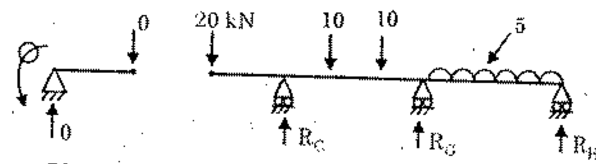
$$V_{free} = 0$$

⇒ V_{fixed} i.e. S.F. at fixed end is calculated

37. (a)



Internal hinge @ B ⇒ seperate @ B.



If AB shares some load, ΣM_A exists.

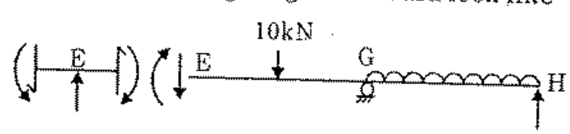
But A is hinge

$$\Rightarrow \quad \Sigma M_A = 0$$

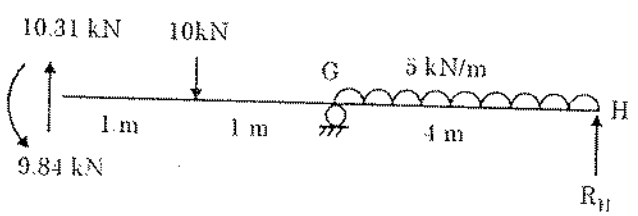
⇒ AB cannot take load.

$$\therefore \quad R_A = 0$$

38. (b) As SF & BM values of section at E is given, the loading diagram should look like



But this diagram gives erroneous answer of R_H, hence the loading diagram should be



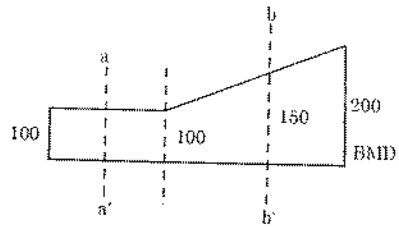
$$\Rightarrow \quad \Sigma M_G = 0$$

$$\Rightarrow -10.31 + 9.84 \times 2 - 10 \times 1 + 5 \times 4 \times 2$$

$$R_H \times 4 = 0$$

$$\Rightarrow \quad R_H = 9.84 \text{ kN (upwards)}$$

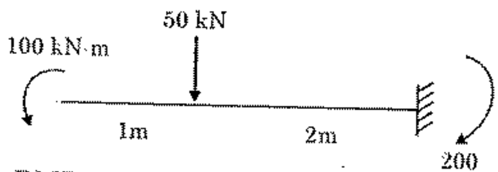
39. (c)



BMD = Constant

⇒ No SFD (A case of moment / couple)

Loading diagram



BMD = Linear ⇒ SFD is constant

⇒ Point load

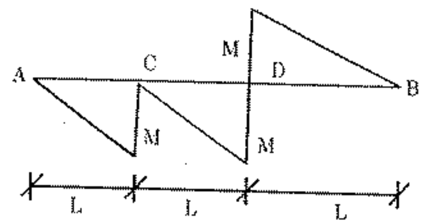
Approach 2

Rate of change of BMD = SF

$$SF\ @\ b - b' = \frac{150-100}{1} = 50\ kN$$

[A problem on moment area theorem]

40. (a)



Property :

Sudden change in BMD

= Moment / couple acting at that point.

@ C → M

@ D → 2M

3

Deflection of Beam

DEFLECTION DIAGRAMS AND THE ELASTIC CURVE

Deflection of structures can occur from various sources, such as loads, temperature, fabrication errors, or settlement. In design, deflections must be limited in order to prevent cracking of attached brittle materials such as concrete or plaster. Hence deflection calculations are required to estimate whether the deflection at any point is less than the permissible value or not. Moreover, in the analysis of indeterminate structures, deflection at a specified points on a structure is required to write down the compatibility conditions.

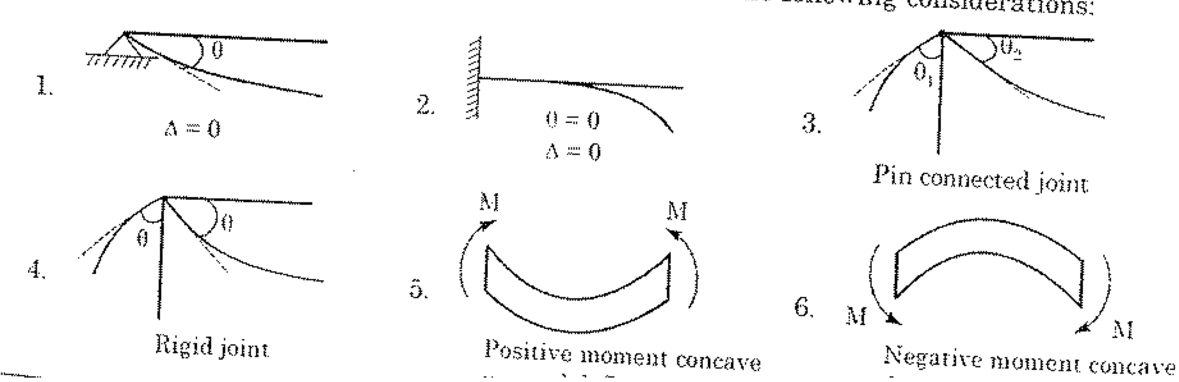
Deflection calculation in this chapter will be done assuming linear elastic material response. [i.e. stress proportional to strain or load is proportional to displacement].

- Deflection of a structure is caused by its <u>internal loadings</u> such as
  Normal force
  Shear force
  Bending moment
  Torsion
- For <u>Beams and frames</u>, major deflection is caused by <u>bending</u>.
- For <u>Trusses</u>, deflection is caused by internal <u>axial forces</u>.

DEFLECTION DIAGRAM FOR BEAMS

To visualize the deflected shape of a structure due to loading, deflection diagram is drawn. Deflection diagram represents the elastic curve for the points at the centroid of the cross-section areas along each of the members.

- Elastic curve can be drawn for a beam or a frame with the following considerations:



7. Point of zero moment is the inflection point [i.e. curvature change point from concave up to concave down or vice versa].

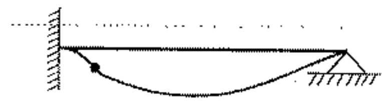
8. To avoid confusion about curvature, it is necessary that BMD be drawn 1st and then using conditions (5) and (6) shown above, deflected shape with appropriate curvature can be plotted.

On the above guidelines, deflected curves for the beams and frames are as shown below.

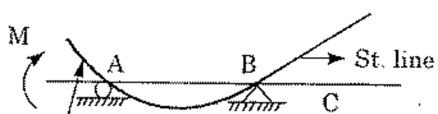
(i)

(ii)

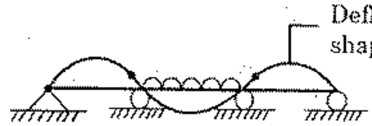
(iii)



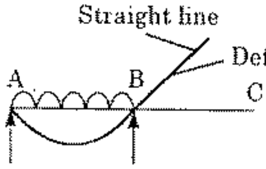
(iv)



(v)

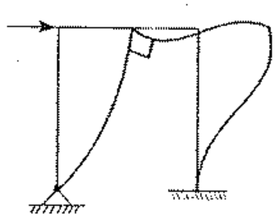


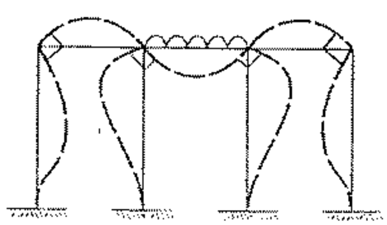
(vi)



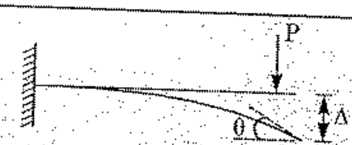
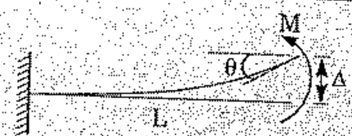
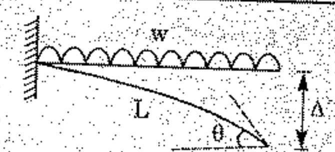
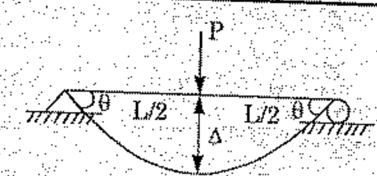
(vii)

Note: *In figure (v) and (vii), if stress is not acting on an element, it will not undergo any distortion.* AB → distortion; BC → beam remains straight

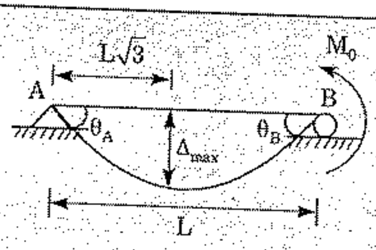
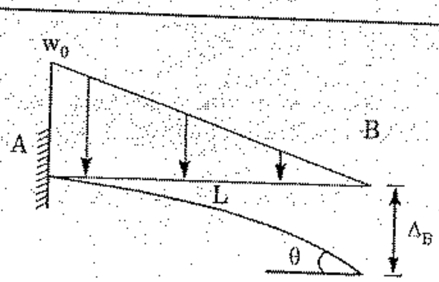
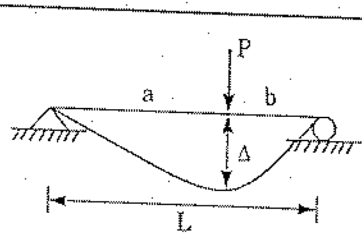


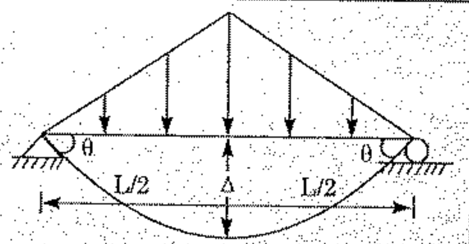
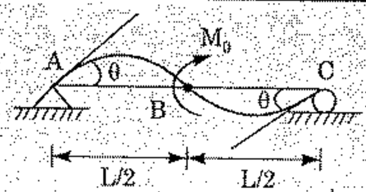
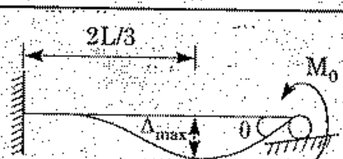
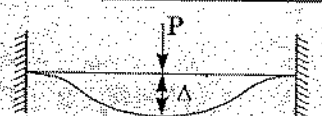


SOME STANDARD RESULTS OF SLOPES AND DEFLECTIONS

	$\Delta = \frac{PL^3}{3EI}$	$\theta = \frac{Pl^2}{2EI}$
	$\Delta = \frac{ML^2}{2EI}$	$\theta = \frac{ML}{EI}$
	$\Delta = \frac{WL^4}{8EI}$	$\theta = \frac{WL^3}{6EI}$
	$\Delta = \frac{PL^3}{48EI}$	$\theta = \frac{PL^2}{16EI}$
	$\Delta = \frac{5}{384}\frac{WL^4}{EI}$	$\theta = \frac{WL^3}{24EI}$

SPECIAL CASES

	$\Delta_{max} = \frac{M_0L^2}{9\sqrt{3}\,EI}$	$\theta_A = \frac{M_0L}{6EI}$ $\theta_B = \frac{M_0L}{3EI}$
	$\Delta_B = \frac{W_0L^4}{30\,EI}$	$\theta_B = \frac{W_0L^3}{24\,EI}$
	$\Delta = \frac{Pa^2b^2}{3EIL}$	

	$\theta = \dfrac{5W_0L^3}{192\,EI}$	$\Delta = \dfrac{WL^4}{120\,EI}$
	$\theta = \dfrac{M_0L}{24\,EI}$	
	$\theta = \dfrac{M_0L}{4EI}$	$\Delta_{max} = \dfrac{ML^2}{27EI}$
	$\Delta = \dfrac{PL^3}{192\,EI} = \dfrac{1}{4}\left(\dfrac{PL^3}{48EI}\right) = \left(\dfrac{1}{4}\times\Delta_{\text{simply supported}}\right)$	
	$\Delta = \dfrac{Pa^3b^3}{3EIL^3}$	
	$\Delta_{max} = \dfrac{WL^4}{384\,EI} = \dfrac{1}{5}\left[\dfrac{5WL^4}{384\,EI}\right] = \left(\dfrac{1}{5}\times\Delta_{\text{simply supported}}\right)$	

APPLICATION OF STANDARD RESULTS BY SUPER-POSITION AND OTHER METHODS

Example 1

Find slope and deflection at point B.

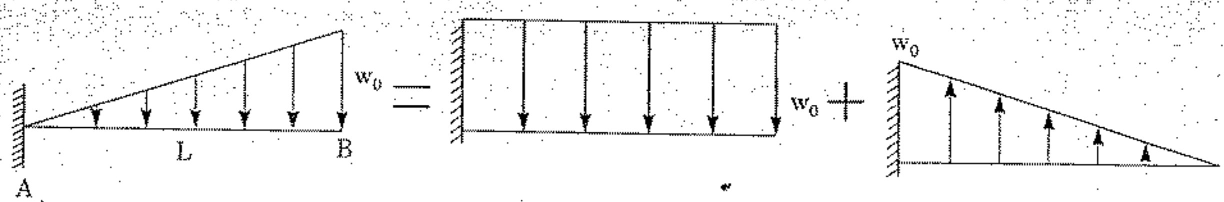


Fig. 1 Fig. 2

Sol:

$$\delta_B = \delta_{B(1)} - \delta_{B(2)} = \frac{w_0L^4}{8EI} - \frac{w_0L^4}{30\,EI}$$

$$\boxed{\delta_B = \frac{11\,w_0L^4}{120\,EI}}$$

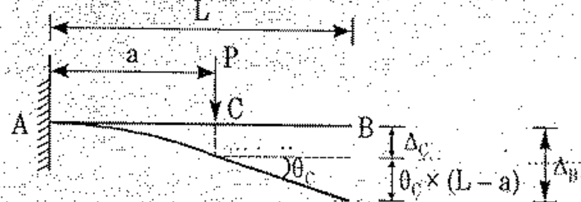
$$\theta_B = \theta_{B_1} - \theta_{B_2}$$

$$= \frac{WL^3}{6EI} - \frac{WL^3}{24EI}$$

$$\boxed{\theta_B = \frac{WL^3}{8EI}}$$

Example 2

Find slope and deflection at B.



Sol: Bending moment in portion BC is zero

⇒ Deformation in beam portion BC = 0

⇒ Slope in BC = Slope at C

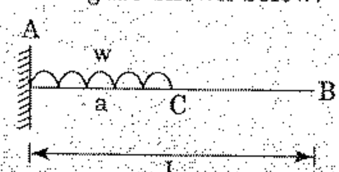
⇒ $\theta_C = \dfrac{Pa^2}{2EI} = \theta_B$

$$\Delta_B = \Delta_C + \theta_C \times (L - a)$$

$$\boxed{\Delta_B = \frac{Pa^3}{3EI} + \frac{Pa^2}{2EI}(L - a)}$$

Example 3

Find slope at C and deflection at B in the figure shown below.



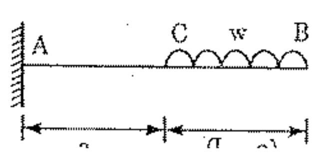
Sol: Deflection and slope at B will be calculated from the same approach as above.

$$\theta_C = \frac{Wa^3}{6EI} = \theta_B$$

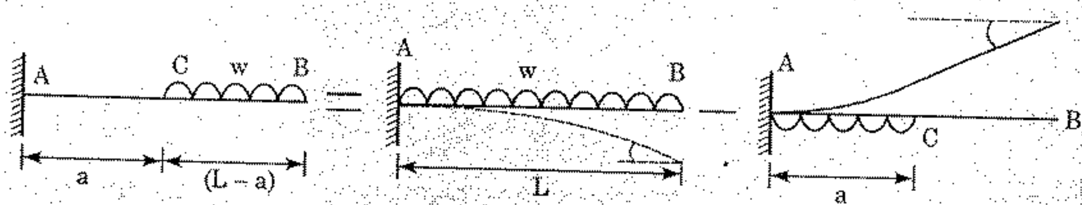
$$\Delta_B = \frac{Wa^4}{8EI} + \frac{Wa^3}{6EI}(L - a)$$

Example 4

Find slope and deflection at B.



Sol:

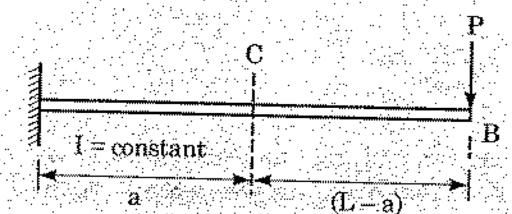


$$\theta_B = \frac{wL^3}{6EI} - \frac{wa^3}{6EI}$$

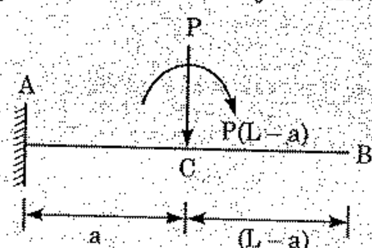
$$\Delta_B = \frac{wL^4}{8EI} - \left[\frac{wa^4}{8EI} + \frac{wa^3(L-a)}{6EI}\right]$$

Example 5

Find θ_C and Δ_C.



Sol: As discussed earlier, θ_C and Δ_C can be calculated by transferring the load effects on C.



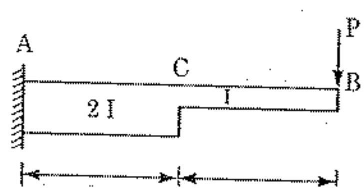
$$\Delta_C = \frac{Pa^3}{3\,EI} + \frac{P(L-a)\,a^2}{2\,EI}$$

$$\theta_C = \frac{Pa^2}{2\,EI} + \frac{P(L-a)\,a}{EI}$$

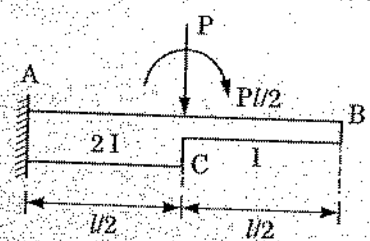
Note: *The above concept of transfer of loading is conveniently applied only in the case of overhangs and cantilevers.*

Example 6

Find Δ_C and θ_C.



Sol: Effect of load at B can be transferred to C as shown in the following figure.



$$\Rightarrow \quad \Delta_C = \frac{P\left(\frac{l}{2}\right)^3}{3E\,(2I)} + \frac{\left(\frac{Pl}{2}\right)\left(\frac{l}{2}\right)^2}{2E\,(2I)}$$

$$\Delta_C = \frac{Pl^3}{48\,EI} + \frac{Pl^3}{32\,EI}$$

$$\boxed{\Delta_C = \frac{5Pl^3}{96\,EI}}$$

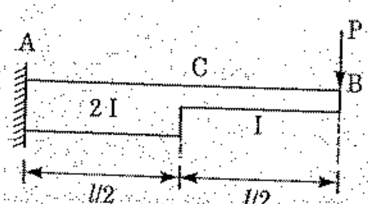
$$\theta_C = \frac{P\left(\frac{l}{2}\right)^2}{2E\,(2I)} + \frac{\left(\frac{Pl}{2}\right)\left(\frac{l}{2}\right)}{E\,(2I)}$$

$$\theta_C = \frac{Pl^2}{16\,EI} + \frac{Pl^2}{8\,EI}$$

$$\theta_C = \frac{3Pl^2}{16\,EI}$$

Example 7

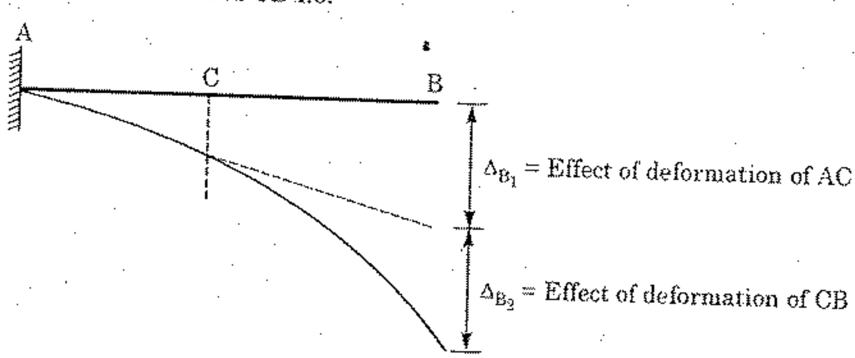
Find Δ_B.



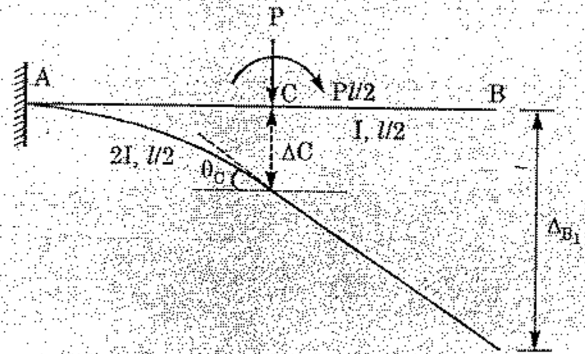
Sol: Deflection at B can be considered to be the effect of combination of

(a) effect due to deformation of AC and

(b) effect due to deformation of CB i.e.



Deformation in AC will occur due to load effects in AC. i.e.



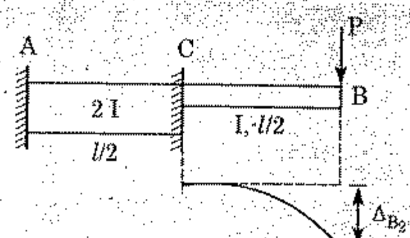
$$\Delta_{B_1} = \Delta_C + \theta_C \frac{l}{2}$$

$$\Delta_{B_1} = \left[\frac{P\left(\frac{l}{2}\right)^3}{3E(2I)} + \frac{\left(\frac{Pl}{2}\right)\left(\frac{l}{2}\right)^2}{2E(2I)}\right] + \left[\frac{P\left(\frac{l}{2}\right)^2}{2E(2I)} + \frac{\left(\frac{Pl}{2}\right)\left(\frac{l}{2}\right)}{E(2I)}\right] \times \frac{l}{2}$$

$$= \frac{5Pl^3}{96EI} + \frac{3Pl^2}{16EI} \times \frac{l}{2} = \frac{5Pl^3}{96EI} + \frac{3Pl^3}{32EI}$$

$$= \frac{14Pl^3}{96EI} = \frac{7Pl^3}{48EI}$$

Effect of deformation of CB can be thought of as shown below i.e. assuming that there is a fixed support at C.



$$\Delta_{B_2} = \frac{P\left(\frac{l}{2}\right)^3}{3EI} = \frac{Pl^3}{24EI}$$

Hence total deflection at B

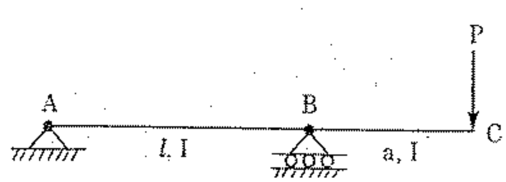
$$\Delta_B = \Delta_{B_1} + \Delta_{B_2}$$

$$\Delta_B = \frac{7Pl^3}{48EI} + \frac{Pl^3}{24EI} = \frac{9Pl^3}{48EI} = \frac{3Pl^3}{16EI}$$

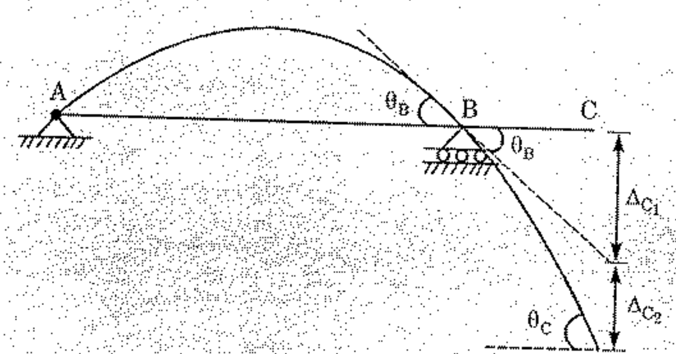
$$\boxed{\Delta_B = \frac{3Pl^3}{16EI}}$$

Example 8

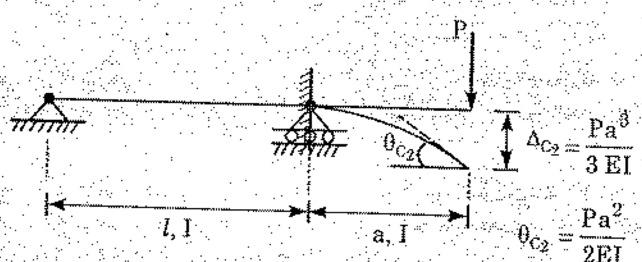
Find deflection and slope at C.

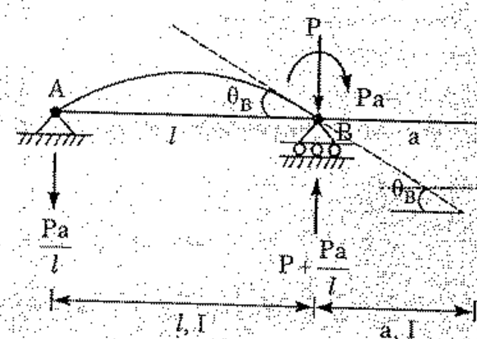


Sol: The deformation of the beam can be through of as:

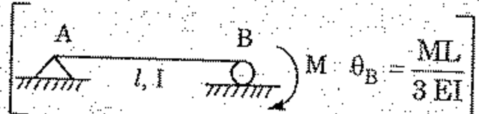


Δ_{C_1} is produced due to deflection of C as caused due to deformation of AB portion. Whereas, Δ_{C_2} is produced due to deformation of BC. Note that when we consider deflection of BC due to deformation of AB, BC have the constant slope of θ_B.





$\theta_B = \frac{Pal}{3\,EI}$



$$\Rightarrow \quad \Delta_C = \Delta_{C_1} + \Delta_{C_2}$$

$$= (\theta_B)a + \Delta_{C_2}$$

$$= \frac{Pal}{3\,EI} \times a + \frac{Pa^3}{3\,EI} = \frac{Pa^2 l + Pa^3}{3\,EI}$$

$$\Delta_C = \frac{Pa^2\,(l + a)}{3\,EI}$$

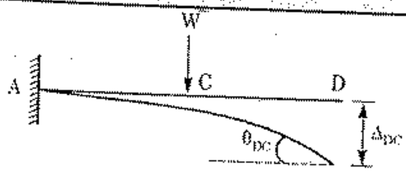
$$\theta_C = \theta_B + \theta_{C_2}$$

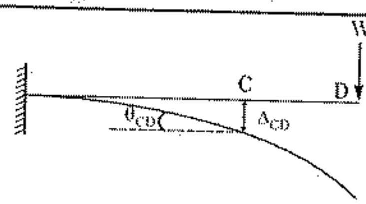
$$= \frac{Pal}{3\,EI} + \frac{Pa^2}{2\,EI}$$

MAXWELL RECIPROCAL THEOREM

Maxwell reciprocal theorem is also used to calculate deflection and slope at a point in a beam, frame or trusses.

> *"In any beam, frame or truss, the deflection at any point 'D' due to load W at any point C is equal to deflection at point 'C' due to load W at point D"*



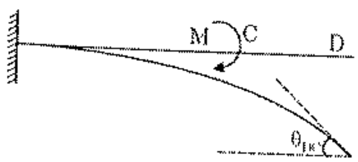


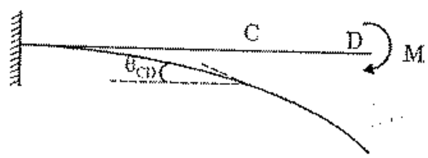
$\Delta_{DC} = \Delta_{CD}$ [As per Maxwell reciprocal theorem]

Δ_{DC} = deflection at point D due to load W at point C.

Δ_{CD} = deflection of point C due to load W at point D.

Similarly if in place of load W, moment M is acting then,



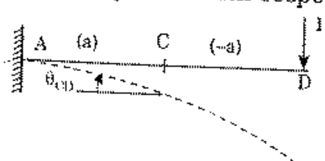


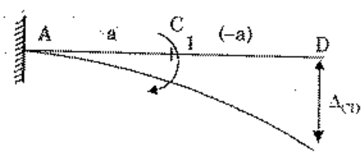
$\theta_{DC} = \theta_{CD}$

θ_{DC} = slope at D due to a couple M applied at C

θ_{CD} = slope at C due to a couple M applied at D

Similarly we can also say that with respect to the fig shown below.



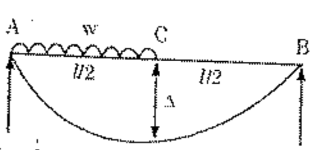


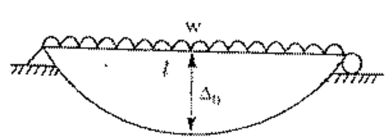
$\theta_{CD} = \Delta_{CD}$

i.e clockwise rotation at C due to downward unit load at D is equal to downward deflection at D due to clockwise unit couple at C

$$\theta_{CD} = \frac{a^2}{2EI} + \frac{(L-a)a}{EI} \qquad \Delta_{DC} = \frac{1\times a^2}{2EI} + \frac{a(L-a)}{EI}$$

SPECIAL CASES

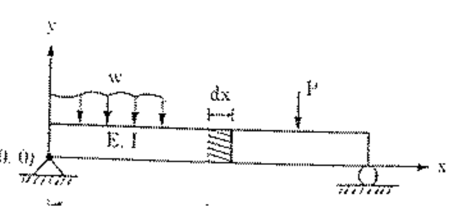




Δ at C will be same whether udl is in AC or CB. Hence due to udl on whole span $\Delta_0 = 2\Delta$.

METHODS FOR DETERMINING SLOPE AND DEFLECTION AT A POINT

(A) Double Integration Method:



$$\frac{M}{EI} = \frac{\frac{d^2y}{dx^2}}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}$$

Since slope of elastic curve $\left(\frac{dy}{dx}\right)$ is very small, the above equation can be reduced to

$$\Rightarrow \quad \boxed{\frac{d^2y}{dx^2} = \frac{M}{EI}} \quad \text{-------------------- (i)}$$

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{M}{EI}$$

$$\Rightarrow \quad \boxed{\frac{d\theta}{dx} = \frac{M}{EI}} \quad \text{-------------------- (ii)}$$

where M = BM at any section x as shown in figure

θ = Slope at any section x as shown in figure

E = modulus of elasticity

I = Moment of inertia of the section.

Double integration of above equation yields the equation of elastic deflected curve. If the beam is having some discontinuity in the elastic deflected curve, double integration technique can not be used directly. The discontinuity could be due to internal hinge (moment release) or slider (shear release).

Note: *The deflection obtained from (i) gives the deflection only due to bending. In beams we generally neglect deflection due to shear, axial force and torsion because these deflections are small as compared to that due to bending.*

Note: *EI = Flexural rigidity*

$$\textit{Flexural Stiffness} = \frac{\textit{Flexural rigidity}}{\textit{length}}$$

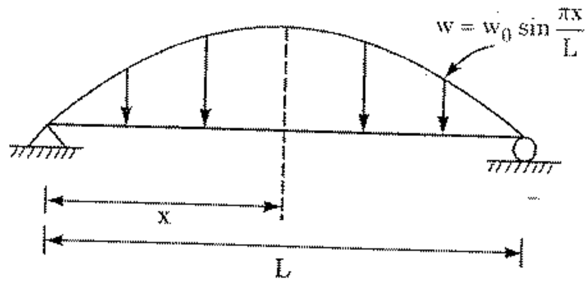
Also, the following relationship must be remembered

$$\frac{d^2y}{dx^2} = \frac{M}{EI}$$

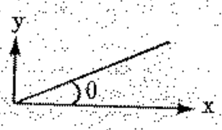
$$\Rightarrow \quad \frac{d^3y}{dx^3} = \frac{dM/dx}{EI} = \frac{V}{EI}$$

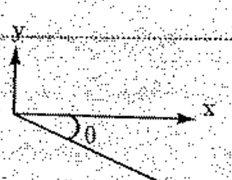
$$\frac{d^4y}{dx^4} = \frac{dV}{dx}\cdot\frac{1}{EI} = \frac{W}{EI} \quad \text{-------------------- (iii)}$$

Deflection can be directly calculated from loading intensity by integrating equation (iii), i.e. $\boxed{\frac{d^4y}{dx^4} = \frac{W}{EI}}$ 4 times. This relationship is used when loading is such that it becomes cumbersome to calculate the bending moment like a sinusoidal loading as shown below in the figure.

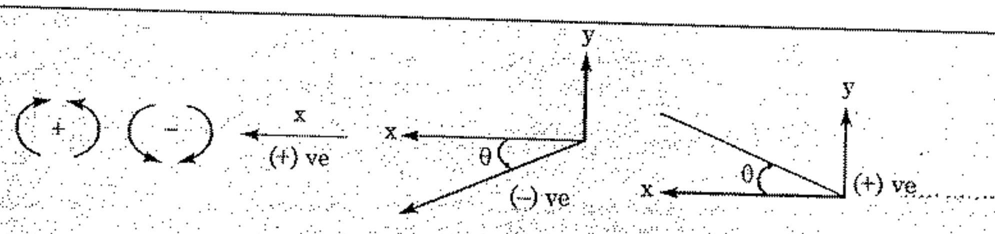


Sign Convention (I)

1.  Sagging moment is taken as (+) ve.
2.  Hogging moment is taken as (–) ve.
3. (Origin) ⟶ x (+) ve [x is taken (+) ve when measured from left to right].
4. Upward deflection is taken as (+) ve.
5. Downward deflection is taken as (–) ve.
6.  Slope is taken as (+) ve, if θ is measured counter clockwise from x-axis.

 Slope is taken as (–) ve, if θ is measured clockwise from x-axis.

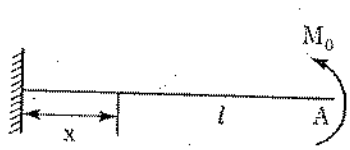
Alternative Sign Convention (II)



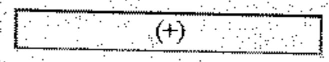
Note that if we see the above sign convention (I) from back of page by holding paper in light, we see it like that shown in the alternative sign conventional. Note that angle measured from (+) ve x-direction to (+) y-direction is taken as (+) ve.

Example 9

Find slope and deflection at A.



Sol:



M_0 BMD.

$$EI\frac{d^2y}{dx^2} = +M_0$$

$$EI\frac{dy}{dx} = M_0x + C_1$$

$$EI\,y = \frac{M_0x^2}{2} + C_1x + C_2$$

Boundary conditions are

at $x = 0$, $y = 0$ and

at $x = 0$, $\frac{dy}{dx} = 0$

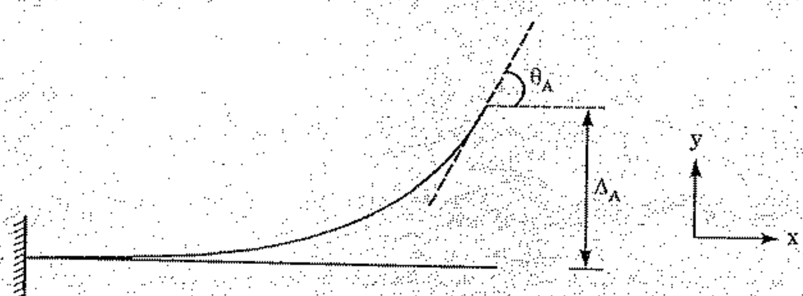
$$\Rightarrow \boxed{C_2 = 0}, \quad \boxed{C_1 = 0}$$

$$\Rightarrow EI\,y = \frac{M_0x^2}{2}$$

$$\boxed{y = \frac{M_0x^2}{2EI}}$$

$$EI\frac{dy}{dx} = M_0x$$

$$\boxed{\frac{dy}{dx} = \frac{M_0x}{EI}}$$

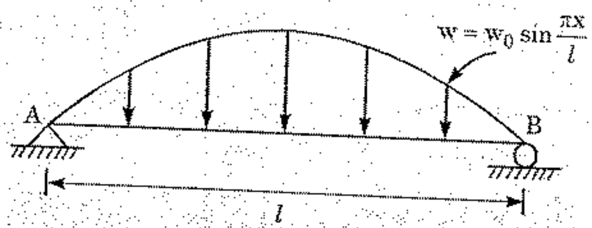


$\theta_A = \frac{M_0L}{EI}$ (+) ve means anticlockwise

$\Delta_A = \frac{M_0L^2}{2EI}$ (+) means upward

Example 10

Find slope at (A) and max. deflection.



Sol: Here load intensity is such that calculation of bending moment will be cumbersome. Thus we work directly in terms of load intensity.

$$EI\frac{d^4y}{dx^4} = -w_0\sin\frac{\pi x}{l} \quad \text{[(–) because loading is downwards]}$$

On integration

$$EI\frac{d^3y}{dx^4} = \frac{w_0 l}{\pi}\cos\frac{\pi x}{l} + C_1$$

$$EI\frac{d^2y}{dx^2} = \frac{w_0 l^2}{\pi^2}\sin\frac{\pi x}{L} + C_1x + C_2 = M \quad \text{(Bending moment)} \text{ ---------- (A)}$$

Boundary conditions are: At x = 0, M = 0,

$$\Rightarrow \boxed{C_2 = 0}$$

and at $x = l$, M = 0

$$0 = \frac{W_0 l^2}{\pi^2}(\sin\pi) + C_1 l$$

$$\Rightarrow \boxed{C_1 = 0}$$

⇒ Thus, equation (A) reduces to

$$EI\frac{d^2y}{dx^2} = \frac{W_0 l^2}{\pi^2}\sin\frac{\pi x}{L}$$

On integration

$$EI\frac{dy}{dx} = \frac{-W_0 l^3}{\pi^3}\cos\frac{\pi x}{L} + C_3 \text{ ---------- (B)}$$

$$EI\,y = \frac{-W_0 l^4}{\pi^4}\sin\frac{\pi x}{L} + C_3x + C_4 \text{ ---------- (C)}$$

Boundary conditions are

at x = 0, y = 0 $\Rightarrow \boxed{C_4 = 0}$

at $x = l$, y = 0 $\Rightarrow 0 = \frac{-W_0 l^4}{\pi^4}\sin\pi + C_3 l$

$$\Rightarrow \boxed{C_3 = 0}$$

Thus equation (B) and (C) reduces to

$$\Rightarrow \boxed{EI\frac{dy}{dx} = \frac{-W_0 l^3}{\pi^3}\cos\frac{\pi x}{L}}$$

$$\boxed{EI\,y = \frac{-W_0 l^4}{\pi^4}\sin\frac{\pi x}{L}}$$

Slope at end A (i.e. for x = 0)

$$EI\,\theta_A = \frac{-W_0 l^3}{\pi^3}$$

$$\boxed{\theta_A = \frac{-W_0 l^3}{\pi^3 EI}}$$ (–) means clockwise

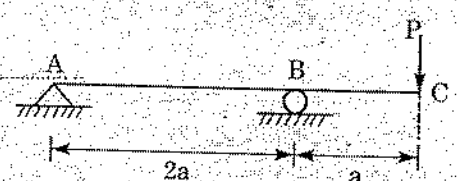
Max. deflection occurs at $x = \frac{l}{2}$ [i.e. when $\frac{dy}{dx} = 0$]

$$EI\,y_{max} = \frac{-W_0 l^4}{\pi^4}\sin\frac{\pi}{2}$$

$$\boxed{y_{max} = \frac{-W_0 l^4}{\pi^4 EI}}$$ (–) mean downward deflection

Example 11

Determine displacement at 'C' assuming EI = constant.



Sol: In this problem equation for moment 'M' to be used in equation $\left[\frac{d^2y}{dx^2} = \frac{M}{EI}\right]$ will be different in regions AB and BC.

- Hence let us write x in AB as x_1 with limit of x_1 from $x_1 = 0$, $x_1 = 2a$.
- x in BC = x_2 with limit of x_2 from $x_2 = 2a$, to $x_2 = 3a$

$$R_A + R_B = P$$
$$R_B \times 2a = P \times 3a$$
$$\Rightarrow \quad R_B = \frac{3P}{2}$$
$$R_A = \frac{-P}{2}$$
$$EI\frac{d^2y_1}{dx_1^2} = -\frac{Px_1}{2}, \quad 0 \le x_1 \le 2a$$

On integration

$$EI\frac{dy_1}{dx_1} = \frac{-Px_1^2}{4} + C_1 \text{ ---------- (iv)}$$

$$EI\,y_1 = \frac{-Px_1^3}{12} + C_1x_1 + C_2 \text{ ---------- (v)}$$

Similarly,

$$EI\frac{d^2y_2}{dx_2^2} = -P(3a - x_2)$$

$$EI\frac{dy_2}{dx_2} = \frac{Px_2^2}{2} - 3Pax_2 + C_3 \text{ ---------- (vi)}$$

$$EI\,y_2 = \frac{Px_2^3}{6} - \frac{3Pax_2^2}{2} + C_3x_2 + C_4 \text{ ---------- (vii)}$$

Applying boundary conditions now

$$\begin{cases} \text{at } x_1 = 0,\ y_1 = 0 \Rightarrow C_2 = 0 & \text{(A)} \\ \text{at } x_1 = 2a,\ y_1 = 0 \Rightarrow C_1 = \dfrac{P(8a^3)}{12\times 2a} = \dfrac{Pa^2}{3} & \text{(B)} \end{cases}$$

$$\text{at } x_2 = 2a,\ y_2 = 0 \text{ ---------- (C)}$$

$$\text{at } x_1 = x_2 = 2a,\ \frac{dy_1}{dx_1} = \frac{dy_2}{dx_2} \text{ ---------- (D)}$$

From (C)

$$\frac{P(8a^3)}{6} - \frac{3Pa}{2}\times 4a^2 + 2C_3a + C_4 = 0 \text{ ---------- (E)}$$

and from (iv) and (vi)

$$\frac{-4Pa^2}{4} + C_1 = \frac{4Pa^2}{2} - 3Pa\,(2a) + C_3 \text{ ---------- (F)}$$

$$\Rightarrow \frac{-4Pa^2}{4} + C_1 = \frac{4Pa^2}{2} - 6Pa^2 + C_3$$

$$\Rightarrow -Pa^2 + \frac{Pa^2}{3} = \frac{4Pa^2}{2} - 6Pa^2 + C_3 \quad \left[\because C_1 = \frac{Pa^2}{3} \text{ from (B)}\right]$$

$$\Rightarrow \boxed{\frac{10\,Pa^2}{3} = C_3} \text{ ---------- (G)}$$

$$\frac{8Pa^3}{6} - \frac{12Pa^3}{2} + 2C_3a + C_4 = 0 \quad \text{[From E]}$$

$$\frac{8Pa^3 - 36Pa^3 + 2a\,(20Pa^2)}{6} = -C_4$$

$$\Rightarrow \frac{12Pa^3}{6} = -C_4$$

$$\boxed{C_4 = -2Pa^3} \text{ ---------- (H)}$$

Thus putting the value of C_3 and C_4 in eq. (vii)

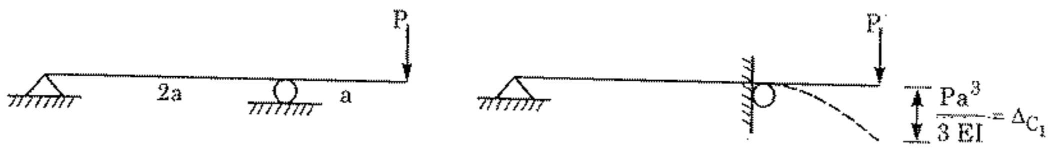
$$EI\,y_2 = \frac{Px_2^3}{6} - \frac{3Pa\,x_2^2}{2} + \frac{10Pa^2x_2}{3} - 2Pa^3$$

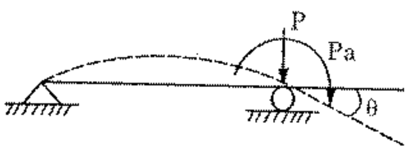
$$\boxed{EI\,y_2 = \frac{P}{6}(x_2^3 - 9ax_2^2 - 20a^2x_2 - 12a^3)}$$

at $\quad x_2 = 3a, \quad EI\,y_2 = \frac{P}{6}[27a^3 - 81a^3 + 60a^3 - 12a^3]$

$\Rightarrow \quad \boxed{y_2 = \frac{-Pa^3}{EI}} \quad y_2$ is (–) ve $\Rightarrow$ downward deflections.

Note: *This deflection can also be calculated as*



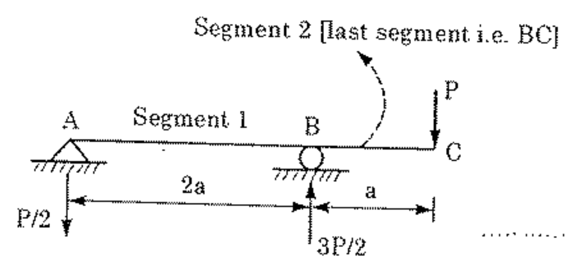


$$\theta.a = \frac{Pa\,(2a)}{3\,EI} \times a = \Delta_{C_2}$$

$$\Delta_c = \Delta_{c_1} + \Delta_{c_2} = \frac{Pa^3}{3EI} + \frac{2Pa^3}{3EI} = \frac{Pa^3}{EI}$$

MACAULAY'S METHOD

When BM equation changes along the length of the beam due to various loads, a convenient method of writting BM for entire beam is as follows.



$$EI\,\frac{d^2y}{dx^2} = M = \frac{-P}{2}x + \frac{3P}{2}\langle x - 2a\rangle \text{ .... (A)}$$

Equation A is the BM written for last segment. However, this equation will be valid for the entire span of the beam under the following condition $\langle x - 2a\rangle$ term is zero if it is (–) ve. i.e. for $x < 2a$, here x is measured from A i.e. left end support

$$\Rightarrow \qquad EI\,\frac{dy}{dx} = -\left(\frac{Px^2}{4} + C_1\right) + \frac{3P}{2}\,\frac{\langle x - 2a\rangle^2}{2} \text{ .... (B)}$$

$$EI\,y = \left(\frac{-Px^3}{12} + C_1x + C_2\right) + \frac{3P}{2}\,\frac{\langle x - 2a\rangle^3}{6} \text{ .... (C)}$$

Boundary conditions are

at x = 0, y = 0

From (C) $\Rightarrow$ $0 = 0 + 0 + C_2 + 0$

$\Rightarrow$ $\boxed{C_2 = 0}$

at x = 2a, y = 0

From (C) $\Rightarrow$ $0 = \frac{-P(8a^3)}{12} + C_1(2a) + 0 + 0$

$$\Rightarrow \quad C_1 = \frac{+P \times 4a^2}{12} \quad \Rightarrow \quad \boxed{C_1 = \frac{Pa^2}{3}}$$

$$EI\,y = \frac{-Px^3}{12} + \frac{3P}{12}\langle x - 2a\rangle^3 + \frac{Pa^2}{3}x$$

Deflection at x = 3a

$$EI\,y = \frac{-27a^3P}{12} + \frac{3Pa^3}{12} + \frac{3Pa^3}{3}$$

$$= (-27 + 3 + 12)\frac{Pa^3}{12}$$

$$EI\,y = \frac{-12Pa^3}{12}$$

$\boxed{y = \frac{-Pa^3}{EI}}$ (–) ve sign means downward deflection

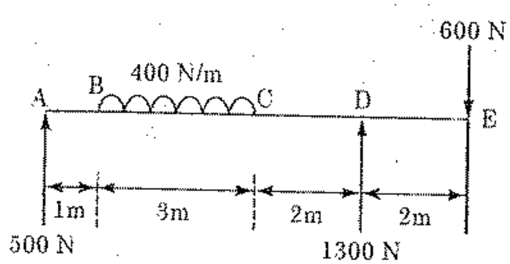
$\Rightarrow$ Deflection is downwards at 'C', equal to $\frac{Pa^3}{EI}$.

Note: *Constant terms (C_1 and C_2) should be included in the 1st term.*

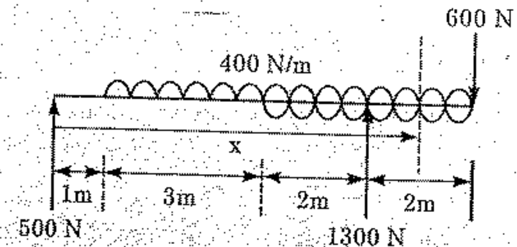
- *Difficulty in this method appear when we have (udl) or (uvl) in part of the beam.*
- *In this case modification to the loading pattern is done so that udl or uvl becomes continuous upto the last segment.*
- *Finally BM equation is written for last segment. So that all loadings are automatically included in general moment equation by writing it for last segment.*

Example 12

Find deflection at a distance of 5 m from support A.



Sol: Modified form of loading is as given below, so that part udl in span be converted into udls continuous upto the last segment.



BM equation written for the last segment becomes

$$\boxed{M = 500\,x - 400\,\frac{\langle x-1\rangle^2}{2} + 400\,\frac{\langle x-4\rangle^2}{2} + 1300\,\langle x-6\rangle}$$

$$\Rightarrow \quad EI\,\frac{d^2y}{dx^2} = 500\,x - \frac{400\,\langle x-1\rangle^2}{2} + \frac{400\,\langle x-4\rangle^2}{2} + 1300\,\langle x-6\rangle$$

On integration

$$EI\,\frac{dy}{dx} = \left\{\frac{500x^2}{2} + C_1\right\} - \frac{400\,\langle x-1\rangle^3}{6} + \frac{400\langle x-4\rangle^3}{6} + \frac{1300\,\langle x-6\rangle^2}{2}$$

$$EI\,y = \left\{\frac{500x^3}{6} + C_1x + C_2\right\} - \frac{400\,\langle x-1\rangle^4}{24} + \frac{400\langle x-4\rangle^4}{24} + \frac{1300\langle x-6\rangle^3}{6}$$

Boundary conditions are

at x = 0, y = 0

$\Rightarrow \quad \boxed{C_2 = 0}$ [Because all terms inside < > bracket becomes (–) ve for x = 0 and hence becomes zero.]

at x = 6m, y = 0

$$\Rightarrow \quad 0 = \frac{500(6)^3}{6} + 6C_1 - \frac{400(6-1)^4}{24} + \frac{400(6-4)^4}{24} + 0$$

$$= 500\times 36 + 6C_1 - \frac{400\times 5^4}{24} + \frac{400(2^4)}{24}$$

$$\Rightarrow \quad \boxed{C_1 = -1308.33}$$

$$\Rightarrow \quad EI\,y = \frac{500x^3}{6} - 1308.33\,x - \frac{400\langle x-1\rangle^4}{24} + \frac{400\langle x-4\rangle^4}{24} + \frac{1300\langle x-6\rangle^3}{6} \qquad \text{(A)}$$

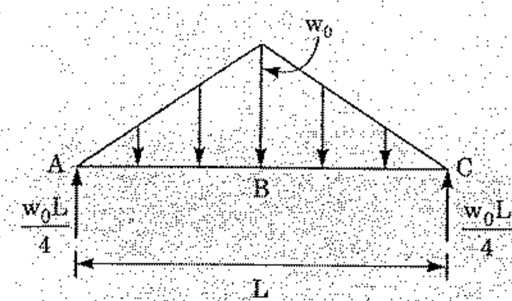
Deflection at x = 5 m, y_5, is calculated by putting x = 5 in equation (A)

$$EI\,y_5 = \frac{500(5)^3}{6} - 1308.33\times 5 - \frac{400}{24}(5-1)^4 + \frac{400(5-4)^4}{24} + 0$$

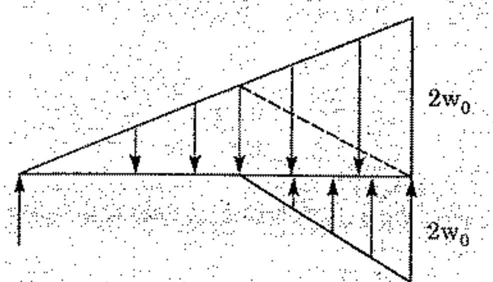
$\Rightarrow \quad \boxed{y_5 = \frac{-375}{EI}}$ [(–) ve sign means downward deflection]

Example 13

Determine deflection equation and magnitude of max. deflection in the beam shown below.



Sol: In the above problem equation for bending moment in span AB and BC will be different when we measure x from left end only. Hence as per Maculey's method, we need to modify the loading. The modified loading could be like as shown below.



But due to symmetry, we can reduce the problem by writting BM equation only for $\frac{1}{2}$ portion of beam i.e. AB only. The deflection and slope calculated at any point in AB will have corresponding location in BC i.e. if deflection in AB is calculated at $\frac{L}{4}$ distance from A, the deflection at a distance $\frac{L}{4}$ from C will also have the same value.

$$EI\frac{d^2y}{dx^2} = M_{AB} = \frac{W_0Lx}{4} - \frac{W_0x}{\frac{L}{2}} \times \frac{1}{2} \times x \times \frac{x}{3}$$

$$EI\frac{d^2y}{dx^2} = \frac{W_0Lx}{4} - \frac{W_0x^3}{3L}$$

$$EI\frac{dy}{dx} = \frac{W_0Lx^2}{8} - \frac{W_0x^4}{12L} + C_1$$

$$EI\,y = \frac{W_0Lx^3}{24} - \frac{W_0x^5}{60L} + C_1x + C_2$$

The boundary conditions are

at $x = 0,\ y = 0 \Rightarrow C_2 = 0$

at $x = \frac{L}{2}, \frac{dy}{dx} = 0$ [due to symmetry slope at B will be zero]

$$\Rightarrow \quad C_1 = \frac{W_0\left(\frac{L}{2}\right)^4}{12L} - \frac{W_0L\left(\frac{L}{2}\right)^2}{8}$$

$$= \frac{W_0 L^4}{192\, L} - \frac{W_0 L^3}{32}$$

$$\frac{W_0 L^3}{192} - \frac{6 W_0 L^3}{192} = \frac{-5 W_0 L^3}{192}$$

$$\Rightarrow \quad \boxed{C_1 = \frac{-5 W_0 L^3}{192}}$$

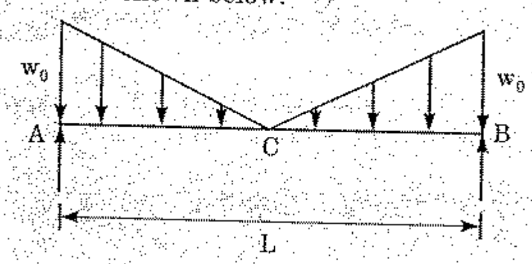
$$\Rightarrow \quad EI\, y = \frac{W_0 L x^3}{24} - \frac{W_0 x^5}{60\, L} - \frac{5 W_0 L^3 x}{192}$$

$$EI\, y_{max}\Big|_{at\, x = \frac{L}{2}} = \frac{-W_0 L^4}{120}$$

$$\boxed{y_{max} = \frac{-W_0 L^4}{120\, EI}} \quad [(-) \text{ means downward deflection}]$$

Example 14

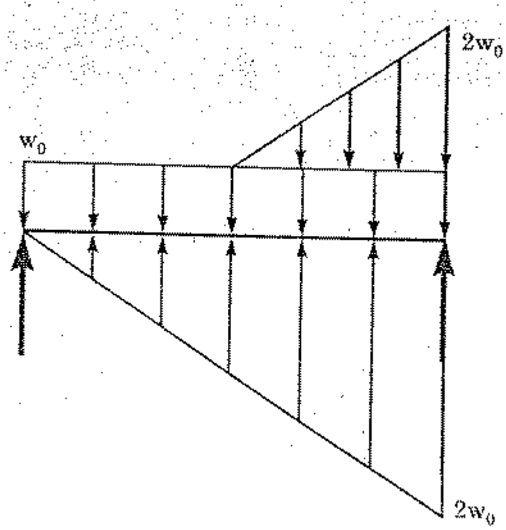
Find maximum deflection in the beam shown below.



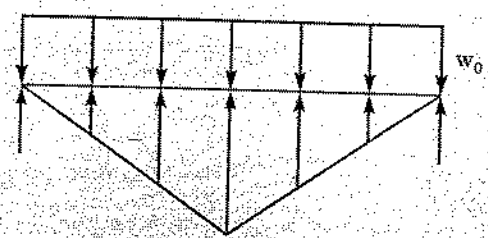
Sol: This problem can be solved by two approaches:

(i) By writting bending moment equation, only for $\frac{1}{2}$ of the beam i.e. AC [due to symmetry] and applying double integration technique.

(ii) By modifying the loading for application of Macaulay's method as shown below.



Alternatively, by using standard results the max deflection at mid span can be calculated as per the loading shown below:



The final results are:

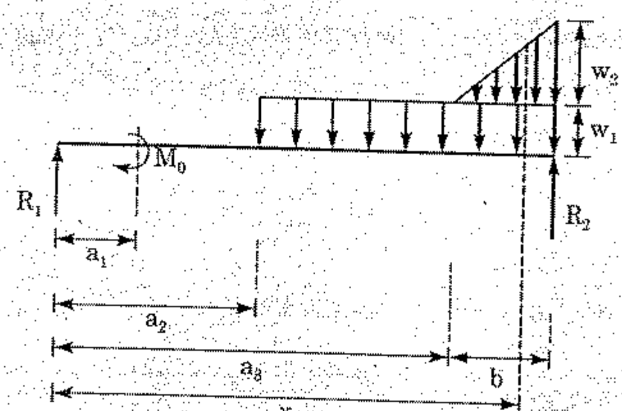
$$\Delta_{max} = \frac{5}{384}\frac{w_0L^4}{EI} - \left(\frac{w_0L^4}{120EI}\right)$$

$$= \frac{w_0L^4}{EI}\left[\frac{5}{384} - \frac{1}{120}\right]$$

$$\boxed{\Delta_{max} = \frac{w_0L^4}{213.33\,EI}} = \frac{9}{1920}\frac{w_0L^4}{EI}$$

Example 15

Write down the bending moment equation for use in Macaulay's method for the loading shown below:



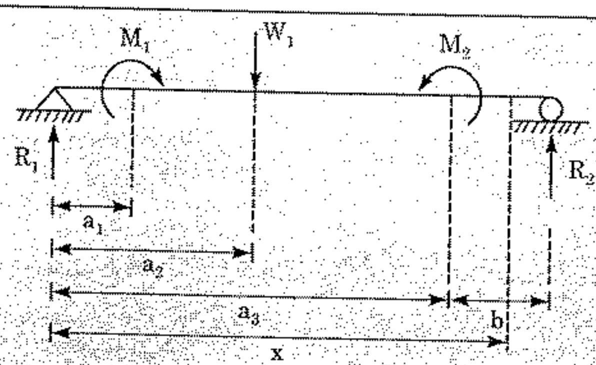
Sol: BM equation to be used in Macaulay's method is written for the last segment as seen from the left support.

$$M = R_1x + M_0\langle x - a_1\rangle^0 - \frac{w_1\langle x - a_2\rangle^2}{2} - \frac{w_2}{6b}\langle x - a_3\rangle^3$$

Note: For concentrated couple, we write $M_0\langle x - a_1\rangle^0$ instead of M_0.

Example 16

Write equation for BM to be used in Macaulay's method.



$$EI\frac{d^2y}{dx^2} = M = R_1x + M_1\langle x-a_1\rangle^0 - W_1\langle x-a_2\rangle - M_2\langle x-a_3\rangle^0$$

On integration:

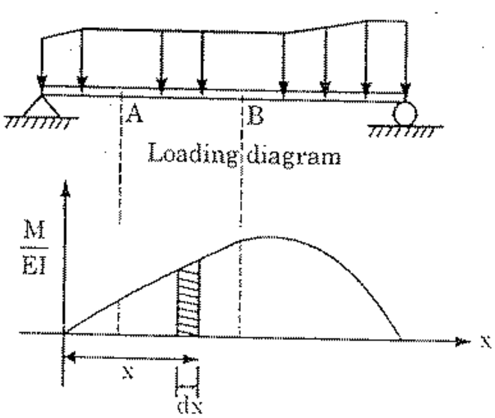
$$EI\frac{dy}{dx} = \frac{R_1x^2}{2} + C_1 + M_1\langle x-a_1\rangle - \frac{W_1\langle x-a_2\rangle^2}{2} - M_2\langle x-a_3\rangle$$

$$EI\,y = \frac{R_1x^3}{6} + C_1x + C_2 + \frac{M_1\langle x-a_1\rangle^2}{2} - \frac{W_2\langle x-a_2\rangle^3}{6} - \frac{M_2\langle x-a_3\rangle^2}{2}$$

Note that Macaula's method is just a modification of double integration method. Hence it should also not be applied if there is discontinuity in the deflected shape of beams like due to internal hinge or slider.

MOMENT AREA METHOD (MOHR'S METHOD)

- This method determines slope and deflection due to bending only.
- It is useful particularly for beams which is subjected to series of concentrated loads or having segments with different moment of inertia.
- It should not be used in case of internal hinge because continuity of slope is not maintained by elastic curve in that case.



$\frac{M}{EI}$ diagram

We know that

$$\frac{d\theta}{dx} = \frac{M}{EI}$$

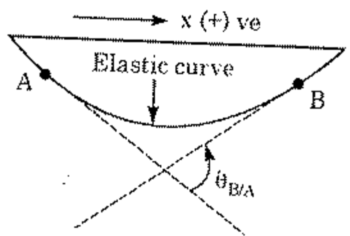
$$\Rightarrow \quad \int d\theta = \int \frac{M}{EI}\,dx$$

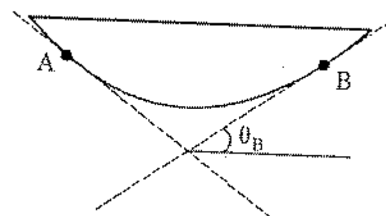
$$\Rightarrow \quad q_B - \theta_A = \int_A^B \frac{M}{EI}\,dx = \text{area under } \frac{M}{EI} \text{ diagram between A and B.}$$

$$\theta_{BA} = \theta_B - \theta_A = \text{Slope of B with respect to tangent drawn at A.}$$

Similarly, θ_{AB} = Slope at A with respect to tangent drawn at B = $\theta_A - \theta_B$.

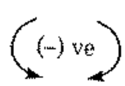
Theorem 1: The change in slope between two points on the elastic curve equals the area of $\frac{M}{EI}$ diagram between these two points.



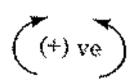


Sign Convention (for slope)

- Angle is measured anticlockwise from tangent at A (i.e. point of smaller x) to tangent at B (i.e. point of larger x) if area of $\frac{M}{EI}$ diagram is (+) ve.
- Angle is measured clockwise from tangent at A to tangent at B if area of $\frac{M}{EI}$ diagram is (–) ve.

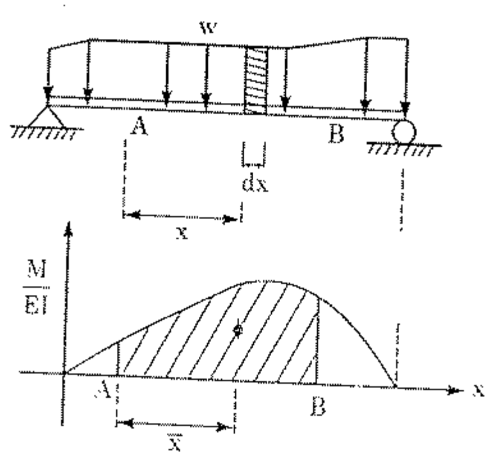


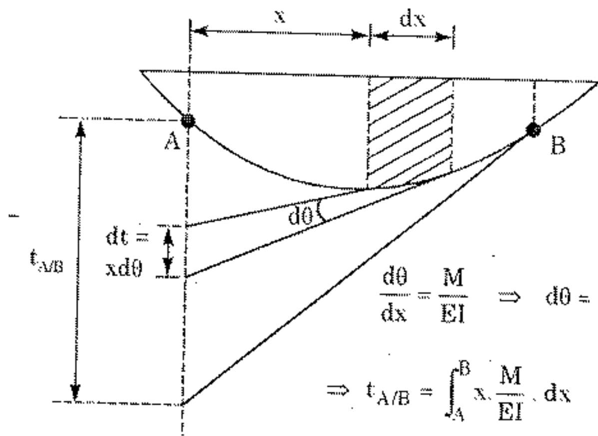
i.e. Hogging moment (–) ve



Sagging moment (+) ve

- Slope at 'B' can be directly known if reference 'A' is chosen such that tangent at 'A' is horizontal. i.e. For example, reference point A could be point at fixed support, mid point of uniformly loaded simply supported beam etc.





$$\frac{d\theta}{dx} = \frac{M}{EI} \Rightarrow d\theta = \frac{M}{EI}dx$$

$$\Rightarrow t_{A/B} = \int_A^B x \cdot \frac{M}{EI} \cdot dx$$

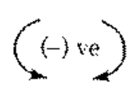
We know that

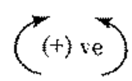
$$\boxed{\int x\,dA = \bar{x}\int dA}$$

$$\Rightarrow \boxed{t_{A/B} = \bar{x}\int_A^B \frac{M}{EI}dx}$$

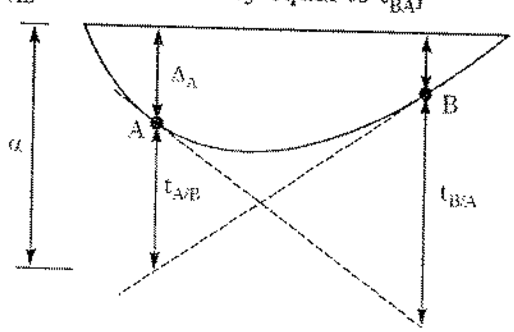
Theorem 2: Deflection of any point 'A' on elastic curve with respect to tangent drawn at another point B (t_{AB}) equals moment of area under $\frac{M}{EI}$ diagram between A and B, about point A.

Sign Convention (for deflection)

- Point 'A' one elastic curve is above tangent extended from 'B' up to A if moment of $\frac{M}{EI}$ diagram between A and B about A is (+) ve.
- Point at 'A' one elastic curve is below tangent extended from 'B' if moment of $\frac{M}{EI}$ diagram between A and B about A is (–) ve.
- 



  Hogging moment is taken as (–) ve   Sagging moment is taken as (+) ve
- $t_{A/B} \neq t_{B/A}$ (generally) [i.e. t_{AB} is not necessarily equal to t_{BA}]

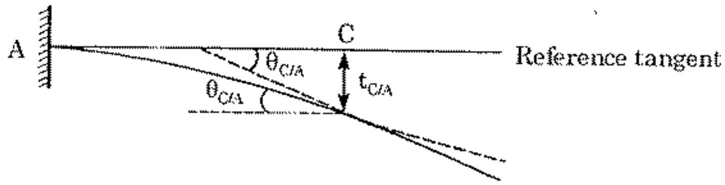


$t_{A/B}$ is not necessarily equal to deflection of beam at 'A'. In the above figure if 'α' is know, $\boxed{|\Delta_A| = \alpha - |t_{A/B}|}$ can be calculated.

Δ_A = deflection of beam at 'A', Numerically.

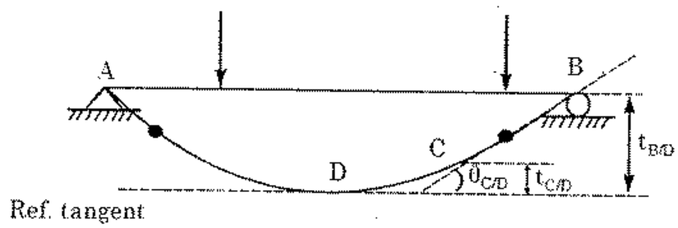
Application of Moment Area Theorem for Cantilever Beams and Beams with Symmetric Loading

- In cantilever beams, at fixed support, slope of deflected curve is zero. Hence it is taken as reference tangent.

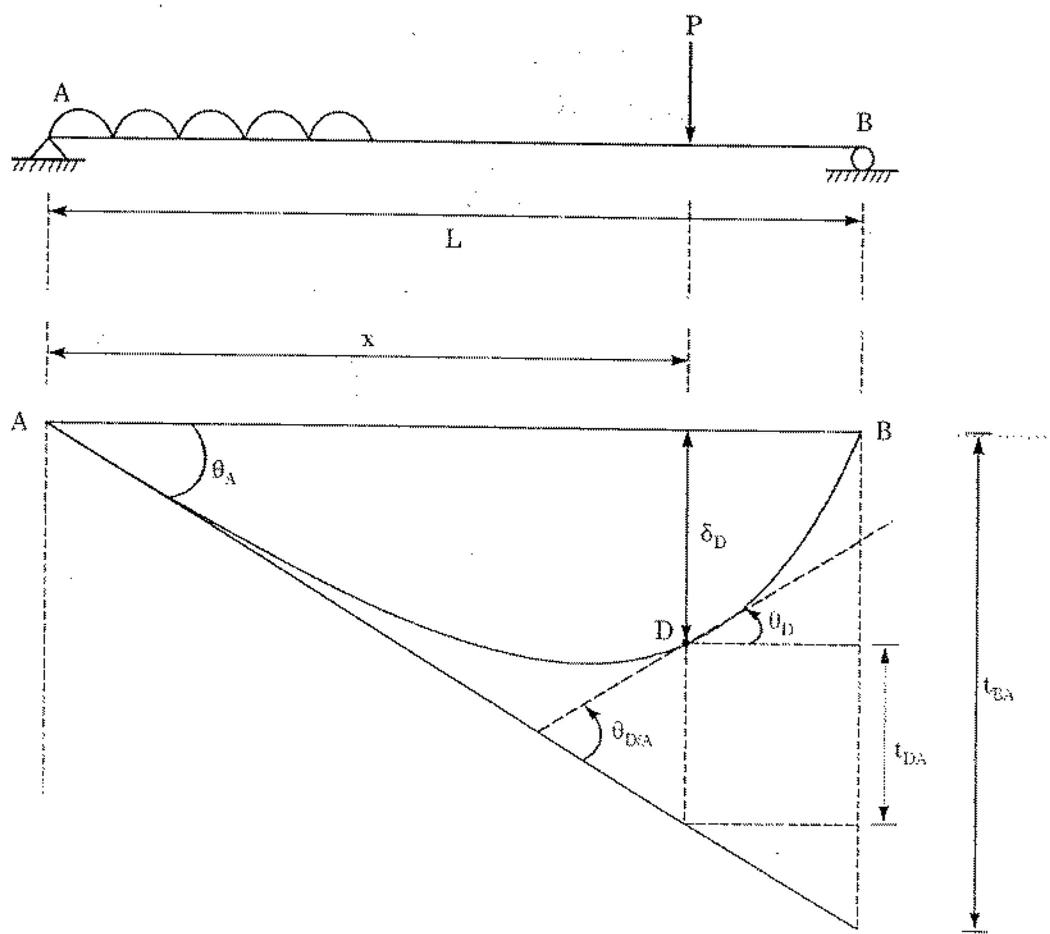


In this case as the reference tangent is horizontal, hence $t_{CA} = \Delta_C$ and slope at C i.e. $\theta_C = \theta_{CA}$.

- For symmetrical loading in a simply supported beam, the slope of deflected curve at mid span is zero. hence it is taken as reference tangent.



Beam with Unsymmetrical Loading



$$\boxed{\theta_A = \frac{-t_{BA}}{L}}$$

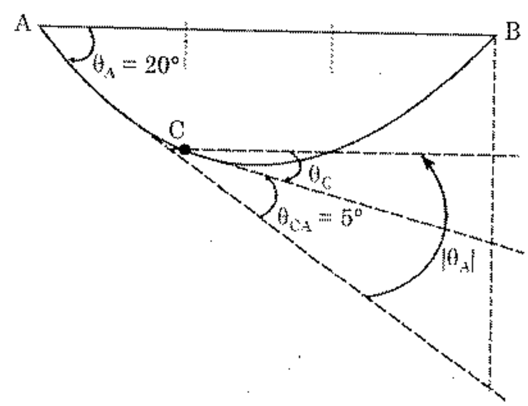
$\boxed{\theta_{DA} = \theta_D - \theta_A}$ [Valid for all points from A to B]

$$\delta_D = \frac{-x}{L} \times t_{BA} + t_{DA}$$

$\delta_D = (-)$ ve $\Rightarrow$ downward

$\delta_D = (+)$ ve $\Rightarrow$ upward

Note:



$\theta_C = \theta_A - \theta_{CA}$, Numerically

$\boxed{\theta_C = \theta_A + \theta_{CA}}$ Algebraically

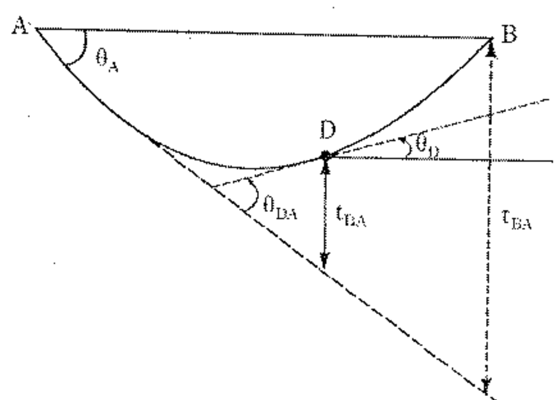
If $\theta_A = -20°$, because angle is clockwise

$\theta_{CA} = 5°$ because it is measured anticlockwise from tangent

then $\theta_C = -20° + 5°$

$\theta_C = -15°$ $\Rightarrow$ Clockwise

θ_C is $(-)$ ve $\Rightarrow$ it is clockwise.



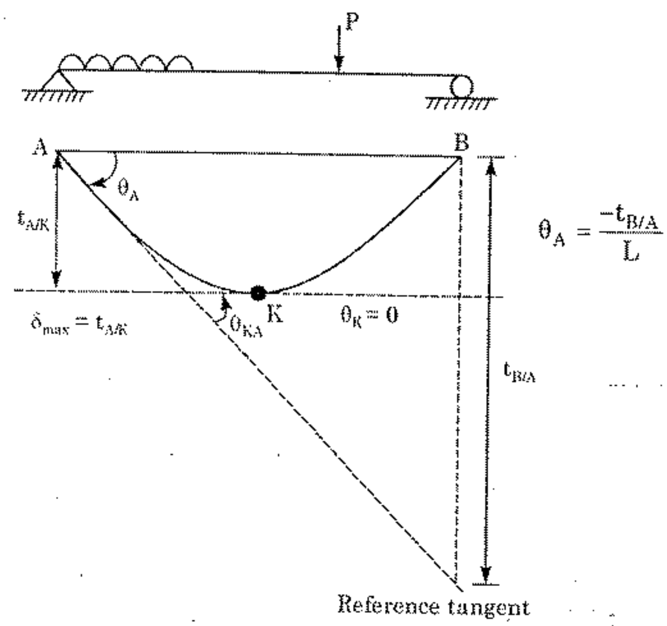
$\boxed{\theta_D = \theta_A + \theta_{D/A}}$ Algebraically

$= -20° + 25°$

$= 5°$ ((+) ve mean anticlockwise)

Maximum Deflection

To compute maximum deflection in a simply supported or overhanging beam under unsymmetrical loading, point where tangent to the elastic curve is horizontal is found out and deflection is calculated at that point.



$$\boxed{\theta_K = \theta_A + \theta_{KA}}$$

but $\theta_K = 0$

$$\Rightarrow \quad \theta_A = -\theta_{KA}$$

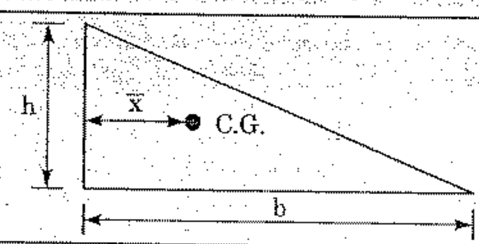
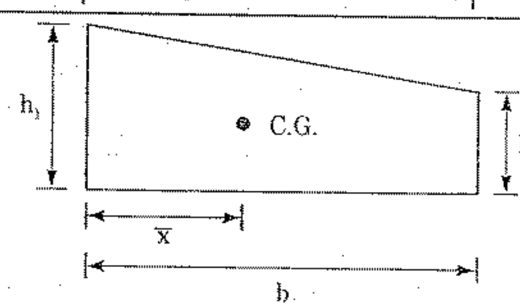
$$\Rightarrow \quad \theta_{KA} = -\theta_A = -\left(-\frac{t_{BA}}{L}\right) = \frac{t_{BA}}{L}$$

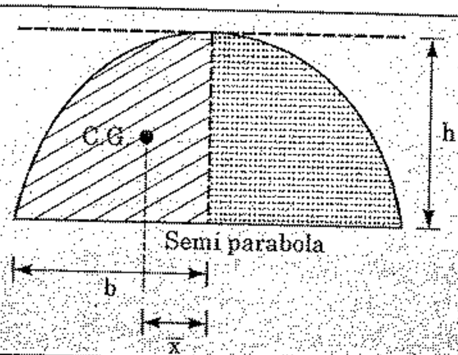
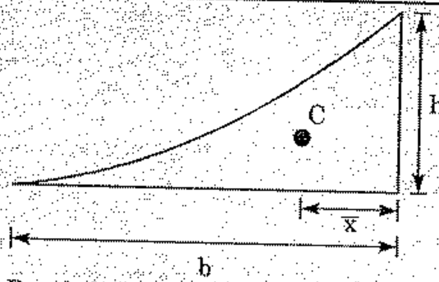
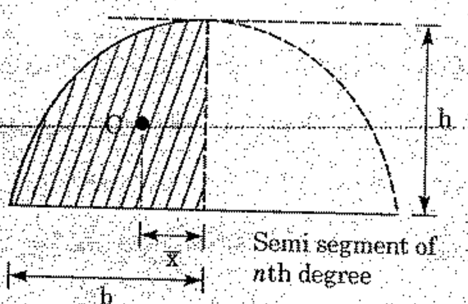
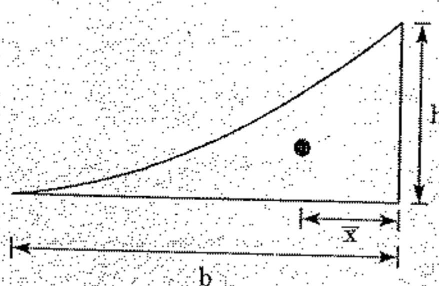
θ_{KA} = Area of $\frac{M}{EI}$ diagram between A and K.

Thus, area of $\frac{M}{EI}$ diagram between A and K = $\frac{t_{BA}}{L}$. From this location of K is found out.

Hence, finally $\delta_{max} = |t_{AK}|$ can be calculated once the location of K is known.

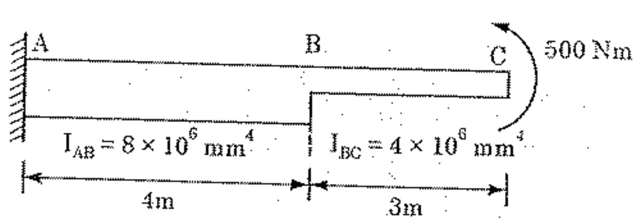
GEOMETRICAL PROPERTIES OF AREA

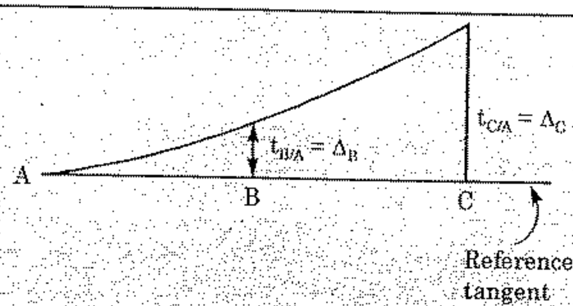
	$A = \frac{bh}{2}$	$\bar{x} = \frac{b}{3}$
	$A = \frac{b(h_1 + h_2)}{2}$	$\bar{x} = \left(\frac{2h_2 + h_1}{h_2 + h_1}\right) \times \frac{b}{3}$

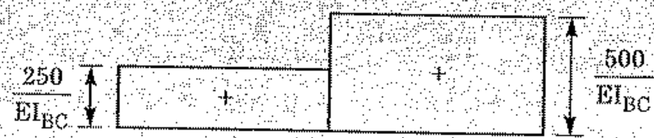
 Semi parabola	$A = \frac{2}{3}bh$	$\bar{x} = \frac{3b}{8}$
 Parabolic sprandle	$A = \frac{bh}{3}$	$\bar{x} = \frac{b}{4}$
 Semi segment of *n*th degree	$A = bh\left(\frac{n}{n+1}\right)$	$\bar{x} = \frac{b}{2}\left(\frac{n+1}{n+2}\right)$
 Sprandle of n^{th} degree	$A = \frac{bh}{n+1}$	$\bar{x} = \frac{b}{n+2}$

Example 17

Find Δ_B and Δ_C.







$\frac{M}{EI}$ diagram

Note $I_{BC} = \frac{I_{AB}}{2}$

$$\Delta_C = t_{C/A} = \text{moment of } \frac{M}{EI} \text{ diagram between A and C about C}$$

$$= \frac{500}{EI_{BC}} \times 3 \times 1.5 + \frac{250}{EI_{BC}} \times 4 \times 5 = \frac{7250}{EI_{BC}}$$

$$= \frac{7250}{\left(2 \times 10^5 \frac{N}{mm^2} \times 4 \times 10^6 mm^4\right) \times 10^{-6}} m$$

$$= 9.06 \times 10^{-3} \text{ m} = 9.06 \text{ mm}$$

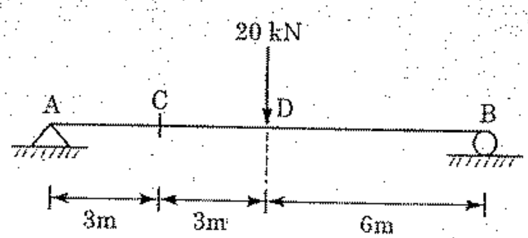
t_{CA} is (+) ve $\Rightarrow$ point C is above tangent from A $\Rightarrow$ upward deflection

$$t_{B/A} = \Delta_B = \frac{250}{EI_{BC}} \times 4 \times 2 = 2.5 \times 10^{-3} \text{m}$$

$$= 2.5 \text{ mm}$$

t_{BA} is (+) ve $\Rightarrow$ above tangent from A $\Rightarrow$ upward deflection

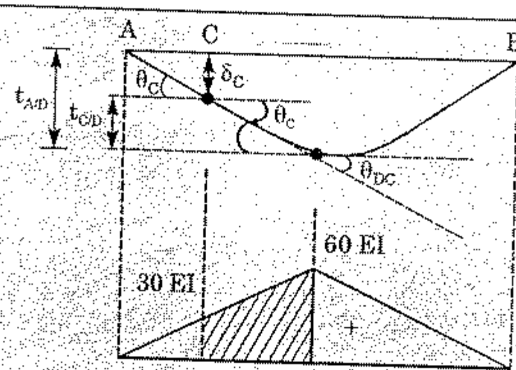
Example 18



$E = 2 \times 10^5$ N/mm^2

$I = 6 \times 10^6$ mm^4 = constant throughout

Find slope at 'C' and deflection at C



$\frac{M}{EI}$ diagram

Sol: Slope at D = horizontal

From the sketch shown above, It is clear that, $|\delta_C| = |t_{AD} - t_{CD}|$ and $|\theta_C| = |\theta_{DC}|$ hence, we need to calculate t_{AD}, t_{CD} and θ_{DC}.

θ_{DC} = Area under the shaded diagram as above

$$\Rightarrow \quad \theta_{DC} = \frac{1}{2}\left(\frac{30+60}{EI}\right)\times 3$$

$$\theta_{DC} = \frac{135\,\text{kNm}^2}{EI} = \frac{135\times10^6\,\text{Nmm}\times10^3\,\text{mm}}{2\times10^5\times6\times10^6\,\text{N mm}^2}$$

$\Rightarrow \quad \theta_{DC} = 0.1125$ rad, [(+) ve means anticlockwise from tangent at C]

From the figure it is clear that $|\theta_C| = |\theta_{DC}| \Rightarrow \theta_C$ is 0.1125 rad, but it is clockwise

$$\delta_C = t_{AD} - t_{CD}$$

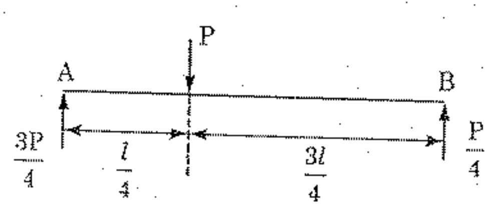
$$= \left[\left(\frac{1}{2}\times\frac{60}{EI}\times6\right)\times\frac{2}{3}\times6\right] - \left[\frac{1}{2}\left(\frac{30+60}{EI}\right)\times3\times\left[\frac{(2\times60+30)}{90}\times\frac{3}{3}\right]\right]$$

$$= \frac{720}{EI} - \frac{225}{EI} = \frac{495}{EI} = \frac{495\text{kNm}^2\text{m}}{EI}$$

$$= \frac{495\times10^{12}\,\text{Nmm}^3}{2\times10^5\times6\times10^6\,\text{Nmm}^2} = 41.25\times10 = 412.5 \text{ mm}$$

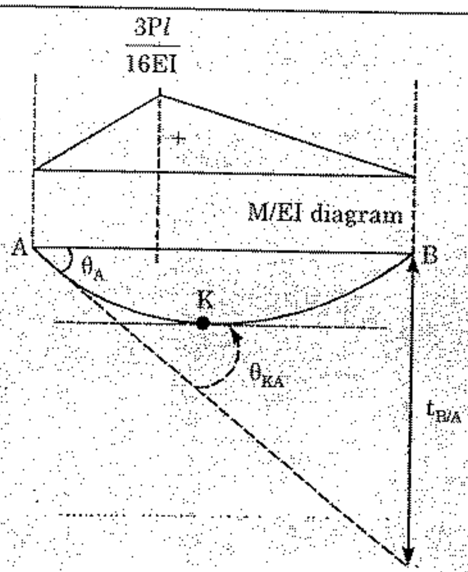
Example 19

Calculate maximum deflection for the loading shown in the figure below.



EI = constant

Sol:



Let the slope be horizontal at location K.

This location is the location of max deflection and our aim is to find out the location of K.

$$t_{B/A} = \frac{1}{2}\left(\frac{3Pl}{16EI}\right)\times\frac{l}{4}\times\left[\frac{3L}{4}+\frac{l}{4}\times\frac{1}{3}\right]+\frac{1}{2}\left(\frac{3Pl}{16EI}\right)\times\frac{3l}{4}\times\left[\frac{3l}{4}\times\frac{2}{3}\right]$$

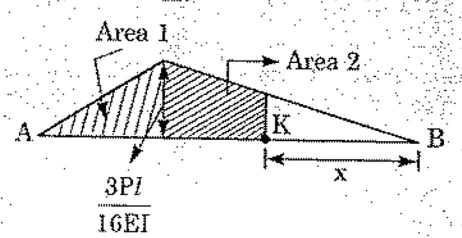
$$t_{B/A} = \frac{7Pl^3}{128EI}$$

Magnitude wise $\theta_{KA} = \theta_A$ (from the figure)

$$\Rightarrow \quad \theta_{KA} = \frac{7Pl^3}{128\,EI}\times\frac{1}{l} = \frac{7Pl^2}{128\,EI}$$

$\Rightarrow$ Area under moment diagram between 'K' and 'A' is $\frac{7Pl^2}{128\,EI}$

Now θ_{KA} will be calculated from moment area theorem with the concept that θ_{KA} = Area of $\frac{M}{EI}$ diagram between K and A and will be equated to the θ_{KA} already found out. Thus we will get the location of K.



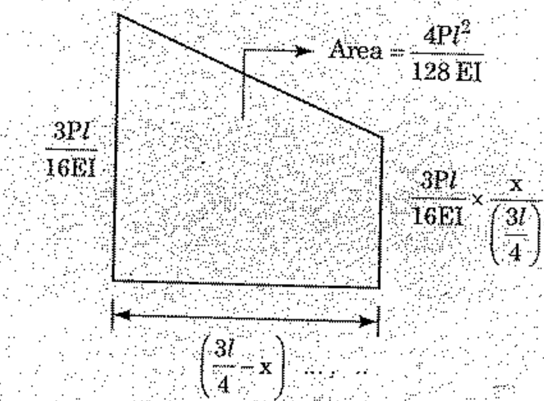
Let the location of K be 'x' distance from B

$$\text{Area } 1 = \frac{1}{2}\times\frac{3Pl}{16\,EI}\times\frac{l}{4}$$

$$= \frac{3Pl^2}{128\,EI}$$

$$\Rightarrow \quad \text{Area } 2 = \frac{7Pl^2}{128\,EI} - \frac{3Pl^2}{128\,EI}$$

$$= \frac{4Pl^2}{128\,EI}$$



$$\Rightarrow \quad \frac{4Pl^2}{128\,EI} = \left[\frac{\frac{3Pl}{16} + \frac{3Pl}{16}\left(\frac{x}{\frac{3l}{4}}\right)}{2}\right] \times \left(\frac{3l}{4} - x\right)$$

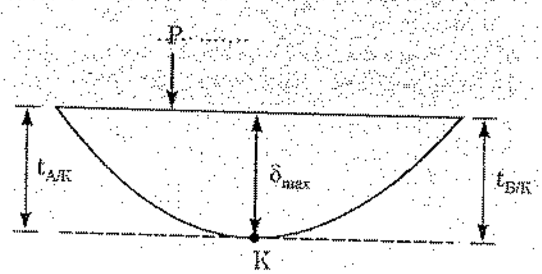
$$\frac{4Pl^2}{128EI} = \frac{3Pl}{16\,EI}\left(\frac{3l + 4x}{3l}\right)\left(\frac{3l - 4x}{8}\right)$$

$$\Rightarrow \quad 4l = \frac{9l^2 - 16x^2}{l}$$

$$\Rightarrow \quad 16x^2 = 5l^2$$

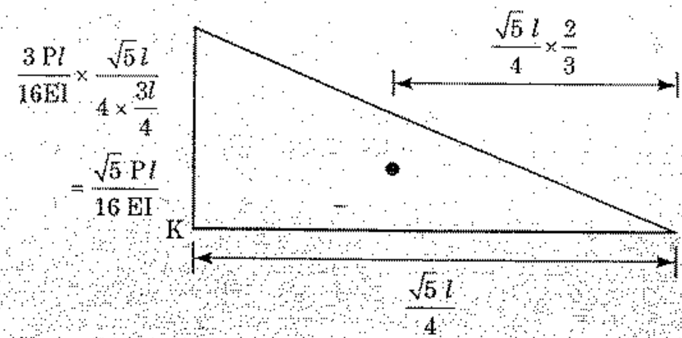
$$\boxed{x = \frac{\sqrt{5}}{4}\,l}$$

Note that max deflection is not below point load. i.e. max deflection need not be at the point of max bending moment.



$$\delta_{max} = t_{A/K} = t_{B/K}$$

It is easier to calculate δ_{max} from t_{BK} because $\frac{M}{EI}$ diagram will be a triangle. Hence calculation of area of triangle and its C.G. location will be easier.



$\frac{M}{EI}$ diagram between K and B.

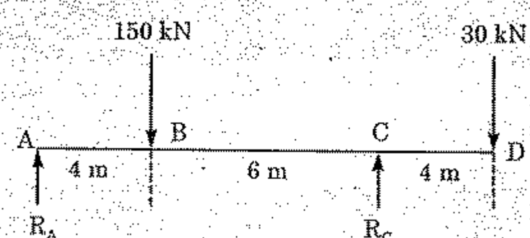
$$\Rightarrow t_{B/K}=\frac{1}{2}\times\frac{\sqrt{5}Pl}{16EI}\times\frac{\sqrt{5}l}{4}\times\frac{\sqrt{5}l}{4}\times\frac{2}{3}$$

$$=\frac{\sqrt{5}Pl}{48EI}\times\left(\frac{\sqrt{5}l}{4}\right)^2$$

$$\boxed{\delta_{max}=\frac{P}{12EI}\left(\frac{\sqrt{5}l}{4}\right)^3=0.01456\frac{Pl^3}{EI}}$$

Example 20

Find deflection at D.



Sol: Let us calculate reactions first

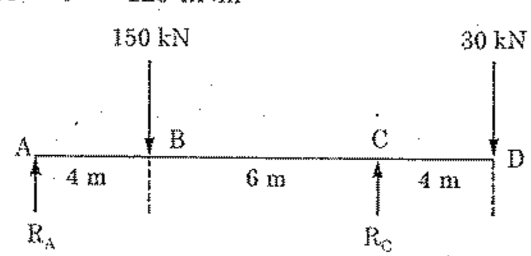
$$R_A + R_C = 180 \text{ kN} \quad \text{------ (i)}$$

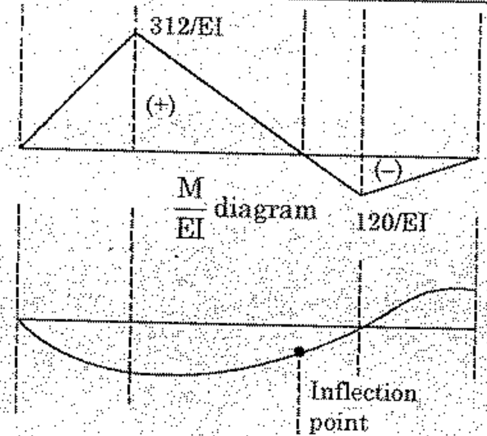
$$\Sigma M_A = 0 \quad \Rightarrow (-30\times 14)+(R_C\times 10)-(150\times 4)=0$$

$$R_C=\frac{600+420}{10}=102 \text{ kN}$$

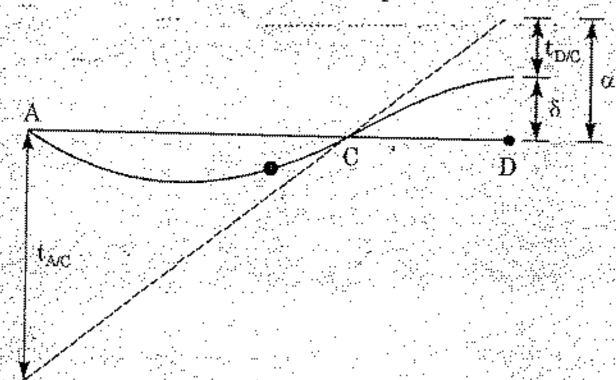
$$R_A = 78 \text{ kN}$$

$$M_C = -30\times 4 = -120 \text{ kNm}$$

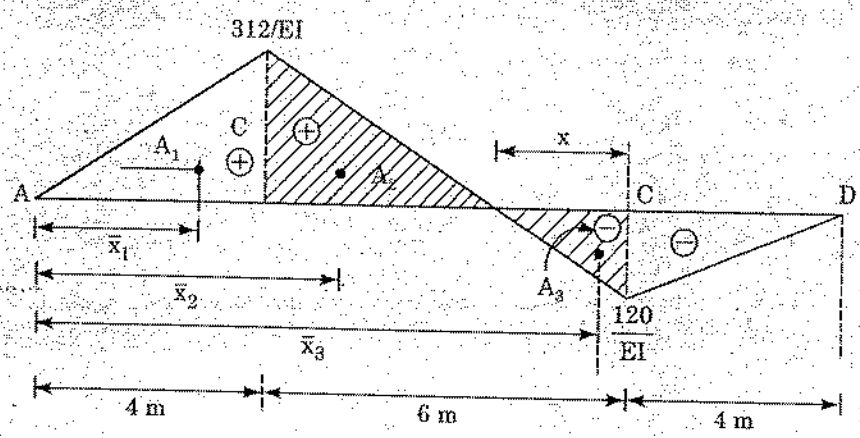




Deflected shape



$t_{A/C}$ = moment of area under $\frac{M}{EI}$ diagram between A and C about A



$$x = \frac{120}{\frac{(120+312)}{6}} = \frac{5}{3}\text{ m}$$

$$t_{A/C} = A_1\bar{x}_1 + A_2\bar{x}_2 + A_3\bar{x}_3$$

$$A_1 = \frac{312}{EI} \times \frac{1}{2} \times 4 = \frac{624}{EI}$$

$$\bar{x}_1 = \frac{2}{3} \times 4 = \frac{8}{3}\text{ m}$$

$$A_2 = \frac{1}{2} \times \frac{312}{EI} \times \left(6 - \frac{5}{3}\right)$$

$$= \frac{1}{2} \times \frac{312}{EI} \times \frac{13}{3} = \frac{676}{EI}$$

$$\bar{x}_2 = 4 + \frac{1}{3} \times \left(6 - \frac{5}{3}\right)$$

$$= 4 + \frac{13}{9} = \frac{49}{9}$$

$$A_3 = \frac{1}{2}\left(\frac{-120}{EI}\right)\left(\frac{5}{3}\right) = \frac{-100}{EI}$$

$$\bar{x}_3 = 10 - \frac{1}{3} \times \frac{5}{3} = \frac{85}{9}$$

$$\Rightarrow \quad t_{A/C} = \left[\frac{624}{EI} \times \left(\frac{8}{3}\right)\right] + \left[\frac{676}{EI} \times \frac{49}{9}\right] - \frac{100}{EI} \times \frac{85}{9}$$

$$\Rightarrow \quad \boxed{t_{A/C} = \frac{4400}{EI}} \quad (+)\text{ ve} \Rightarrow \text{ point A above tangent from C}$$

$$t_{D/C} = \left((-)\frac{120}{EI}\right)\left(\frac{1}{2}\right)(4) \times \frac{2}{3} \times 4$$

$$\Rightarrow \quad \boxed{t_{D/C} = \frac{-640}{EI}} \quad \Rightarrow \text{ Point D below tangent from C}$$

$$\alpha = \frac{t_{AC}}{10} \times 4 = \frac{4400}{10\,EI} \times 4 = \frac{1760}{EI}$$

$$\Rightarrow \quad \delta = \frac{1760}{EI} - \frac{640}{EI} = \frac{1120}{EI}$$

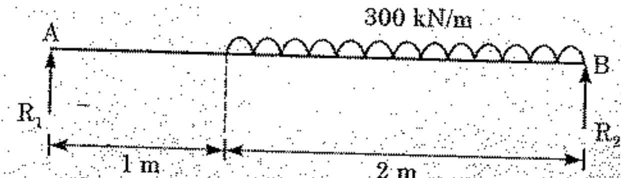
$$\boxed{\delta = \frac{1120}{EI}}$$

MOMENT DIAGRAM BY PARTS

- When no. of loadings on a beam is large, it is difficult to calculate the area under resultant $\frac{M}{EI}$ diagram and corresponding c.g. of M/EI diagram.
- We know that resultant BM at any section is the algebraic sum of bending moments at that section caused by each loading separately [either from left or right of that section]. Hence, effect of individual load can be considered, instead of taking effects of all the loads together for drawing BMD. By doing so, we get simpler $\frac{M}{EI}$ diagrams and it is easier to calculate Area and C.G. of the individual moment diagrams. This method of drawing BM diagram due to individual load is called **BMD by parts**.

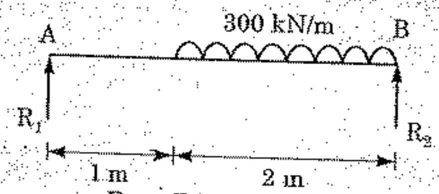
Example 21

Draw BM diagram by parts for the following beam.



I = constant

Sol: Let us calculate reactions 1st

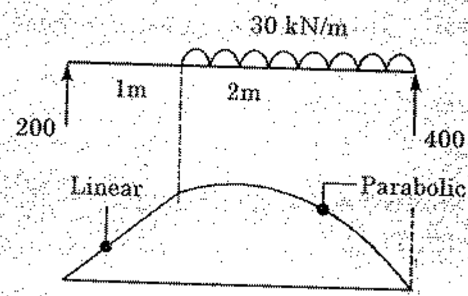


$$R_1 + R_2 = 600 \text{ kN}$$

Taking moment about A, $R_2 \times 3 - 300 \times 2 \times 2 = 0$

$$\Rightarrow \quad R_2 = 400 \text{ kN}$$

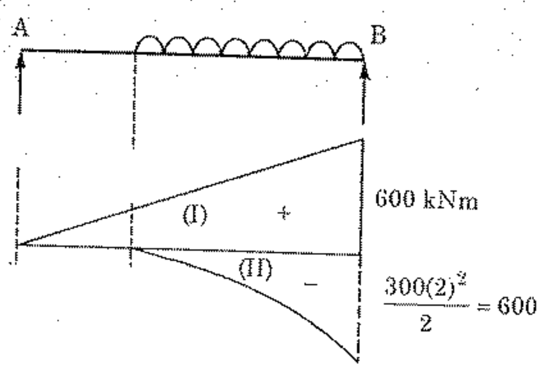
$$\Rightarrow \quad R_1 = 200 \text{ kN}$$



$\frac{M}{EI}$ diagram

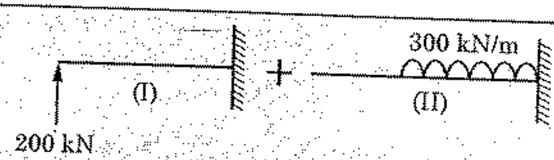
Calculation of area and C.G. of this $\frac{M}{EI}$ diagram area is difficult. To obviate this problem, we will use BM diagram by parts.

To draw BMD by parts, any section on the beam is chosen and it is considered as fixed. BMD upto that section is drawn for all forces on the left and right of the section.



BMD by parts

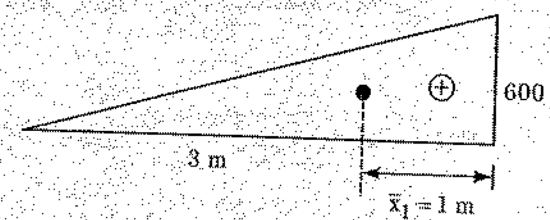
The moment diagram by parts in the above figure has been drawn by assuming as if there is a fixed support at B. i.e. bending moment diagram has been drawn corresponding to the following loading condition



Note that Area and C.G. of BM diagram for (i) and (ii) is easier to calculate.
Also note that the moment diagram by parts corresponds to

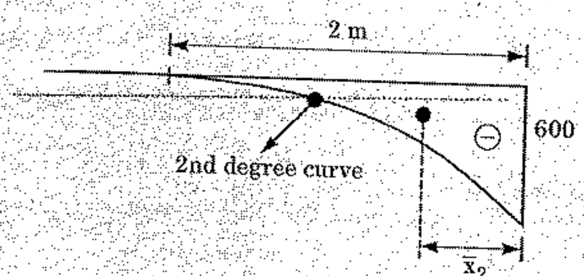
$$EI\frac{d^2y}{dx^2} = R_1x - \frac{300(x-1)^2}{2}$$

where moment due to each loading has been separately calculated.



$A = 900\ kNm^2 \quad \bar{x}_1 = 1\ m$

BMD for Fig. (I)



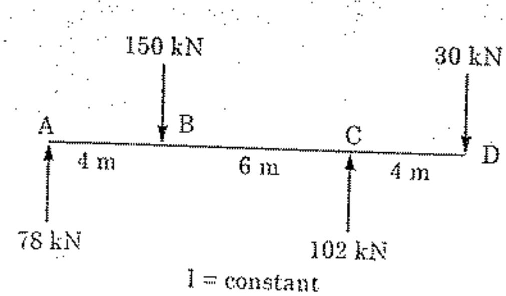
BMD for Fig. (II)

$$A = \frac{bh}{n+1} = \frac{600 \times 2}{2+1} = 400\ kNm^2$$

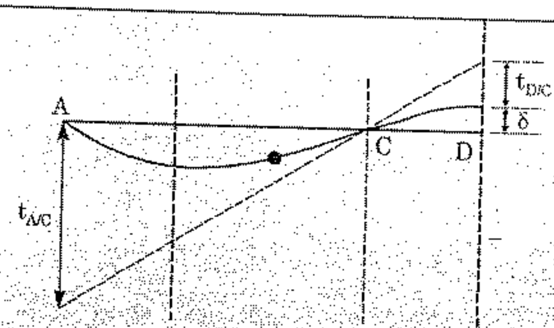
$$\bar{x}_2 = \frac{b}{n+2} = \frac{2}{2+2} = 0.5\ m$$

Example 22

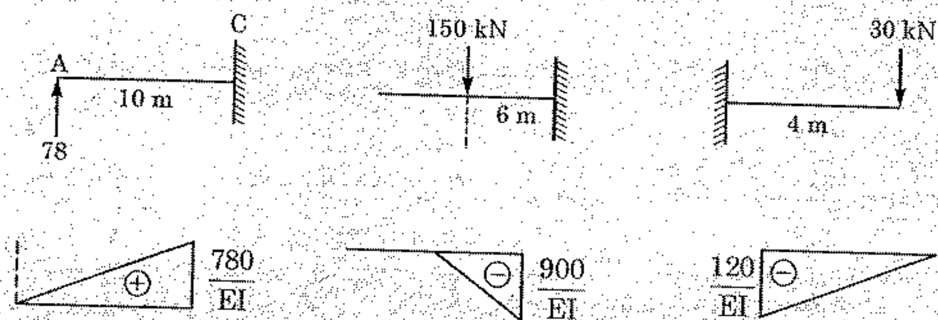
Calculate deflection at D.



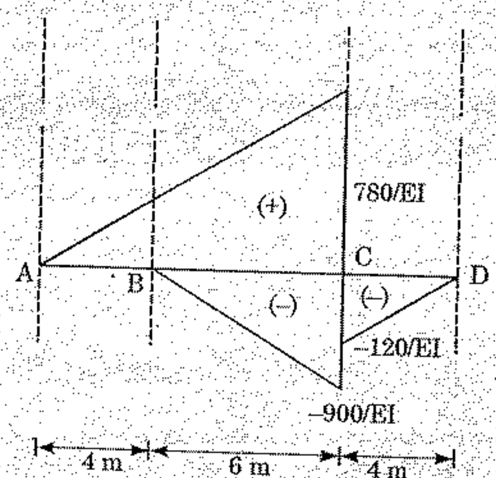
Sol:



To calculate $t_{A/C}$ and $t_{D/C}$ area of M/EI diagram between AC and CD and the moment of $\frac{M}{EI}$ diagram is to be determined. Hence we draw moment diagram by parts at C from both sides.



Hence resultant BMD by parts is as shown below.



$$t_{A/C} = \left(\frac{1}{2} \times \frac{780}{EI} \times 10 \times \frac{2}{3} \times 10\right) - \frac{1}{2} \times \frac{900}{EI} \times 6 \times \left(4 + \frac{2}{3} \times 6\right)$$

$$= \frac{26000}{EI} - \frac{21600}{EI} = \frac{4400}{EI}$$

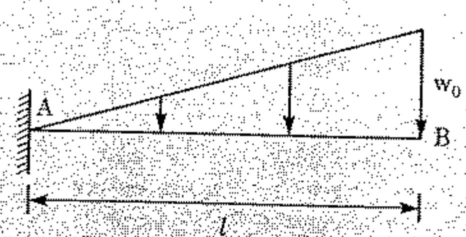
$$t_{D/C} = \frac{-1}{2} \times \frac{120}{EI} \times 4 \times \frac{2}{3} \times 4 = -\frac{640}{EI}$$

$$\delta = |t_{AC}| \times \frac{4}{10} - |t_{DC}| = \frac{4400}{EI} \times \frac{4}{10} - \frac{640}{EI}$$

$$\Rightarrow \delta = \frac{1120}{EI}$$

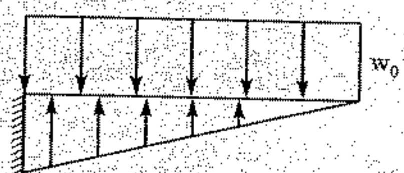
Example 23

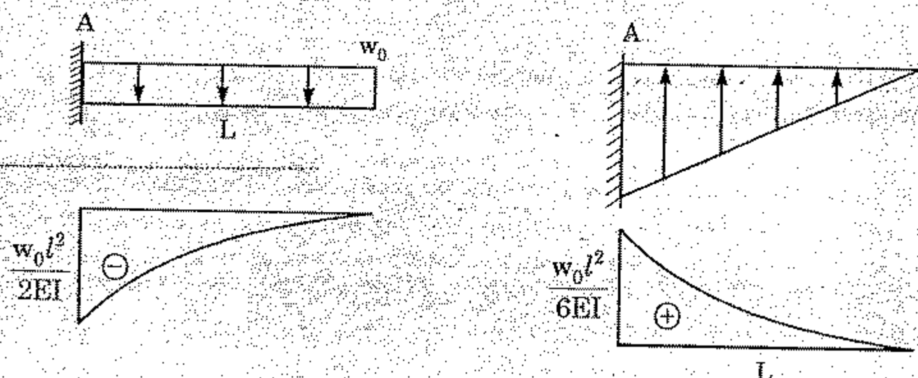
Calculate deflection at B.



Sol: We will draw bending moment by parts about section at A.

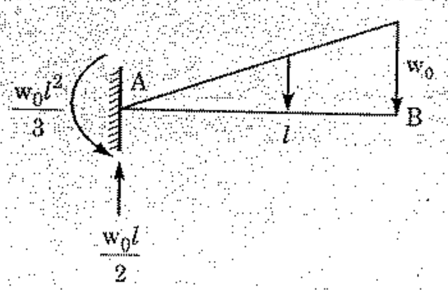
The loading can be modified as follows such that standard results of area and its C.G. can be used



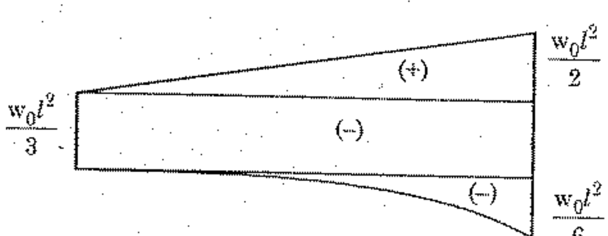


BMD by parts about A.

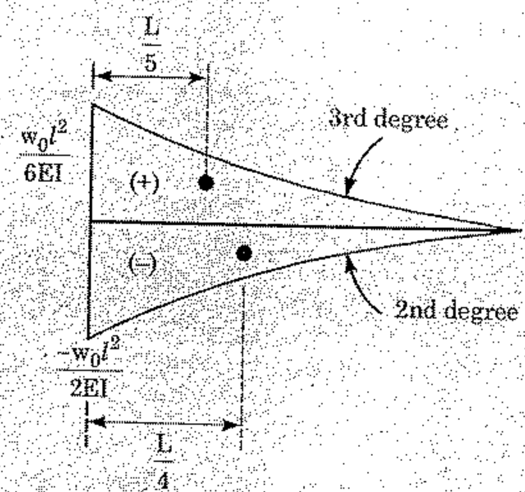
Note: We have drawn moment diagram by parts by writing BM by parts about A. We can also write down moment diagram by parts by writing BM about B. But for this reaction at A and moment at A needs to be calculated 1st, because all forces to the left of B will be required to draw BM about B. Thus

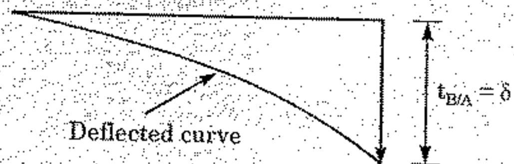


BM diagram by parts about B will be



The resultant BMD by parts about A is as under.





$$t_{B/A} = \frac{\frac{W_0 l^2}{6EI} \times l}{4} \times \frac{4L}{5} - \frac{\frac{W_0 l^2}{2EI} \times l}{3} \times \frac{3L}{4}$$

$$= \frac{W_0 l^4}{30\ EI} - \frac{W_0 l^4}{8\ EI}$$

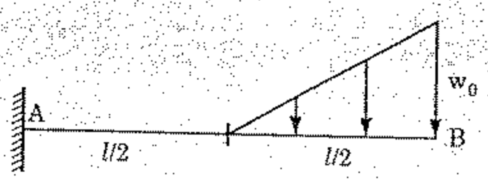
$$\delta = t_{B/A} = \frac{4Wl^4}{120\ EI} - \frac{15 W_0 l^4}{120\ EI} = \frac{-11\ W_0 l^4}{120\ EI}$$

$$\boxed{\delta = \frac{-11 W_0 l^4}{120\ EI}}$$

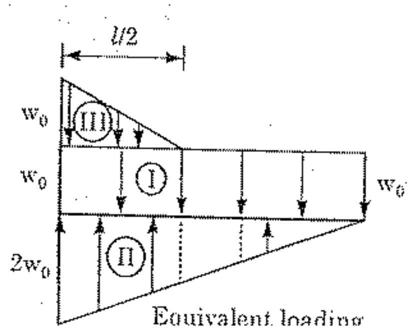
(–) means B on elastic curve is below the ref. tangent from A.

Example 24

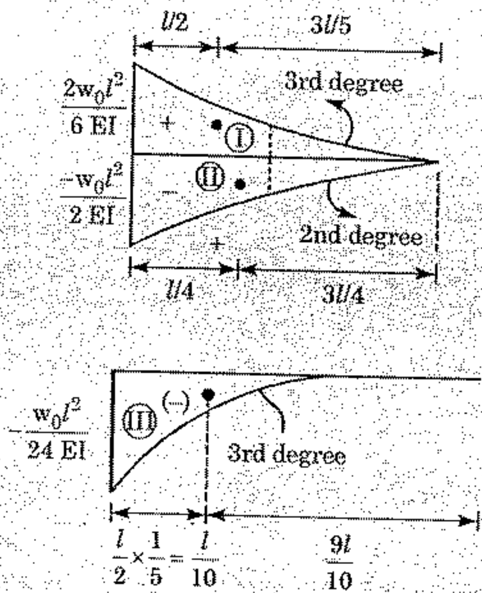
Find δ_B.



Sol:



By drawing BMD by parts about A we get



$\frac{M}{EI}$ diagram

$$\delta_B = t_{B/A}$$

$$= \frac{\frac{2W_0l^2}{6EI}}{(3+1)} \times l \times \left(l - \frac{l}{5}\right) - \frac{W_0l^2}{2EI} \times l \times \frac{1}{3} \times \frac{3l}{4} - \frac{W_0l^2}{24EI} \times \frac{l}{2} \times \frac{1}{4} \times \frac{9L}{10}$$

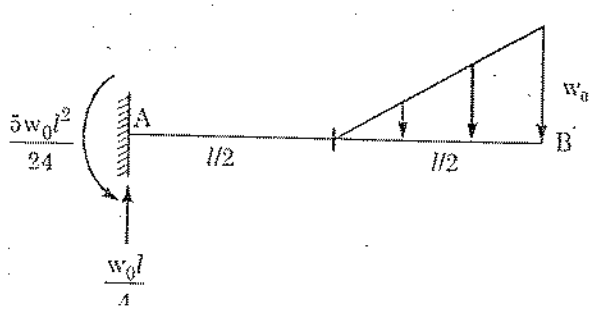
$$= \frac{W_0l^4}{15\,EI} - \frac{W_0l^4}{8\,EI} - \frac{9W_0l^4}{1920\,EI}$$

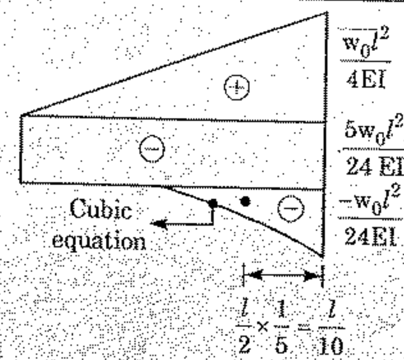
$$= \frac{128W_0l^4}{1920EI} - \frac{240\,W_0l^4}{1920\,EI} - \frac{9W_0l^4}{1920\,EI}$$

$$= \frac{-121\,w_0l^4}{1920\,EI}$$

$\boxed{\delta_B = -\frac{121W_0l^4}{1920\,EI}}$, δ_B negative $\Rightarrow$ Point B on elastic curve is below tangent from A

Alternate approach: If we draw BMD by parts about B, we do not need to go in for the equivalent loading. In this case we will draw BM by parts due to the reaction $\frac{w_0l}{4}$, the end moment $\frac{5w_0l^2}{24}$ and the uvl loading on part of beam about B.



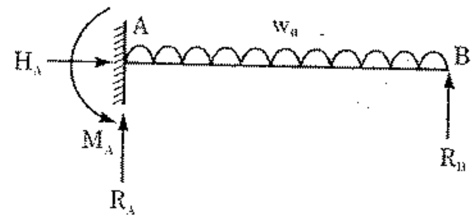


$$t_{B/A} = \delta_B = \left(\frac{1}{2} \times \frac{W_0 l^2}{4EI} \times l \times \frac{l}{3}\right) - \left(\frac{5W_0 l^2}{24EI} \times l \times \frac{l}{2}\right) - \left(\frac{W_0 l^2}{24EI} \times \frac{l}{2} \times \frac{1}{4} \times \frac{l}{2 \times 5}\right)$$

$$\Rightarrow \quad t_{B/A} = \delta_B = \frac{W_0 l^4}{24\,EI} - \frac{5W_0 l^4}{48\,EI} - \frac{W_0 l^4}{1920\,EI}$$

$$\Rightarrow \quad \boxed{\delta_B = \frac{-121 W_0 l^4}{1920\,EI}}$$

CALCULATION OF REDUNDANTS USING MOMENT AREA THEOREM



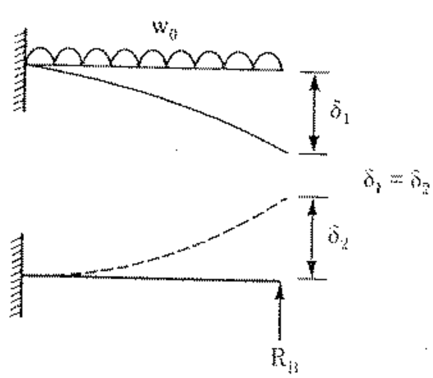
No. of external reactions = 4

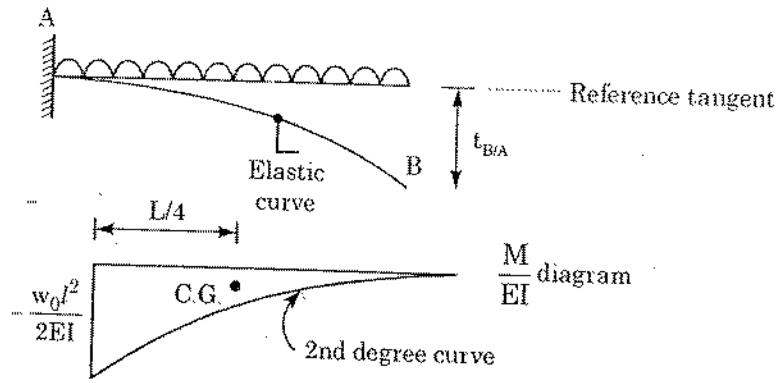
No. of equations of static equilibrium = 3.

The beam is redundant by 1 – degree.

Hence one more equation is required to calculate the external reactions. This is obtained from compatibility........ condition.

The compatibility condition is that downward deflection of B due to w_0 load is equal to upward deflection due to load R_B.





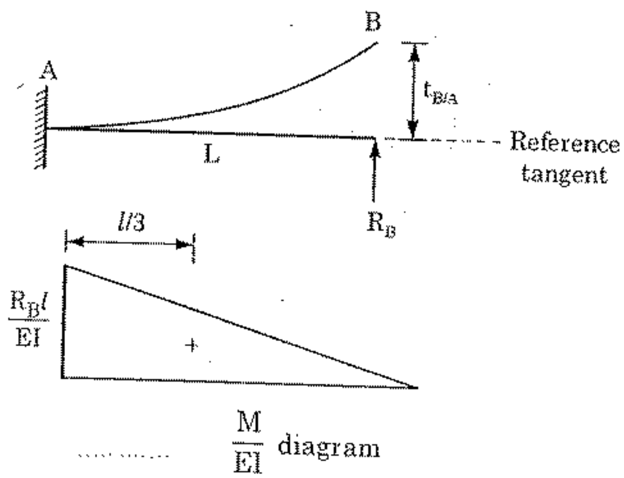
$$t_{B/A} = -\frac{1}{3}\left(\frac{W_0 l^2}{2EI}\right) \times l \times \frac{3l}{4}$$

$$= \frac{-W_0 l^4}{8EI}$$

(–) ve means that point on elastic curve at B is below ref. tangent at A.

⇒ deflection is downwards.

$$\Rightarrow \boxed{\delta_1 = \frac{W_0 l^4}{8\,EI}}$$



$$t_{B/A} = \frac{1}{2} \times \frac{R_B \times l \times l}{EI} \times \frac{2l}{3}$$

$$= \frac{R_B l^3}{3EI}$$

$$\Rightarrow \boxed{\delta_2 = \frac{R_B l^3}{3EI}}$$

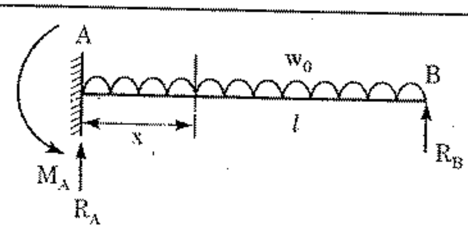
(+) ve sign means that point B on elastic curve is above reference tangent at A i.e. upwards.

Thus from compatibility condition

$$\Rightarrow \quad \frac{W_0 l^4}{8EI} = \frac{R_B l^3}{3EI}$$

$$\Rightarrow \quad \boxed{R_B = \frac{3W_0 l}{8}}$$

Note:



The same problem as above can also be solved using double integration method. (Taking R_B as redundant)

$$EI\frac{d^2y}{dx^2} = R_B\,(l-x) - \frac{W_0(l-x)^2}{2}$$

On integration

$$EI\frac{dy}{dx} = -\frac{R_B(l-x)^2}{2} + \frac{W_0(l-x)^3}{6} + C_1$$

$$EI\,y = +\frac{R_B(l-x)^3}{6} - \frac{W_0(l-x)^4}{24} + C_1x + C_2$$

No. of unknowns are R_B, C_1 and C_2 and the boundary conditions are

(a) Deflection at B = 0,

(b) Deflection at A = 0,

(c) Slope of A = 0

$\Rightarrow$ at $x = l$, $y = 0$

$\Rightarrow$ $\boxed{C_1 l + C_2 = 0}$ .... (i)

$\Rightarrow$ at $x = 0$, $y = 0$

$$\Rightarrow \quad 0 = \frac{R_B l^3}{6} - \frac{W_0 l^4}{24} + C_2 \quad \text{............ (ii)}$$

$\Rightarrow$ at $x = 0$, $\frac{dy}{dx} = 0$

$$0 = -\frac{R_B l^2}{2} + \frac{W_0 l^3}{6} + C_1 \quad \text{............ (iii)}$$

From the above three equations (i), (ii) and (iii) we have

$$\left(\frac{R_B l^3}{6} - \frac{R_B l^3}{2}\right) + \left(\frac{W_0 l^4}{6} - \frac{W_0 l^4}{24}\right) = 0$$

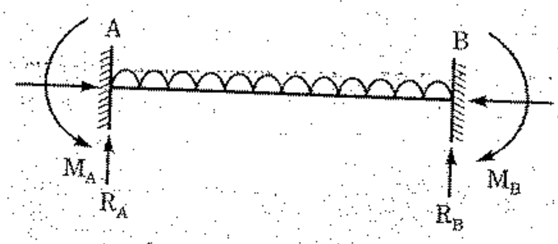
$$-\frac{2R_B l^3}{6}+\frac{3W_0 l^4}{24}=0$$

$$R_B=\frac{3W_0 l^4}{24}\times\frac{6}{2l^3}$$

$$\boxed{R_B=\frac{3W_0 l}{8}}$$

Example 25

Analyse the beam as shown below.



Sol: In this case the horizontal reactions are not existing, hence no. of unknown reactions are 4(M_A, R_A, M_B, R_B) and no. of equations of static equillibrium are 2 ($\Sigma F = 0$, $\Sigma M = 0$)

No. of redundant reactions are two.

No. of unknowns in double integration technique will be

C_1, C_2, M_B, R_B [we have taken R_B and M_B as unknowns]

No. of available conditions are

(a) Slope at A = 0 .... (i)

(b) Deflection at A = 0 .... (ii)

(c) Slope at B = 0 .... (iii)

(d) Deflection at B = 0 .... (iv)

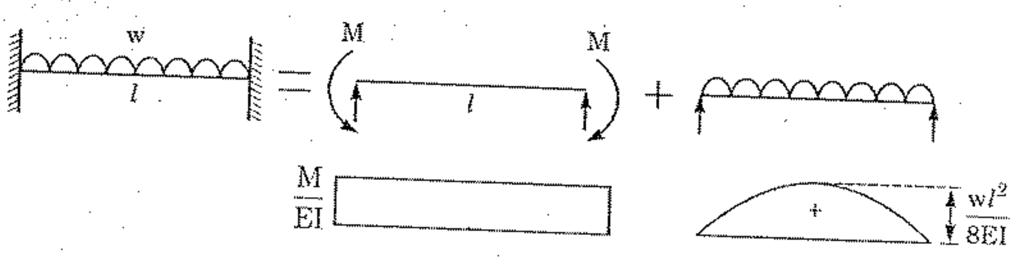
As 4 unknowns can be obtained from 4-known conditions. Hence solution can be obtained.

Simpler Method (assume M_A and M_B are redundants)

Due to symmetry $M_A = M_B$

Also $\theta_{B/A} = 0$

$$\Rightarrow \frac{\text{Area of moment diagram}}{EI} \text{ between A and B} = 0$$



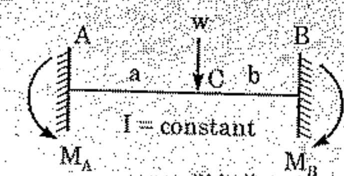
$\frac{M}{EI}$ diagrams

$$\Rightarrow \quad (-)\frac{Ml}{EI} + \frac{2}{3} \times \frac{Wl^2}{8EI} \times l = 0$$

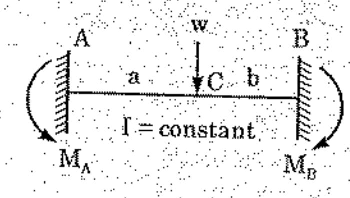
$$\Rightarrow \quad \boxed{M = \frac{W_0 l^2}{12}}$$

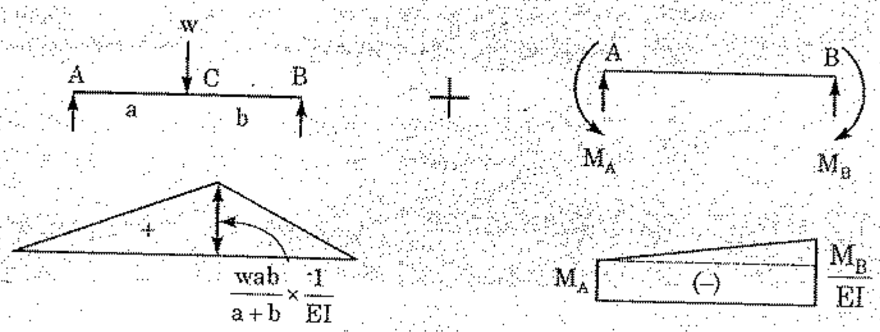
Example 26

Find redundant reactions M_A and M_B.



Sol:





$\frac{M}{EI}$ diagram

By moment area theorem

$$\theta_{BA} = 0 \quad \text{-------------------- (A)}$$

$\Rightarrow$ Area of $\frac{M}{EI}$ diagram between A and B = 0

$$t_{BA} = 0 \quad \text{-------------------- (B)}$$

$\Rightarrow$ Moment of area of $\frac{M}{EI}$ diagram betwen A and B about B = 0

From A:

$$\Rightarrow \quad \left[-\frac{M_A + M_B}{2} \times l + \left(\frac{W\,ab}{a+b}\right) \times \frac{1}{2} \times (a+b)\right]\frac{1}{EI} = 0$$

$$\Rightarrow \quad \boxed{M_A + M_B = \frac{Wab}{a+b}} \quad \text{-------------------- (i)}$$

Note: Summation of end moments is equal to BM below the point load in free BMD.

From B:

$$-\frac{M_A l^2}{2}-(M_B-M_A)\times\frac{1}{2}\times l\times\frac{l}{3}+\frac{1}{2}\times\frac{Wab}{a+b}\left[a\times\left(b+\frac{a}{3}\right)+b\left(\frac{2b}{3}\right)\right]=0$$

$$\Rightarrow -\frac{l^2}{2}\left(M_A+\frac{M_B}{3}-\frac{M_A}{3}\right)+\frac{Wab}{2(a+b)}\left[\frac{(2b+a)(a+b)}{3}\right]=0$$

$$\Rightarrow \boxed{2M_A+M_B=\frac{Wab(2b+a)}{l^2}} \quad \text{------ (ii)}$$

From (i) and (ii)

$$M_A=\frac{Wab\,(2b+a)}{l^2}-\frac{Wab}{l}$$

$$M_A=\frac{Wab}{l}\left[\frac{2b+a}{a+b}-1\right]=\frac{Wab^2}{l^2}$$

$$\Rightarrow \boxed{M_A=\frac{Wab^2}{l^2}}$$

and

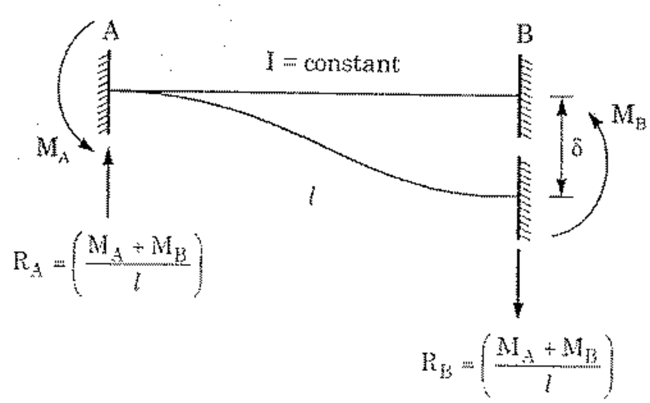
$$\boxed{M_B=\frac{Wa^2b}{l^2}}$$

Remember this relationship

Fixed Beam with Sinking Support

Sinking of support in redundant structures introduces stress in the members of the structure and hence reactions are generated at supports.

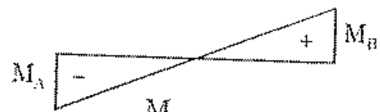
Let, due to sinking of fixed support B by an amount 'δ' as shown in figure, moments M_A and M_B and reactions R_A and R_B are generated. Let M_A and M_B be the redundant reactions.



From the geometry of the deflected shape it is clear that

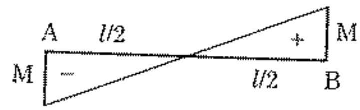
$$\boxed{\theta_{BA}=0} \qquad \boxed{t_{B/A}=-\delta}$$

$t_{BA}=-\delta$, because as per our sign convention, point B on deflected curve is below reference tangent at A.



$\frac{M}{EI}$ diagram

Since $\theta_{B/A} = 0 \Rightarrow M_A = M_B$ [Net area has to be zero]



Let $M_A = M_B = M$

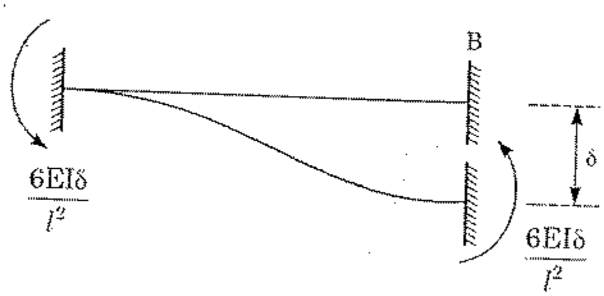
$$t_{BA} = -\delta \Rightarrow \frac{1}{EI}\left[\frac{Ml}{2} \times \frac{1}{2} \times \frac{1}{3} \times \frac{l}{2} - \frac{Ml}{2} \times \frac{1}{2}\left(l - \frac{l}{6}\right)\right] = -\delta$$

$$\frac{Ml^2}{24\,EI} - \frac{5Ml^2}{24\,EI} = -\delta$$

$$\frac{-4\,Ml^2}{24\,EI} = -\delta$$

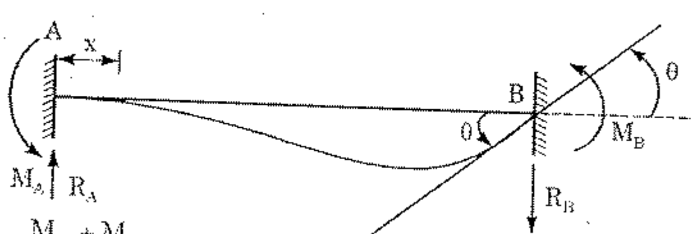
$$\Rightarrow \boxed{M = \frac{6\,EI\,\delta}{l^2}}$$

Hence the final result is as shown below.



Fixed Beam with Rotational Slip

Let the fixed support B has rotational slip of anticlockwise nature of magnitude θ.
Rotational slip will also introduce fixed end moment as in the case of settlement of support.



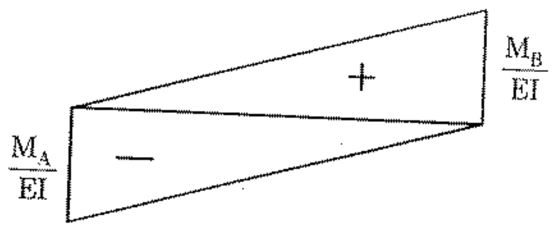
$$R_A = \frac{M_A + M_B}{l} \qquad R_B = \frac{M_A + M_B}{l}$$

The boundary moment at a distance x from end A is given by M, where

$$M = -M_A + \frac{M_A + M_B}{l}x$$

$$= \left(-M_A + \frac{M_A x}{l}\right) + \frac{M_B x}{l}$$

Hence $\frac{M}{EI}$ diagram can be drawn as



Now from the geometry of deflected curve it is clear that

$$t_{B/A} = 0$$

and

$$\theta_{B/A} = \theta$$

Angle θ is measured anti clockwise from tangent at A (i.e. point of smaller x) to tangent at B (i.e. point of larger x). Hence it is (+) ve.

$$t_{B/A} = \frac{1}{2}\frac{M_B}{EI} \times l \times \frac{l}{3} - \frac{1}{2}\frac{M_A l}{EI} \times \frac{2l}{3} = 0$$

$$M_B - 2M_A = 0$$

$$\Rightarrow \boxed{M_A = \frac{M_B}{2}}$$

$$\theta_{BA} = \theta = \frac{1}{2}\frac{M_B}{EI} \times l - \frac{1}{2}\frac{M_A}{EI} \times l = \text{area under BMD between A and B}$$

$$= \frac{(M_B - M_A)\,l}{2EI} = \frac{\left(M_B - \frac{M_B}{2}\right)l}{2EI}$$

$$\boxed{\theta = \frac{M_B l}{4EI}}$$

$$\Rightarrow \boxed{M_B = \frac{4EI\theta}{l}}$$

$$\Rightarrow \boxed{M_A = \frac{2EI\theta}{l}}$$

CONJUGATE BEAM METHOD

The calculation of slope and deflection using area moment theorem required the understanding of geometry of deflected shape and was applicable only when the deflected shape was continuous (i.e. no discontinuity in the deflected shape). As a beam with internal hinge has discontinuity in deflected shape, area moment theorem could not be applied to it.

However in conjugate beam method, we rely only on principle of statics and hence geometry of deflected shape need not be taken into account. Thus, beam with discontinuity in deflected shape (as due to internal hinge or slider) can be easily analysed. On account of these, application of conjugate beam method is simpler.

We know that

$$\left[\frac{dV}{dx} = w \qquad \text{or} \qquad \frac{d^2M}{dx^2} = w\right] \text{.................. (A)}$$

$$\left[\frac{d\theta}{dx} = \frac{M}{EI}\right] \quad \text{or} \quad \left[\frac{d^2y}{dx^2} = \frac{M}{EI}\right] \quad \text{------- (B)}$$

On integration

$$\begin{bmatrix} V = \int w\,dx \\ \downarrow \qquad \uparrow \\ \theta = \int \frac{M}{EI}dx \end{bmatrix} \quad \text{------- (C)}$$

$$\begin{bmatrix} M = \iint \left[\int w\,dx\right] dx \\ \downarrow \qquad \uparrow \\ y = \iint \left[\frac{M}{EI}dx\right] dx \end{bmatrix} \quad \text{------- (D)}$$

The basis of conjugate beam method comes from the similarity of equation (A) and (B) as also the similarity of eq. (C) and (D). Thus, if $\frac{M}{EI}$ diagram is taken as, loading diagram, we get a conjugate beam and then,

slope at any point in real beam = shear at that point in conjugate beam

Similarly,

deflection of any point in real beam = bending moment at that point in the conjugate beam

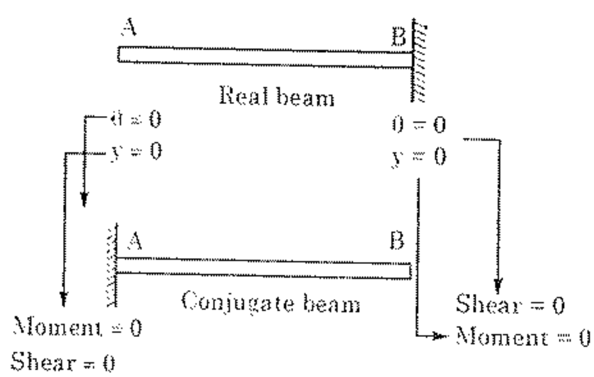
Thus following theorems are used in conjugate beam method of analysis:

Theorem 1: The slope at a point in the real beam is numerically equal to shear at the corresponding point in the conjugate beam.

Theorem 2: The displacement of a point in the real beam is numerically equal to bending moment at the corresponding point in the conjugate beam.

Conjugate Beam Support

While drawing the conjugate beam, it is necessary that *shear and moment developed at the supports of a conjugate beam corresponds to the slope and displacement of the real beam at the location of supports in conjugate beam.*



In the case shown above, we have a cantilever beam AB with fixed support at B and free end at A. Hence at A, slope and deflection exists and at B, slope and deflections are zero. Thus as per conjugate beam theorems, shear and moment must exist at A in conjugate beam and shear and moment must be zero at B in conjugate beam. This can be ensured only by changing the support conditions in conjugate beams as shown in the figure above.

Note that at free end, shear and moment is zero and at fixed end shear and moment will not be zero. Thus, fixed end in beam is replaced by free end in conjugate beam and free end in real beam is replaced by fixed end in conjugate beam.

Hence, the real beam can be converted to a conjugate beam by changing the support conditions as shown below.

Real beam	Conjugate beam
Pin: $\theta \neq 0$, $\Delta = 0$	Pin: $V \neq 0$, $M = 0$
Roller: $\theta \neq 0$, $\Delta = 0$	Roller: $V \neq 0$, $M = 0$
A Fixed: $\theta = 0$, $\Delta = 0$; B Free: $\theta \neq 0$, $\Delta \neq 0$	A Free: $V = 0$, $M = 0$; B Fixed: $V \neq 0$, $M \neq 0$
Internal pin: $\theta \neq 0$, $\Delta = 0$	Hinge: $V \neq 0$, $M = 0$
Internal roller: $\theta \neq 0$, $\Delta = 0$	Hinge: $V \neq 0$, $M = 0$
Internal hinge: θ, Δ exist	Internal roller: V, M exist
Slider: $\Delta \neq 0$, $\theta = 0$	Slider: $M \neq 0$, $V = 0$

Thus, on the basis of conditions discussed above, conjugate beams of real beams can be made as follows.

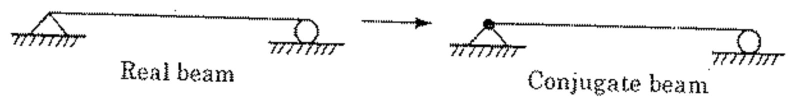


Fig. A

Fig. B

Note from Fig. A that statically determinate and stable real beam has statically determinate and stable conjugate beam. [Neglecting axial forces]

Similarly from Fig. B it can be concluded that statically indeterminate real beam has unstable conjugate beam.

In this case although the conjugate beam is unstable, the $\frac{M}{EI}$ loading will provide the necessary equillibrium to hold the conjugate beam in stable form. Thus the analysis of conjugate beam will give unique solution. **Note that a beam is said to be unstable only when its analysis doesnot yield unique solution].**

Sign Convention

1. If $\frac{M}{EI}$ is (+) ve → conjugate beam loading is upward.

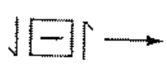
2. If $\frac{M}{EI}$ is (–) ve → conjugate beam loading is downward.

Note: *As we have been plotting (+) ve BM above the beam and (–) ve BM below the beam, we can say that conjugate beam loading will always be alway from the beam.*

3.  Hogging BM = (–) ve

    Sagging BM = (+) ve

4.  Positive shear

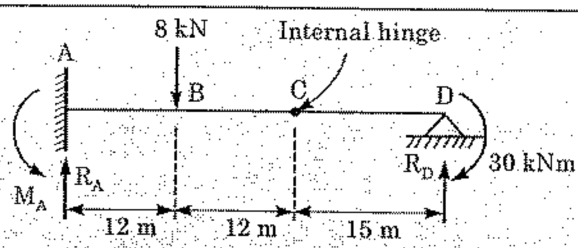
    Negative shear

5. If shear in conjugate beam is (+) ve ⇒ slope is anticlockwise.
6. If moment in conjugate beam is (+) ve ⇒ deflection is upward.
7. Upward deflection is (+) ve and downward deflection is (–) ve.
8. Anticlockwise slope is (+) ve and clockwise slope is (–) ve.

Example 27

Determine slope of each beam segment connected to the hinge i.e. θ_{CA} and θ_{CD} and also calculate deflection at point 'C'.

Area moment Not applicable due to Internal Hinge.



Calculation of reactions

$$R_A + R_D = 8 \text{ kN}$$

$$M_C = 0, \quad R_D \times 15 - 30 = 0$$

$$\Rightarrow \boxed{R_D = 2 \text{ kN}}$$

$$\Rightarrow \boxed{R_A = 6 \text{ kN}}$$

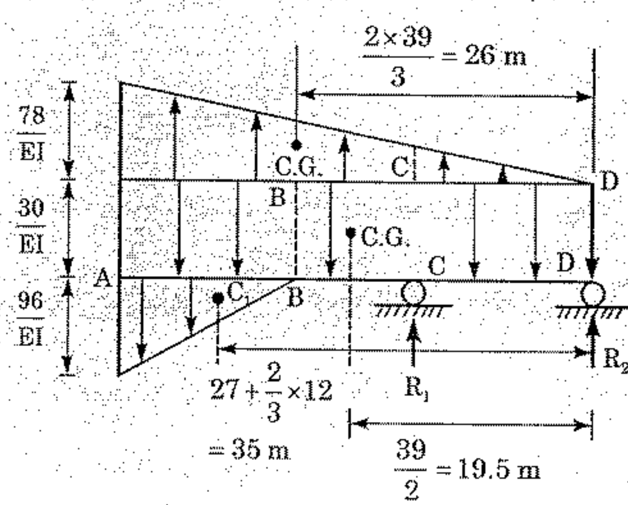
$$M_C = 0 \Rightarrow 6 \times 24 - 8 \times 12 - M_A = 0$$

$$\Rightarrow \boxed{M_A = 48 \text{ kNm}}$$

Conjugate beam is formed, by changing the supports, and loading the resultant beam, with $\frac{M}{EI}$ diagram.

1. Fixed end becomes free end.
2. Internal hinge becomes internal roller.
3. End roller remains end roller.

$\frac{M}{EI}$ diagram is drawn using BM diagram by parts concept. In this case BM by parts has been drawn about section at A.



Note that (+) ve BM leads to upward force and (–) ve BM leads to downward force.

$$\text{Total triangular loading upward} = \frac{1}{2} \times \frac{78}{EI} \times 39$$

$$= \frac{1521}{EI} \text{kN}$$

Uniform loading downward $= \frac{30 \times 39}{EI} = \frac{1170}{EI}$ kNm

Triangular loading downward $= \frac{1}{2} \times \frac{96}{EI} \times 12 = \frac{576}{EI}$

Thus in conjugate beam, for $\Sigma F_V = 0$

$$R_1 + R_2 = \frac{1170}{EI} + \frac{576}{EI} - \frac{1521}{EI}$$

$$\boxed{R_1 + R_2 = \frac{225}{EI}} \text{ kNm}$$

$$\Sigma M_D \circlearrowright = 0 \Rightarrow \frac{1521}{EI} \times 26 - \frac{1170}{EI} \times 19.5 - \frac{576}{EI} \times 35 + R_1 \times 15 = 0$$

$$\Rightarrow \boxed{R_1 = \frac{228.6}{EI} \text{ kNm}}$$

$$\boxed{R_2 = (-)\frac{3.6}{EI}}$$

θ_{CD} = shear force from right at C in conjugate beam

$$= \frac{30}{EI} \times 15 - \left(\frac{78}{39\,EI} \times 15 \times \frac{1}{2} \times 15\right) - R_2 \text{ [Right side downward shear = (+) ve]}$$

$$= \frac{450}{EI} - \frac{225}{EI} - \left(-\frac{3.6}{EI}\right)$$

$\theta_{CD} = \boxed{\frac{228.6}{EI}}$ (+) ve means anticlockwise

As $R_1 = \frac{228.6}{EI}$, hence

$\frac{228.6}{EI} \Rightarrow V = 0$

$R_1 = \frac{228.6}{EI}$

$\Rightarrow$ Shear at C from left = 0

$\Rightarrow$ Slope at C from left = 0

$\Rightarrow \boxed{\theta_{CA} = 0}$

Deflection at C = Bending Moment in Conjugate Beam at C

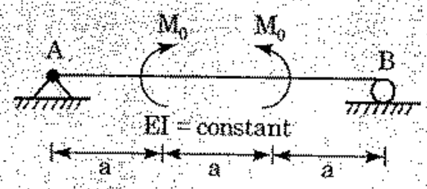
$$\Delta_C = R_2 \times 15 - \frac{30}{EI} \times 15 \times 7.5 + \left(\frac{78}{39\,EI} \times 15\right)\frac{1}{2} \times 15 \times \frac{15}{3}$$

$$\Delta_C = -\frac{3.6\times 15}{EI} - \frac{450\times 7.5}{EI} + \frac{225\times 5}{EI}$$

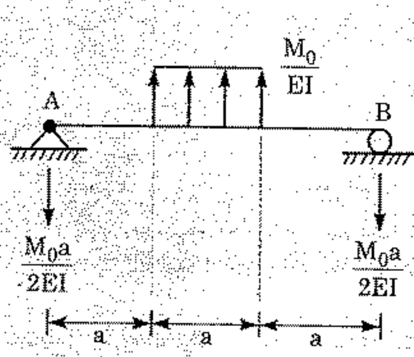
$$\Rightarrow \quad \Delta_C = (-)\frac{2304}{EI}, \quad \text{(-) ve means downward deflection}$$

Example 28

Find slope at A and maximum deflection in the beam shown below.



Sol:



Conjugate beam.

Slope at A $= -\frac{M_0a}{2EI}$ [left side downward shear is (–) ve]

$$\Rightarrow \boxed{\theta_A = -\frac{M_0a}{2\,EI}}$$

Δ_{max} occurs at mid span due to symmetry of loading

$\Rightarrow \Delta_{max}$ = BM in conjugate beam loading at mid span

$$\Delta_{max} = -\frac{M_0a}{2\,EI}\times\frac{3a}{2} + \frac{M_0}{EI}\frac{\left(\frac{a}{2}\right)^2}{2}$$

$$= -\frac{3M_0a^2}{4\,EI} + \frac{M_0a^2}{8\,EI}$$

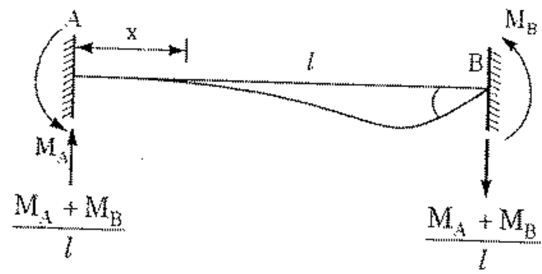
$$\boxed{\Delta_{max} = (-)\frac{5M_0a^2}{8EI}}$$ (–) ve sign means downward deflection.

Note: *Conjugate beam method is very useful in calculating slopes at support. Slope at support in real beam will just be equal to reaction at supports in the conjugate beam.*

Calculation of Redundants using Conjugate Beam Method

Fixed Beam with Rotational Slip

Let there be a rotational slip in fixed beam at support B in anticlockwise direction. Let the fixed end moments generated due to this be M and M as shown in figure below.



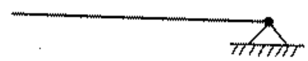
The conjugate beam for this case is made as follows:

1. *Fixed end becomes free end.*
2. At end B, there is rotation existing but there is no deflection.

   ⇒ In conjugate beam, shear must exist and bending moment must be zero at end B.

   ⇒ Fixed end B in real beam will be replaced by a hinged end in the conjugate beam.

   Hence the conjugate beam will be as shown below.

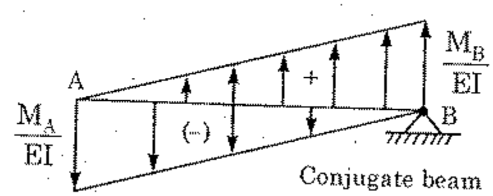


Loading on conjugate beam will be $\frac{M}{EI}$ diagram of real beam.

⇒ M at any x (in real beam)

$$= M = \frac{M_A + M_B}{l} x - M_A$$

$$= \left(\frac{M_A x}{l} - M_A\right) + \left(\frac{M_B x}{l}\right)$$



Conjugate beam

Now

Deflection of B in real beam = 0

⇒ BM of conjugate beam at B = 0

$$\Rightarrow \frac{1}{2}\frac{M_B}{EI} \times l \times \frac{l}{3} - \frac{1}{2}\frac{M_A}{EI} \times l \times \frac{2l}{3} = 0$$

$$\Rightarrow \boxed{M_B - 2M_A = 0}$$

$$\Rightarrow \boxed{M_A = \frac{M_B}{2}} \quad \text{...(i)}$$

Rotation of real beam at B = θ [slope is (+) ve because it is anticlockwise]

⇒ Shear force at B in conjugate beam = θ

$$\Rightarrow -\frac{M_A}{2\,EI} \times l + \frac{M_B}{2\,EI} \times l = \theta \quad \text{[Left side upward shear force = (+) ve]}$$

$$\frac{(-M_A + M_B)\,l}{2\,EI} = \theta$$

$$\frac{\left(-\frac{M_B}{2} + M_B\right)l}{2\,EI} = \theta$$

$$\Rightarrow \frac{M_B l}{4\,EI} = \theta$$

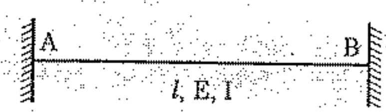
$$\Rightarrow \boxed{M_B = \frac{4EI\,\theta}{l}}$$

$$\boxed{M_A = \frac{2EI\theta}{l}}$$

{M_A and M_B values are (+) ve, this means that the assumed directions of M_A and M_B are correct}

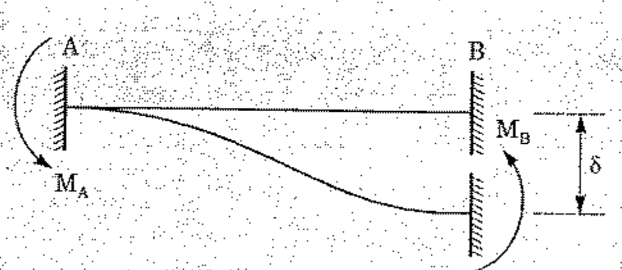
Example 29

Fixed beam with settlement of support.



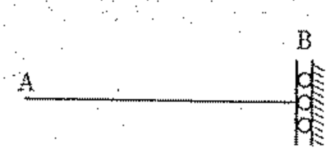
The support B settles by δ downward. Find the moment generated due to this at support A and B.

Sol:



To make conjugate beam for this case:

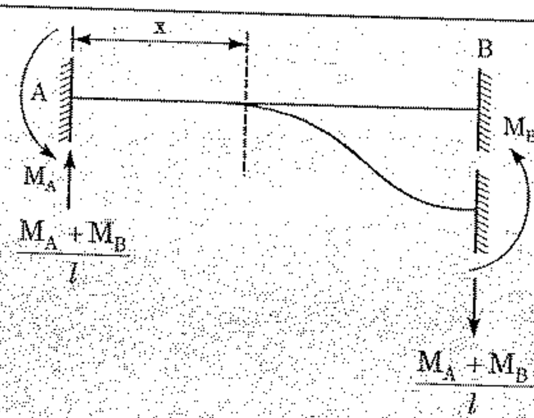
- Fixed support at A in real beam becomes free end in conjugate beam.
- At support B in real beam there is deflection but no rotation.
  ⇒ In conjugate beam there should be bending moment at support B but no shear force.
- This can be modelled by a slider support at B. Hence the support condition in conjugate beam becomes like the fig. shown below.



Loading on Conjugate Beam

Loading on conjugate beam will be M/EI diagram in real beam.

$\frac{M}{EI}$ diagram of real beam

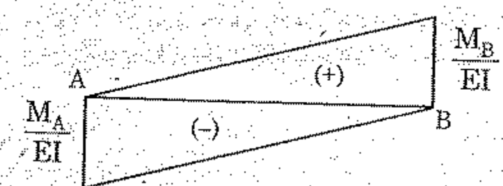


B.M. at section

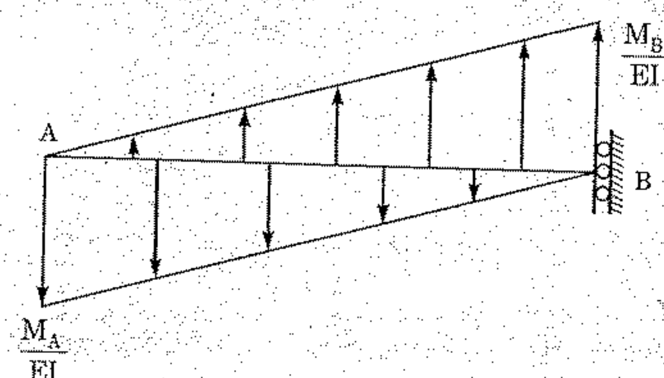
$$x = -M_A + \frac{M_A + M_B}{l}x$$

$$= \frac{M_B x}{l} - M_A\left(1 - \frac{x}{l}\right)$$

Thus, $\frac{M}{EI}$ diagram is



Hence the loaded conjugate beam is as shown below.



Slope of real beam at B = 0

⇒ Shear in conjugate beam at B = 0

⇒ $\frac{1}{2} \times \frac{M_B}{EI} \times l - \frac{1}{2}\frac{M_A}{EI} \times l = 0$

⇒ $\boxed{M_B = M_A}$ ---------- (i)

Deflection of B in real beam = $-\delta$ [(–) because settlement of support is downward]

⇒ BM at B in conjugate beam = $-\delta$

⇒ $\frac{M_B l}{2EI} \times \frac{l}{3} - \frac{M_A l}{2EI} \times \frac{2l}{3} = -\delta$

$$M_B - 2M_A = -\frac{6\,EI\,\delta}{l^2} \quad \text{-------- (ii)}$$

From (i)

$$\Rightarrow \; M_B - 2M_B = -\frac{6\,EI\,\delta}{l^2}$$

$$\Rightarrow \; \boxed{M_B = \frac{6EI\,\delta}{l^2}}$$

$$\boxed{M_A = \frac{6EI\,\delta}{l^2}}$$

{(+)ve sign of M_A and M_B means they are in the same sense as that assumed}

DEFLECTION USING ENERGY METHODS

Principle of work and energy is applied to determine deflection/rotation at a point in a structure. To understand the concept of work and energy, let us discuss certain basic concepts first.

1. Conservation of Energy Principle

The work done by all the external forces acting on a structure (U_e) is transformed into internal work or strain energy (U_i), which is developed when the structure deforms.

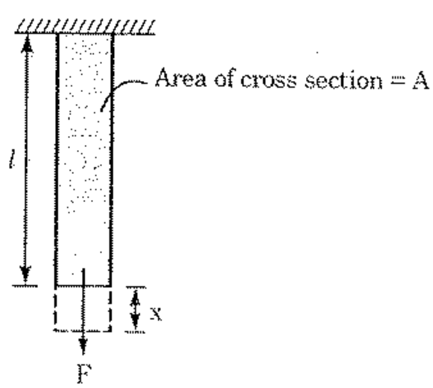
$$\Rightarrow \; \boxed{U_e = U_i}$$

$\Rightarrow$ External work done = internal strain energy stored

2. External Work Done

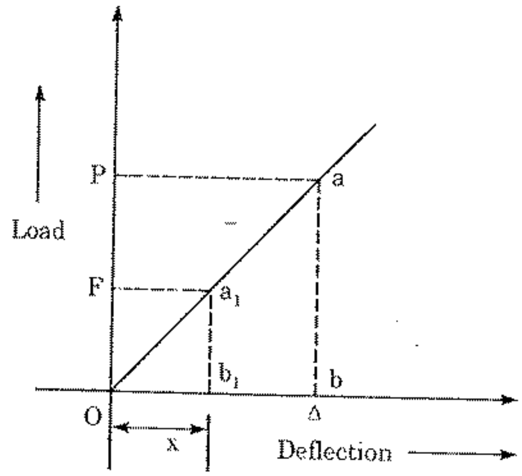
External Work: Due to axial force

Let a force P is applied gradually to a bar. i.e. the load increases from a value zero to P gradually during which deflection changes from zero to Δ gradually.



For a linearly elastic body [a body in which stress is proportional to strain], if the load and deflection are within proportional limit then load deflection curve will be a straight line

i.e. $\frac{F}{x} = \frac{P}{\Delta}$ [where x is the final deflection due to gradually applied load F]



External work done due to gradually applied force P is $\int_0^{\Delta} F dx$

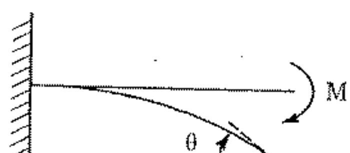
i.e. $U_e = \int_0^{\Delta} F dx$

$$= \int_0^{\Delta} \frac{Px}{\Delta} dx = \frac{P\Delta}{2}$$

$\Rightarrow$ $\boxed{U_e = \frac{P\Delta}{2}}$ = *Area under load displacement curve i.e. area oab*

External work:

Due to Moment



$\Rightarrow$ If external moment is gradually applied then external work done is

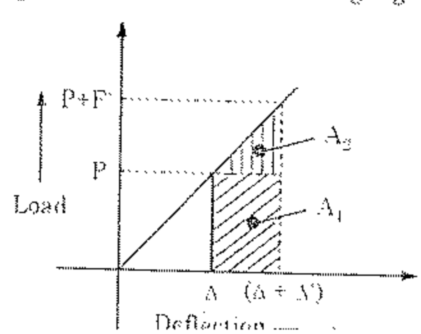
$$U_e = \int_0^{\theta} M d\theta = \frac{1}{2} M\theta$$

$$\boxed{U_e = \frac{1}{2} M\theta}$$

If force P is already applied to the bar and another force F′ is now gradually applied so that bar deforms further by amount Δ′.

Then work done by P(not F′) during deformation of $\Delta' = P \cdot \Delta'$ = Area A_1 as shown in the following figure.

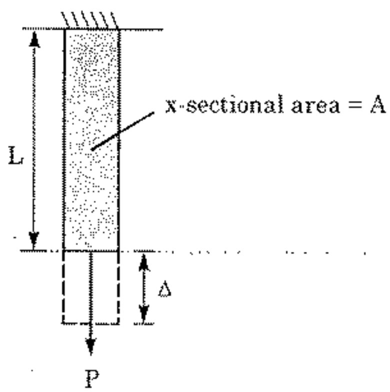
Work done by $F' = \frac{1}{2} F'\Delta'$ = Area A_2 as shown in the following figure.



Similarly if moment 'M' is already applied and any other loading deforms the structure by amount θ', then work done by $M = M\theta'$.

ELASTIC STRAIN ENERGY (U_i)

Due to Axial Force



$$\text{Stress} = \frac{P}{A}$$

$$\text{Strain} = \frac{\Delta}{L}$$

$$\Rightarrow \frac{\frac{P}{A}}{\frac{\Delta}{L}} = E$$

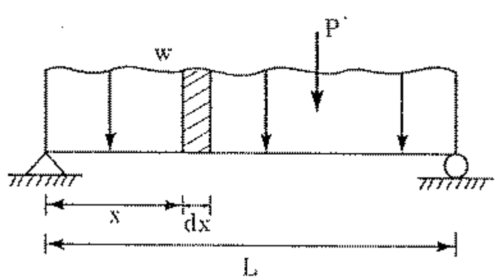
$$\Rightarrow \frac{PL}{AE} = \Delta$$

$$U_e = U_i = \frac{1}{2} P \cdot \frac{PL}{AE} = \frac{P^2L}{2AE}$$

$$\boxed{\text{Strain energy in bar} = \frac{P^2L}{2AE}}$$

ELASTIC STRAIN ENERGY: BENDING

We know that if beam is divided into differential elements then each differential element deforms by some angle which will add to give rotation θ at any point. Like, in case of symmetrically loaded simply supported beams, the mid span has no rotation and slope at any point either to the left or right of mid span is summation of rotation of various elements from mid span upto that point.



Gradually applied load P and w creates internal moment M at a distance x from left support. The rotation of differential element dx is $d\theta$. But we know that

$$\boxed{d\theta = \frac{M}{EI} dx}$$

$\Rightarrow$ Internal work done in rotating this differential element from angle 0° to $d\theta$ is $\frac{1}{2} M d\theta$. (Since internal moment develops gradually)

⇒ Internal strain energy stored in the differential element dx = $dU_i = \frac{1}{2} M \times d\theta = \frac{1}{2} \times M \times \left(\frac{M}{EI} dx\right) = \frac{M^2 dx}{2EI}$

⇒ Total strain energy in the complete beam due to bending = $\boxed{U_i = \int_0^L \frac{M^2 dx}{2EI}}$

Note: On similar line it can be shown that

$$U_i = \int \frac{P^2 dx}{2AE} \text{ for axial load}$$

$$U_i = \int \frac{T^2 dx}{2GJ} \text{ for torsion}$$

$$U_i = K \int \frac{V^2 dx}{2AG} \text{ for shear}$$

K = Form factor

K depends on the shape of x-section

AE = Axial rigidity

GJ = Torsional rigidity

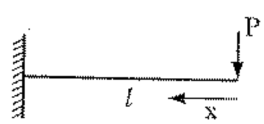
AG = Shear rigidity

EI = Flexural rigidity

$$\text{Stiffness} = \frac{\text{rigidity}}{\text{length}}$$

PRINCIPAL OF WORK AND ENERGY

Principal of work and energy can be applied to determine the displacement at a point in a structure. Suppose we have to determine displacement at the location of P.



In the above figure internal strain energy due to bending =

$$U_i = \int_0^L \frac{M^2 dx}{2EI} = \int_0^L \frac{(-Px)^2 dx}{2EI}$$

$$U_i = \frac{P^2 l^3}{6\,EI}$$

External work done $U_e = \frac{1}{2} P\Delta$, where Δ is the deflection at the location of P due to bending alone.

$$\frac{1}{2} P\Delta = \frac{P^2 l^3}{6\,EI}$$

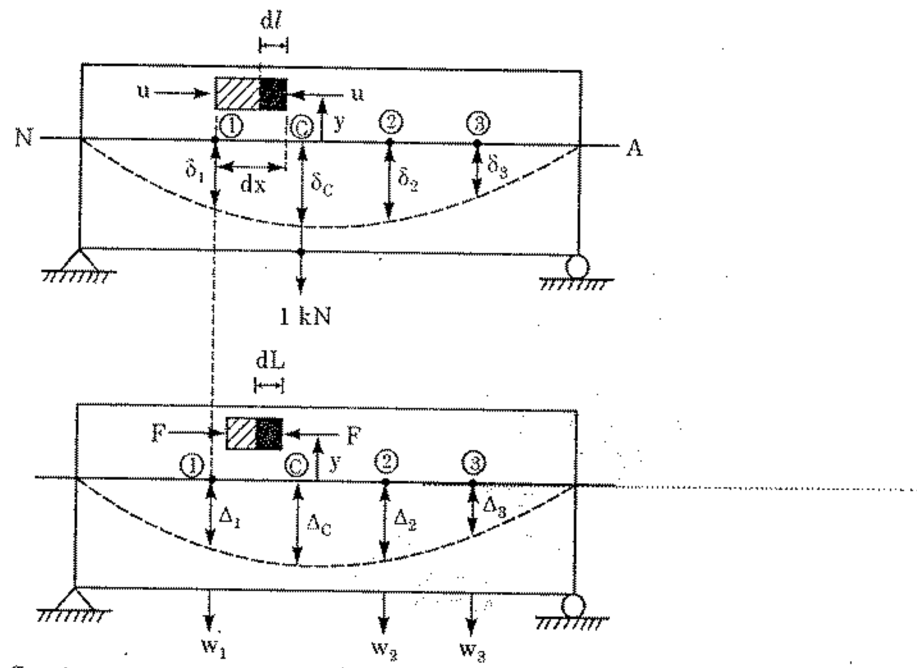
$$\boxed{\Delta = \frac{Pl^3}{3\,EI}}$$

[Note that in case of beams deflection is primarily due to bending only. Hence we are justified in using strain energy due to bending only]

This method can be used when only one load is acting and displacement is to be determined at the location of application of the load; because if many loads are acting, there would be unknown displacements below each load and hence in external work done equation, there would be more than one unknown displacements but to find out unknown displacements only one equation can be written [i.e. $U_e = U_i$]. Hence from one equation one displacement out of many displacements can not be found.

Due to this restriction, method of virtual work (unit load method) or Castigliano's theorem can be used to find out deflection at a point when many loads are acting on a beam.

METHOD OF VIRTUAL WORK (UNIT LOAD METHOD)



Let δ_1, δ_2, δ_3 and δ_C are deflection at point 1, 2, 3 and C due to unit load at C.

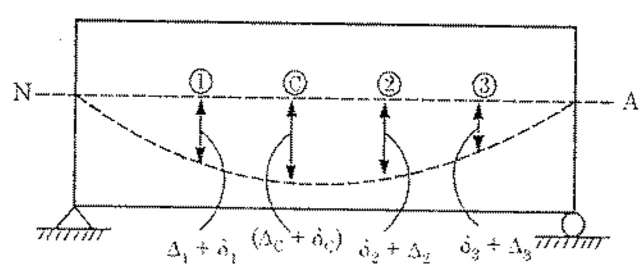
u = Internal force developed at any point in beam due to the effect of external unit load.

This load 'u' causes a displacement of dl in the differential element of lengths dx as shown in figure.

Let Δ_1, Δ_2, Δ_3 and Δ_C are deflection at point 1, 2, 3 and C due to w_1, w_2 and w_3.

F = Internal force developed in the differential element dx in beam due to the effect of external unit load. This load F causes a displacement of dL in the differential element dx.

If unit load is applied 1st and then the external loads (w_1, w_2 and w_3) are applied, then due to combined action deflections are as shown in fig. below.



Total external work done due to unit load and external load combinedly

$$= \left(\frac{1}{2} \times 1 \times \delta_C\right) + (1 \times \Delta_C) + \left(\frac{1}{2} w_1 \Delta_1 + \frac{1}{2} w_2 \Delta_2 + \frac{1}{2} w_3 \Delta_3\right)$$

[Note that when unit load is already acting and then external load system w_1, w_2 and w_3 are applied, work done by unit load will be simply $1 \times \Delta_C$, where Δ_C = deflection at C due to external load system].

Strain Energy Stored (i.e. internal work done)

$$= \left(\frac{1}{2}\sum udl\right) + \left(\frac{1}{2}\sum F.dL\right) + \left(\sum u.dL\right)$$

but $$\frac{1}{2}\sum udl = \frac{1}{2} \times 1 \times \delta_C$$

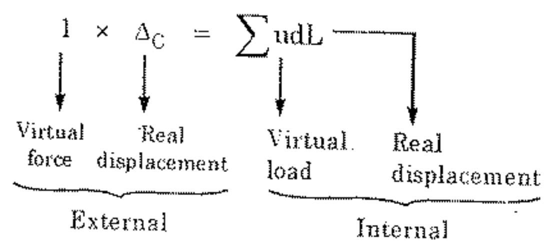
and $$\frac{1}{2}\sum FdL = \frac{1}{2}w_1\Delta_1 + \frac{1}{2}w_2\Delta_2 + \frac{1}{2}w_3\Delta_3$$

$\Rightarrow$ $$\boxed{1 \times \Delta_C = \sum udL} \quad \text{------------ (A)}$$

Note that Δ_C is the deflection at the location of point load due to the effect of external load (not the unit load). Hence $1 \times \Delta_C$ can be thought of as a virtual work. Similarly, ΣudL is the internal virtual work done on differential elements. Thus we can write down

$$\boxed{\text{External virtual work done}} = \boxed{\text{Internal virtual work done}} \quad \text{------------ (B)}$$

This expression (A) or (B) is thus the principle of virtual work.

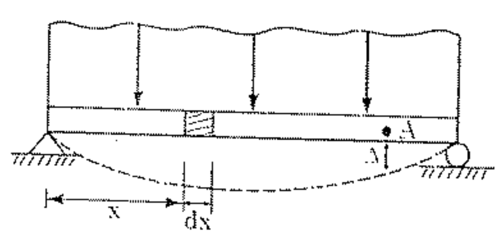


This method can be used to find out deflection of truss joints and deflection and rotation in beams/frames due to external loads.

Note: 1 = Unit load applied at the point at which deflection is to be found out.
Δ_C = Deflection/rotation due to external load (i.e. desired deflection).
u = Internal load developed due to unit load.
dL = Deflection of internal element due to external load.

UNIT LOAD METHOD (METHOD OF VIRTUAL WORK): BEAMS AND FRAMES

Displacement at point 'A' is to be determined:



Strain due to bending is the primary cause of deflection in beams and frames. Hence we take only moment effects here.

For this we follow the steps as described below:

- Apply unit load at 'A' in the direction of desired deflection.
- Due to this unit load, calculate internal virtual moment 'm' in beam at section 'x'.
- When external load is applied, point 'A' moves down by amount Δ and element dx (internal element) deforms or rotates by amount $d\theta$.

For linear elastic material response

$$\boxed{d\theta = \frac{M}{EI}dx}$$

where 'M' is the bending moment at 'x' due to real loads

External virtual work done = $1 \cdot \Delta$

Internal virtual work done = $\int_0^L m \cdot d\theta$

From principle of virtual work $1 \cdot \Delta = \int_0^L m d\theta$

$$\Rightarrow \quad 1 \cdot \Delta = \int_0^L \frac{m \cdot Mdx}{EI}$$

$$\Rightarrow \quad \boxed{\Delta = \int_0^L \frac{m \cdot Mdx}{EI}}$$

m = BM generated at the location of differential element dx in the beam due to unit load applied at the point of desired deflection and in the direction of desired deflection.

M = BM generated at the location of differential element dx due to external loads.

On similar line, it can be shown that if slope at any point of beam or frame is to be found out then instead of applying unit load we apply unit couple and

$$\boxed{\theta = \int_0^L \frac{m_\theta Mdx}{EI}}$$

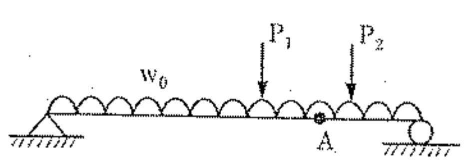
where, m_θ = Bending moment at the location of differential element dx due to unit couple at the point where rotation is to be found. The unit couple will be applied in the direction of desired rotation.

Note: *Unit load method can be applied to plastic range of stress-strain also, but $d\theta$ will not be equal to $\frac{M}{EI}dx$.*

Application of Method of Virtual Work or Unit Load Method [For Beams and Frames]

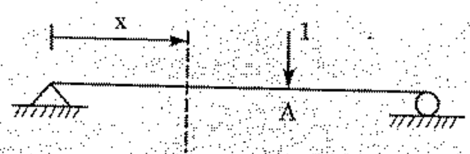
Example 30

Deflection at point 'A' is desired.



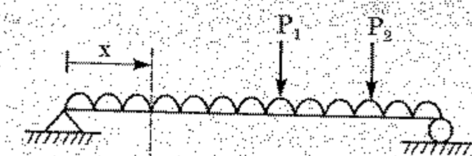
Sol:

- Apply unit load at the point where deflection is to be found out.



m = bending moment at x location

- Calculate moment due to unit load at x say 'm'.
- Apply external load and find BM at x say M.



M = bending moment at x location

From principle of virtual work.

$$\Delta = \int_0^L m \cdot d\theta$$

$\Rightarrow$ $\boxed{\Delta = \int_0^L \frac{m \cdot Mdx}{EI}}$ if Δ is (+), this implies that deflection is in the direction of applied unit load otherwise opposite to unit load.

If Slope at A is to be Found Out

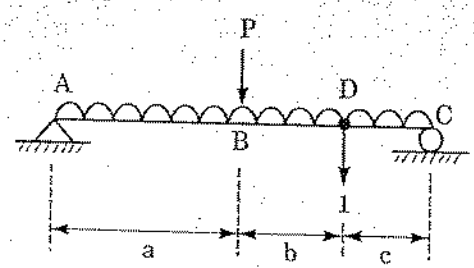
Apply unit couple instead of unit load and follow the same procedure

$$\theta = \int_0^L \frac{m_\theta \cdot Mdx}{EI}$$

If θ is positive, this implies that θ is in the direction of applied couple otherwise opposite to it.

Note: *If expressions of m and M are different in different parts of beam then beam should be divided into various zones such that for each individual zone, expression of 'm' and 'M' do not change.*

The following figure clarifies this concept.



$$\Delta_D = \int_0^a \frac{m_{AB} \cdot M_{AB} \cdot dx}{EI} + \int_0^b \frac{m_{BD} \cdot M_{BD} \cdot dx}{EI} + \int_0^c \frac{m_{DC} \cdot M_{DC}}{EI} dx$$

[because expression for 'm' will be different in AB, BD and DC and expression for M will be different in AB and BC].

Sign convention :

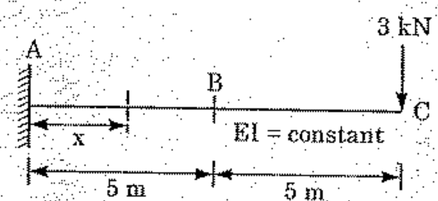
- Hogging moment is taken as (–) ve.
- Sagging moments is taken as (+) ve.
- Deflection and slope will be in the direction of applied unit load or unit moment if

$$\int_0^L \frac{m \cdot Mdx}{EI} \text{ or } \int_0^L \frac{m_\theta \cdot Mdx}{EI} \text{ are (+) ve}$$

  otherwise opposite to it
- In this method it is immaterial in which direction x is taken to increase. Only thing is that both 'm' and M should be calculated with same origin and same direction of x.

Example 31

Find slope and deflection at point B.



Sol:

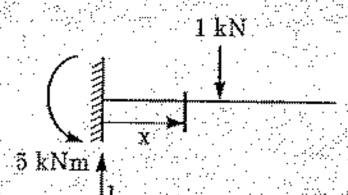
Calculation of deflection:

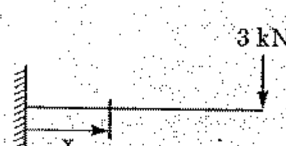
Apply unit load at B. Hence m = –(5 – x), for $0 \le x \le 5$.

m = 0, for x > 5

Apply external load

M = –3 (10 – x) for $0 \le x \le 10$





$$\Rightarrow \quad \Delta_B = \int_0^5 \frac{[-(5-x)][-3(10-x)]\,dx}{EI} + \int_5^{10} 0\,dx$$

$$= \int_0^5 \frac{3(5-x)(10-x)\,dx}{EI} = \int_0^5 \frac{3(x^2-15x+50)\,dx}{EI}$$

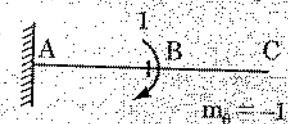
$$\frac{3}{EI}\left[\frac{x^3}{3} - \frac{15x^2}{2} + 50x\right]_0^5$$

$$\frac{3}{EI}\left[\frac{125}{3} - \frac{375}{2} + 250\right]$$

$$\frac{3}{EI}\left[\frac{250-1125+1500}{6}\right] = \frac{312.5}{EI}$$

(+) ve sign means deflection is in the direction of applied unit load i.e. downwards

Calculation of Slope: Apply unit couple in clockwise sense at B



$$\theta = \int \frac{m_\theta M dx}{EI}$$

$$\Rightarrow \quad \theta = \int_0^5 \frac{(-1)(-3)(10-x)\,dx}{EI} + 0$$

$$\frac{3}{EI}\left(10x - \frac{x^2}{2}\right]_0^5\Bigg)$$

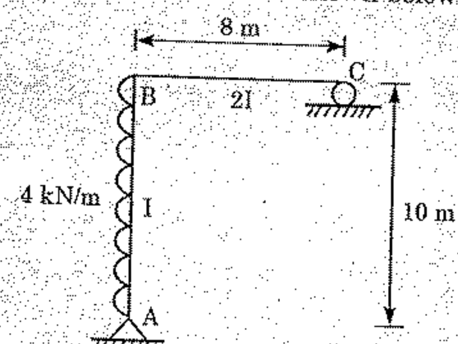
$$= \frac{3}{EI}\left[50 - \frac{25}{2}\right] = \frac{3 \times 75}{2EI} = \frac{112.5\ kNm^2}{EI}$$

$$\boxed{\theta = \frac{112.5}{EI}\ kNm^2}$$

(+) ve sign means slope is in the direction of applied unit couple i.e. clockwise.

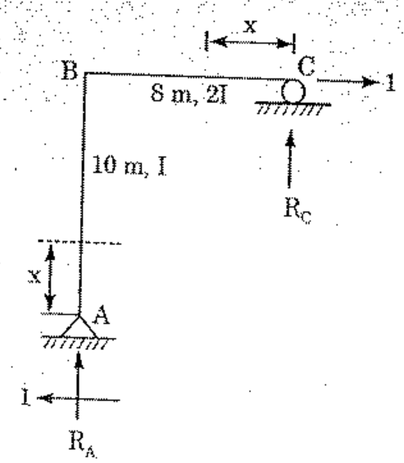
Example 32

Calculate horizontal displacement at C in the frame as shown below.



Sol:

Apply unit load at C in horizontal direction and write expression for 'm' in various zones.



$$\Sigma F = 0, \Rightarrow R_A + R_C = 0$$

$$\Sigma M_A = 0, \Rightarrow 1 \times 10 - R_C \times 8 = 0$$

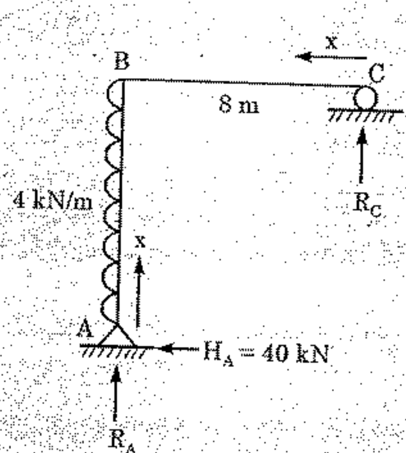
$$\boxed{R_C = \frac{10}{8}}$$

$$\boxed{R_A = -\frac{10}{8}}$$

$$m_{BC} = \frac{10x}{8}, \quad 0 \le x \le 8$$

$$m_{AB} = 1 \times x, \quad 0 \le x \le 10$$

Apply external load and find out expression for 'M' in various zones in which expression of 'm' has been written.



$$\Sigma F_H = 0 \Rightarrow H_A = 40 \text{ kN}$$

$$\Sigma F_V = 0 \Rightarrow R_A + R_C = 0 \quad \text{(i)}$$

$$\Sigma M_A = 0 \Rightarrow R_C \times 8 - \frac{4(10)^2}{2} = 0$$

$$R_C = \frac{400}{2 \times 8} = 25 \text{ kN}$$

$$R_A = -25 \text{ kN}$$

$$M_{AB} = 40x - \frac{4x^2}{2}, \quad 0 \le x \le 10$$

$$M_{BC} = 25x, \quad 0 \le x \le 8$$

$$\Rightarrow \quad \Delta = \int_0^{10} \frac{x\left(40x - \frac{4x^2}{2}\right).dx}{EI} + \int_0^8 \frac{\left(\frac{10x}{8}\right) \times (25x)\, dx}{2\,EI}$$

$$= \frac{40x^3}{3EI} - \frac{4x^4}{2 \times 4EI}\Bigg|_0^{10} + \frac{250x^3}{48\,EI}\Bigg|_0^8$$

$$\frac{40000}{3EI} - \frac{4\times10000}{2\times4\ EI} + \frac{250\times64\times8}{48EI} = \frac{11000}{EI}$$

$\boxed{\Delta = \frac{11000}{EI}}$ (+) ve means deflection at 'C' is in the direction of applied unit load at C i.e. right wards.

Note that it the moment of inertia of the structural element change in the length of beam, then the integration limit has to be chosen such that various zones of different moment of inertia are under different integration limits.

DEFLECTION OF BEAM DUE TO TEMPERATURE CHANGE

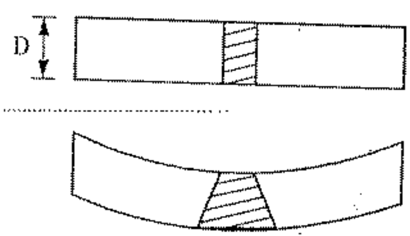
From virtual work principle, we know that

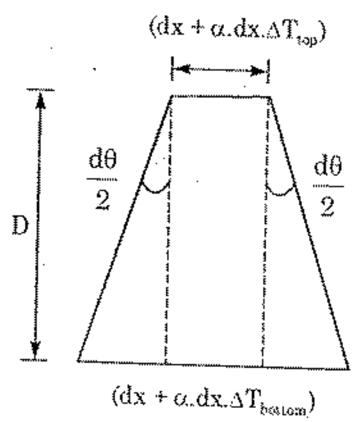
$$1\cdot\Delta = \int m\cdot d\theta$$

where, m = BM in beam due to unit load at the point where deflection is to be found.

$d\theta$ = angle of rotation which is $\left(\frac{M}{EI}dx\right)$ under loading condition.

But under temperature change condition, the deformed shape of a differential element of length dx is as shown in the figure below.





$$\Rightarrow d\theta = \frac{\alpha dx\,(\Delta T_{bottom} - \Delta T_{top})}{D}$$

$$d\theta = \alpha dx\left[\frac{(T_{bottom} - T)}{D} - \frac{(T_{top} - T)}{D}\right]$$

where T is the original temp at top and bottom of beam.

$$d\theta = \frac{\alpha dx\,(\Delta T)}{D}$$

where $\Delta T = (T_{bottom} - T_{top})$.

$$\Rightarrow \quad \boxed{1 \cdot \Delta = \int m \cdot \frac{(\alpha \cdot \Delta T \cdot dx)}{D}}$$

Note that if bottom temperature is more, $d\theta$ will have the same sign as that due to sagging moment but if bottom temperature is less than top temperature, $d\theta$ will have the sign as due to hogging moment.

These effect can be modelled by simply taking

ΔT as (+) ve if $T_{bottom} > T_{top}$ and ΔT as (–) ve if $T_{bottom} < T_{top}$ or simphy defining ΔT as

$$\boxed{\Delta T = T_{bottom} - T_{top}}$$

If Δ comes out to be (+) ve $\Rightarrow$ it is in the direction of applied unit load otherwise opposite to it.

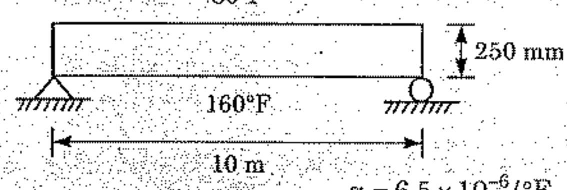
Similarly if slope is to be found, we apply unit couple instead of unit load and find out

$$\theta = \int m.\left(\frac{\alpha.\Delta T\, dx}{D}\right)$$

if θ is (+) ve, it is in the direction of applied couple otherwise opposite to it.

Example 33

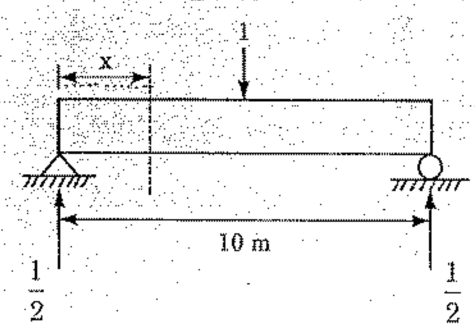
Find mid span deflection due to temperature condition shown in the figure below.



A rectangular beam

Sol: Apply unit load at mid span

$$\Rightarrow \quad m = \frac{1}{2}x \quad \text{for } 0 \le x < 5$$



As bottom temperature is more, hence

$$\Rightarrow \quad d\theta = \frac{(dx)(6.5 \times 10^{-6})(160 - 80)}{0.250}$$

$$\Rightarrow \quad d\theta = 2.08 \times 10^{-3}\, dx$$

$$1 \cdot \Delta = 2\int_0^5 \frac{x}{2} \times 2.08 \times 10^{-3} dx$$

$$= 2.08(10^{-3})\left.\frac{x^2}{2}\right|_0^5 = \frac{2.08\times10^{-3}(5)^2}{2}\text{ m}$$

$$\boxed{\Delta = 0.026\text{ m}} = 26\text{ mm}$$

$\Delta = (+)$ ve $\Rightarrow$ deflection is in the direction of applied unit load i.e. downwards

CASTIGLIANO'S THEOREM (METHOD OF LEAST WORK)

As per Castigliano's theory

Displacement at a point in a structure is equal to the 1st partial derivative of strain energy in the structure with respect to a force acting at the point and in the direction of desired displacement.

This theorem applies to the structures that have:

1. Constant temperature
2. Unyielding support
3. Linear elastic material response

Hence, $$\boxed{\Delta_i = \frac{\partial U}{\partial P_i}}$$

and similarly $$\boxed{\theta_i = \frac{\partial U}{\partial M_i}}$$

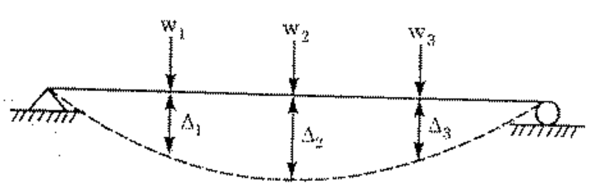
P_i = Force acting at the point where deflection is to be found out

M_i = couple acting at the point where rotation is to be found out

U = Strain energy in the structure due to various forces acting on the structure

Note:

- This method is applicable only when forces on the structure are conservative i.e. they do not cause energy losses.
- As against this theorem, method of virtual work also applies to the inelastic behaviour cases. But, theorem applies only to elastic behaviour case.

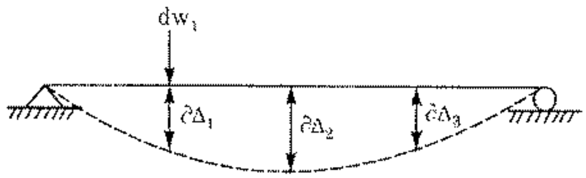


The external loads, w_1, w_2 and w_3 acting on a beam produces deflection Δ_1, Δ_2 and Δ_3 (when applied gradually).

Hence strain energy stored in the beam (U) = external work done

$$\Rightarrow \quad U = \frac{1}{2}w_1\Delta_1 + \frac{1}{2}w_2\Delta_2 + \frac{1}{2}w_3\Delta_3 \quad \text{------------ (i)}$$

while w_1, w_2 and w_3 already acting on the beam, a new increment of load dw_i is applied which causes additional displacements at point 1, 2 and 3 equal to $\partial\Delta_1$, $\partial\Delta_2$ and $\partial\Delta_3$ respectively.



Hence additional strain energy stored in the beam

$$\partial U = \frac{1}{2} dw_1 . \partial\Delta_1 + w_1\partial\Delta_1 + w_2\partial\Delta_2 + w_3\partial\Delta_3 \quad \text{---------- (ii)}$$

If all loads w_1, w_2, w_3 and dw_1 are applied simultaneously, then
Total strain energy stored (∂U + U) is given by

$$U + \partial U = \frac{1}{2}(w_1 + dw_1)(\Delta_1 + \partial\Delta_1) + \frac{1}{2} w_2(\Delta_2 + \partial\Delta_2) + \frac{1}{2} w_3(\Delta_3 + \partial\Delta_3) \quad \text{---------- (iii)}$$

But we know that order of application of load will not have any effect on total strain energy stored in the system.
Hence (i) + (ii) = (iii)

i.e. $$\frac{1}{2} w_1\Delta_1 + \frac{1}{2} w_2\Delta_2 + \frac{1}{2} w_3\Delta_3 + \frac{1}{2} dw_1\partial\Delta_1 + w_1\partial\Delta_1 + w_2\partial\Delta_2 + w_3\partial\Delta_3$$

$$= \frac{1}{2}(w_1 + dw_1)(\Delta_1 + \partial\Delta_1) + \frac{1}{2} w_2(\Delta_2 + \partial\Delta_2) + \frac{1}{2} w_3(\Delta_3 + \partial\Delta_3)$$

$$\Rightarrow \quad \frac{1}{2} dw_1\partial\Delta_1 + w_1\partial\Delta_1 + w_2\partial\Delta_2 + w_3\partial\Delta_3$$

$$= \frac{1}{2} dw_1 d\Delta_1 + \frac{1}{2} w_1\partial\Delta_1 + \frac{dw_1 . \Delta_1}{2} + \frac{1}{2} w_2\partial\Delta_2 + \frac{1}{2} w_3\partial\Delta_3$$

$$\Rightarrow \quad \boxed{\frac{1}{2} w_1\partial\Delta_1 + \frac{1}{2} w_2\partial\Delta_2 + \frac{1}{2} w_3\partial\Delta_3 = \frac{dw_1\Delta_1}{2}} \quad \text{---------- (iv)}$$

Now from (ii) and (iv)

$$\partial U = \frac{1}{2} dw_1 \partial\Delta_1 + dw_1 . \Delta_1$$

Neglecting smaller term $\frac{1}{2} dw_1\partial\Delta_1$, we get

$$\partial U = dw_1 . \Delta_1$$

$$\Rightarrow \quad \boxed{\frac{\partial U}{\partial w_1} = \Delta_1} \quad : \textbf{Castigliano's theorem} \qquad (A)$$

This is also called Castigliano's 2nd-theorem.

The 1st theorem is $\frac{\partial U}{\partial \Delta_n} = P_n$.

Statement 'A' proves the statement that displacement at a point in a structure is equal to the 1st partial derivatives of strain energy in the structure w.r. to force acting at the point and in the direction of desired displacement.

***Note:** In this case the deflection Δ_1 was desired only at the point where external load w_1 was acting. If however, deflection is desired at a point where no external load is acting, then we apply an imaginary load (P) at the point of desired deflection and find out strain energy in the system due to combined action of imaginary load (P) and external load and then*

$$\Delta = \left.\frac{\partial U}{\partial P}\right|_{P=0}, \text{ where P is set to zero.}$$

CASTIGLIANO'S THEOREM FOR BEAM AND FRAMES

$$\Delta = \frac{\partial}{\partial P}\left[\int_0^L \frac{M^2 dx}{2EI}\right]$$

***Note:** Strain energy due to bending has only been considered because in beams and frames deflections are caused mainly by bending.*

$$\boxed{\Delta = \int_0^L \frac{M \cdot \frac{\partial M}{\partial P} \cdot dx}{EI}}$$

Similarly,

$$\boxed{\theta = \int_0^L \frac{M \cdot \frac{\partial M}{\partial M'} dx}{EI}}$$

M' = moment applied at point where θ is to be found

M = Bending moment in the beam both due to external load and imaginary/real load 'P' applied at the point where deflection is to be found out.

[Imaginary load needs to be applied only when there is no external load/moment acting at the point and in the direction of desired deflection/rotation.]

Sign Convention

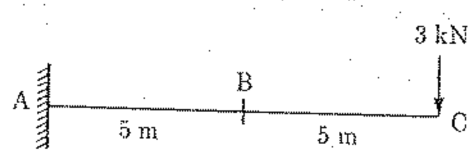
Sagging moment = (+) ve.

Hogging moment = (–) ve.

Positive value of Δ/θ means deflection/rotation is in the direction of applied load/moment at the point of desired displacement/rotation.

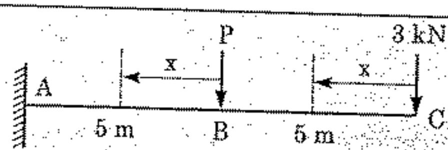
Example 34

Calculate slope and deflection at point B.



Sol:

Deflection: As there is no external load acting at point B, we apply an imaginary load at B equal to P as shown in figure below.



Segment	M	$\frac{\partial M}{\partial P}$	x range
BC	$-3x$	0	0 – 5
AB	$-3(5+x)-Px$	$-x$	0 – 5

$$\Rightarrow \quad \Delta = \int_0^5 \frac{-3x(0)\cdot dx}{EI} + \int_0^5 \frac{(-(15+3x)-Px)(-x)\,dx}{EI}$$

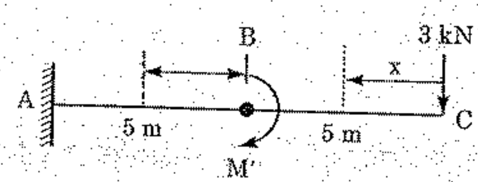
$$\Delta = \int_0^5 \frac{(15+3x+Px)\,x\,dx}{EI} = \left.\frac{\frac{15x^2}{2}+\frac{(3+P)x^3}{3}}{EI}\right|_0^5$$

Set P = 0

$$\Rightarrow \quad \Delta = \frac{\left.\frac{15x^2}{2}+\frac{3x^3}{3}\right|_0^5}{EI} = \frac{\frac{15\times 25}{2}+125}{EI}$$

$\boxed{\Delta = \frac{625}{2\,EI}}$ kNm^3 (+) ve value of Δ means that it has same direction as that of applied load P i.e. downwards.

Slope: To calculate slope at B, apply imaginary couple M' at B as shown in figure below.



Segment	M	$\frac{\partial M}{\partial M'}$	limit of x
BC	$-3x$	0	0–5
AB	$-15-3x-M'$	-1	0–5

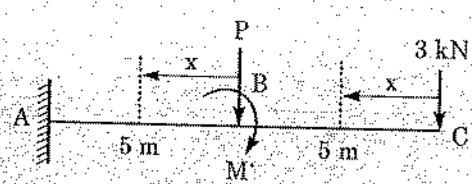
$$\theta = \left.\int_0^5 \frac{(15+3x+M')\,dx}{EI}\right|_{M'=0}$$

$$\theta = \frac{1}{EI}\left[15x+\frac{3x^2}{2}\right]_0^5$$

$$\theta = \frac{1}{EI}\left[75+\frac{75}{2}\right] = \frac{112.5}{EI}$$

$$\boxed{\theta = \frac{112.5}{EI}\,kNm^2}$$ (+) ve ⇒ same direction as applied M′, i.e. clockwise

The above problem can also be solved in single step as follows.

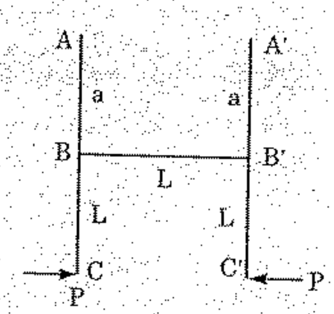


Segment	M	$\frac{\partial M}{\partial P}$	$\frac{\partial M}{\partial M'}$	limit of x
BC	$-3x$	0	0	0 – 5
AB	$-3(5+x) - Px - M'$	$-x$	-1	0 – 5

$$\Rightarrow \quad \Delta = \int_0^5 \frac{-(15+3x)(-x)\,dx}{EI} \qquad \text{[Set P and } M' = 0]$$

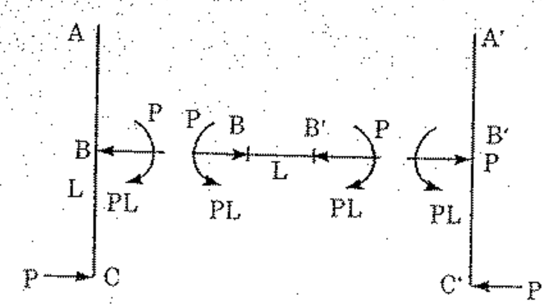
$$\theta = \int_0^5 \frac{-(15+3x)(-1)\,dx}{EI} \qquad \text{[Set P and } M' = 0]$$

Example 35

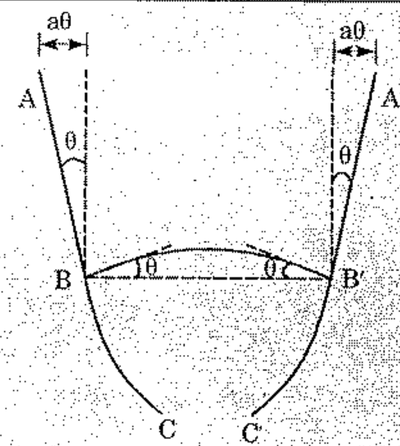


All members are flexible and prismatic. Consider BB′ as axially rigid. EI = constant for all members. If load 'P', as shown, is applied at C and C′, find by what distance AA′ will move apart from each other.

Sol: The free body diagram of the structure is shown as below.

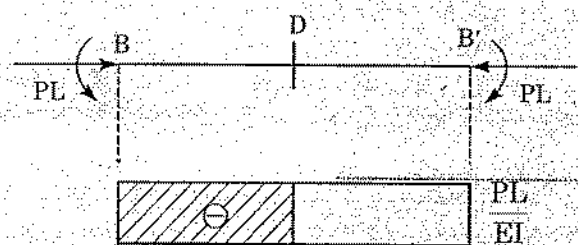


The deformed shape of the structure is



If slope of BB' at B is known, 'aθ' can be calculated. AA' will move apart by 2aθ.
θ at B_1 in BB_1' can be calculated using various methods. Let us use area-moment method.

By moment area theorem



D is the mid span of BB'. Due to symmetry of loading, slope at D = 0

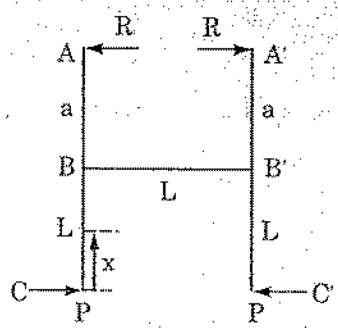
$$\Rightarrow \ |\theta_{DB}| = |\theta| = \left|\text{Area of } \frac{M}{EI} \text{ diagram between D and B}\right| = \frac{PL^2}{2EI}$$

AA' moves apart by 2aθ

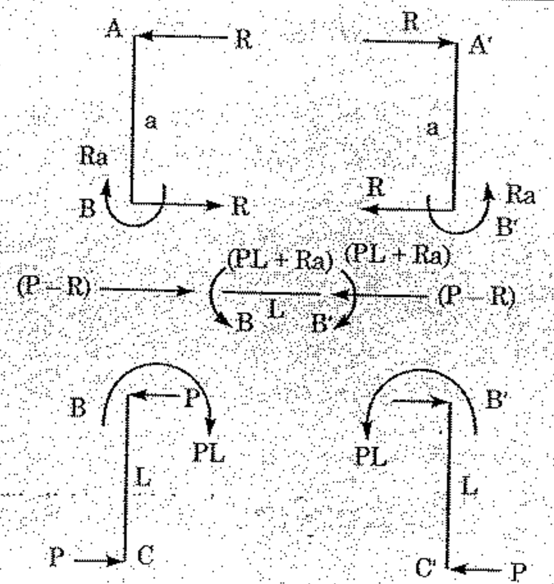
$$= 2\left(\frac{PL^2}{2EI}\right)a = \frac{PaL^2}{EI}$$

Solution By Castigliano's Theorem

Apply imaginary load (R) as shown.



The structure can be broken into five parts.



Total strain energy of the system $= U = \int \frac{M^2 dx}{2EI}$

$$U = 2\int_0^L \frac{P^2x^2dx}{2EI} + 2\int_0^a \frac{R^2x^2dx}{2EI} + \int_0^L \frac{(PL+Ra)^2dx}{2EI}$$

***Note:** Axial strain energy in BB' (neglected) [Because BB' has been assumed to be axially rigid]*

$$U = \frac{P^2L^3}{3EI} + \frac{R^2a^3}{3EI} + \frac{(PL+Ra)^2L}{2EI}$$

$$\Delta = \left.\frac{\partial U}{\partial R}\right|_{R=0} = 0 + \frac{2Ra^3}{6EI} + \left.\frac{2(PL+Ra)\,L}{2EI}\times a\right|_{at\ R=0}$$

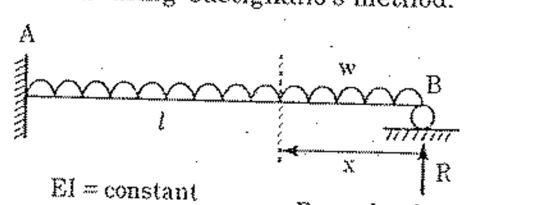
$$= 0 + 0 + \frac{2PL^2a}{2EI} = \boxed{\frac{PL^2a}{EI} = \Delta}$$

***Note:** If member BB' had not been axially rigid, we will have had to use the strain energy due to axial force (P – R) also in the total strain energy.*

Castigliano's Theorem can also be used to calculate redundants

Example 36

Calculate redundant reaction R at B using Castigliano's method.



Sol: We know that deflection at support B = 0

$$\Rightarrow \quad \frac{\partial U}{\partial R} = 0$$

$$U = \int_0^l \frac{\left(Rx - \frac{wx^2}{2}\right)^2 dx}{2EI}$$

$$= \int_0^l \frac{\left(R^2x^2 + \frac{w^2x^4}{4} - Rwx^3\right) dx}{2\,EI}$$

$$U = \frac{\frac{R^2l^3}{3} + \frac{w^2l^5}{20} - \frac{Rwl^4}{4}}{2EI}$$

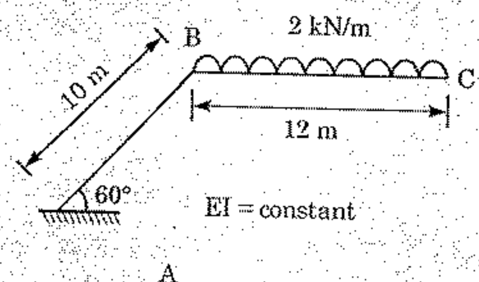
$$\frac{\partial U}{\partial R} = \frac{2Rl^3}{6EI} + 0 - \frac{wl^4}{8EI} = 0$$

$$\Rightarrow \quad R = \frac{wl^4 \times 6EI}{8EI \times 2l^3} = \frac{6wl^4}{16l^3}$$

$$\boxed{R = \frac{3wl}{8}}$$

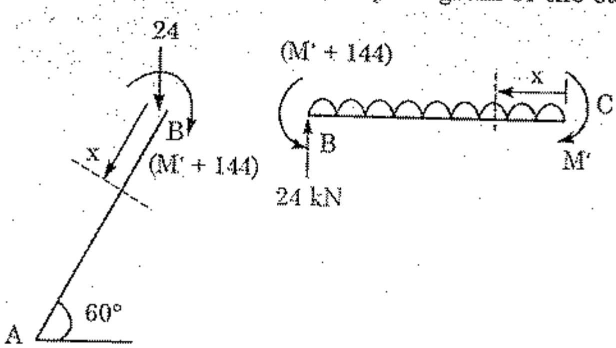
Example 37

Find slope at point C.

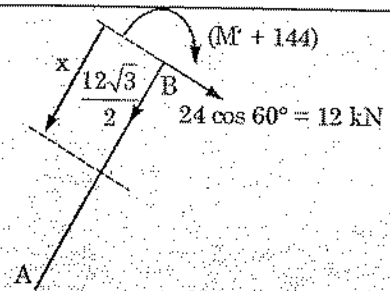


Sol:

Apply a couple M′ in clockwise sense at C. The free body diagram of the structure is as shown below.



To write down bending moment in AB, let us calculate the component of forces ⊥r to AB



Segment	M	$\frac{\partial M}{\partial M'}$	limit of x'
BC	$-\left(M'+\frac{2x^2}{2}\right)$	-1	$x = 0-12$
AB	$-(M'+144)-12x$	-1	$0-10$

$$\theta = \int_0^{12} \frac{(-x^2)(-1)\,dx}{EI} + \int_0^{10} \frac{(-12x-144)(-1)\,dx}{EI} \quad \text{(after setting } M' = 0\text{)}$$

$$\theta = \frac{x^3}{3EI}\Big|_0^{12} + \frac{12x^2}{2\,EI} + \frac{144x}{EI}\Big|_0^{10}$$

$$\theta = \frac{x^3}{3EI}\Big|_0^{12} + \frac{\left(\frac{12x^2}{2}+144x\right)}{EI}\Bigg|_0^{10}$$

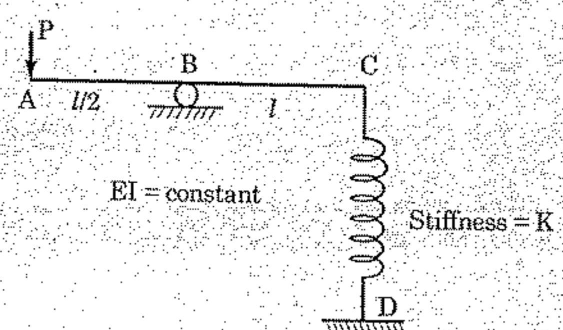
$$= \frac{(12)^3}{3EI} + \frac{6(10)^2+144\times 10}{EI}$$

$$\boxed{\theta = \frac{2616}{EI}}$$

(+) ve value of θ means it is in the same direction as that of applied moment M' i.e. clockwise

Example 38

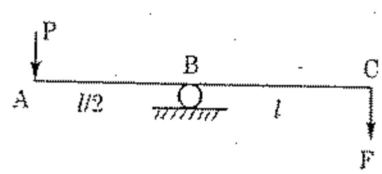
Find Δ_A.



Sol:

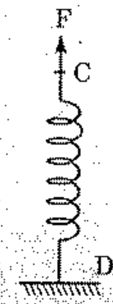
$$\boxed{\text{Strain energy in spring} = \frac{1}{2}Kx^2}$$

where, x = deformation in spring



$$\Rightarrow \quad \frac{Pl}{2} = Fl$$

$$\boxed{F = \frac{P}{2}}$$



$$Kx = F$$

$$x = \frac{F}{K} = \frac{P}{2K}$$

External work done $= \frac{1}{2} P\Delta$

$$\text{Internal strain energy} = \int_0^{\frac{l}{2}} \frac{(Px)^2 dx}{2EI} + \int_0^{l} \frac{\left(\frac{Px}{2}\right)^2 dx}{2EI} + \frac{1}{2} Kx^2$$

$$= \frac{P^2\left(\frac{l}{2}\right)^3}{6EI} + \frac{P^2}{4}\frac{(l)^3}{6EI} + \frac{1}{2}Kx^2$$

$$= \frac{P^2 l^3}{48EI} + \frac{P^2 l^3}{24EI} + \frac{1}{2}K \times \frac{P^2}{4K^2}$$

$$= \frac{P^2 l^3}{48EI} + \frac{P^2 l^3}{24EI} + \frac{P^2}{8K}$$

$$U = \frac{3P^2 l^3}{48EI} + \frac{P^2}{8K}$$

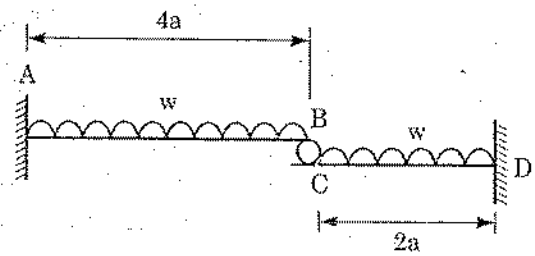
$$\frac{1}{2}P \cdot \Delta = \frac{3P^2 l^3}{48EI} + \frac{P^2}{8K}$$

$$\Rightarrow \quad \Delta = \frac{3Pl^3}{24EI} + \frac{P}{4K}$$

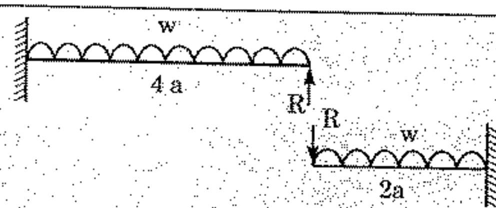
$$\boxed{\Delta = \frac{3Pl^3}{24EI} + \frac{P}{4K}}$$

Example 39

Find deflection at B.



Sol: At roller, we will have only one reaction. It is R as shown in the figure below.



Downward deflection of B

$$\Delta_B = \frac{w(4a)^4}{8EI} - \frac{R(4a)^3}{3EI}$$

Downward deflection of C

$$\Delta_C = \frac{w(2a)^4}{8EI} + \frac{R(2a)^3}{3EI}$$

Both point B and C will move by the same amount hence $\Delta_B = \Delta_C$

$$\frac{32wa^4}{EI} - \frac{64Ra^3}{3EI} = \frac{2wa^4}{EI} + \frac{8Ra^3}{3EI}$$

$$\Rightarrow \frac{30wa^4}{EI} = \frac{72Ra^3}{3EI}$$

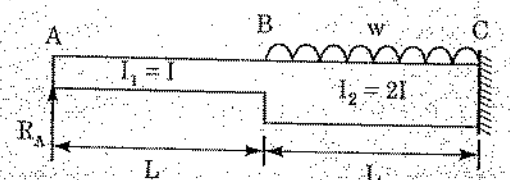
$$R = \frac{30\ wa}{72} \times 3 = \frac{5\ wa}{4}$$

$$\Delta_B = \Delta_C = \frac{2wa^4}{EI} + \frac{5wa \times 8a^3}{4 \times 3EI}$$

$$= \frac{2wa^4}{EI} + \frac{10wa^4}{3EI} = \frac{16wa^4}{3EI}$$

Example 40

Find R_A.



Sol: Deflection at A = 0

$$\frac{\partial U}{\partial R_A} = 0$$

$$U = \int_0^L \frac{(R_A x)^2 dx}{2EI_1} + \int_0^L \frac{\left[R_A(x+l) - \frac{wx^2}{2}\right]^2 dx}{2EI_2}$$

$$= \frac{R_A^2 l^3}{6EI_1} + \int_0^L \frac{\left[R_A^2(x+L)^2 + \frac{w^2x^4}{4} - R_A wx^2(x+l)\right] dx}{2EI_2}$$

$$= \frac{R_A^2 L^3}{6EI_1} + \left[\frac{R_A^2 (x+L)^3}{6EI_2} + \frac{w^2 x^5}{40EI_2} - \frac{R_A w x^4}{8EI_2} - \frac{R_A w L x^3}{6EI_2}\right]_0^L$$

$$U = \frac{R_A^2 L^3}{6EI_1} + \frac{8R_A^2 L^3}{6EI_2} + \frac{w^2 l^5}{40EI_2} - \frac{R_A w l^4}{8EI_2} - \frac{R_A w l^4}{6EI_2} - \frac{R_A^2 L^3}{6EI_2}$$

$$\frac{\partial U}{\partial R_A} = \frac{2R_A L^3}{6EI_1} + \frac{16R_A L^3}{6EI_2} + 0 - \frac{wl^4}{8EI_2} - \frac{wl^4}{6EI_2} - \frac{2R_A L^3}{6EI_2}$$

$$\frac{\partial U}{\partial R_A} = \frac{R_A L^3}{3EI_1} + \frac{16R_A L^3}{6EI_2} - \frac{7wl^4}{24EI_2} - \frac{2R_A L^3}{6EI_2} = 0$$

$$\Rightarrow \quad \frac{R_A L^3}{3EI} + \frac{8R_A L^3}{6EI} - \frac{7wl^4}{48EI} - \frac{2R_A L^3}{12EI} = 0$$

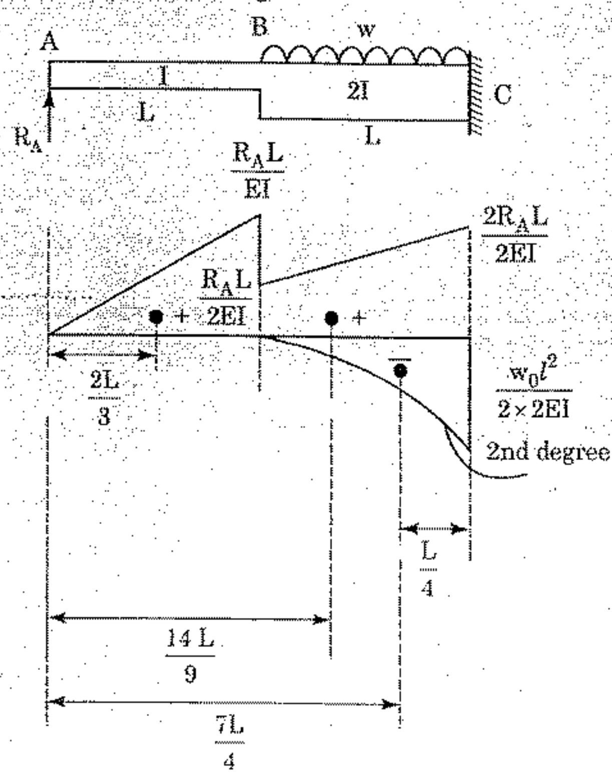
$$\frac{R_A L^3}{3EI}\left(1 + 4 - \frac{1}{2}\right) - \frac{7wl^4}{48EI} = 0$$

$$\frac{9R_A L^3}{6EI} - \frac{7wl^4}{48EI} = 0$$

$$R_A = \frac{7wl^4}{48EI} \times \frac{6EI}{9L^3}$$

$$\boxed{R_A = \frac{42wl}{48 \times 9} = \frac{7wl}{72}}$$

The same problem as above can be solved using area-moment theorem as shown below.



$$t_{A/C} = 0 \Rightarrow \left(\frac{1}{2} \times \frac{R_A L}{EI} \times L\right) \times \frac{2L}{3} + \frac{\left(\frac{R_A L}{EI} + \frac{R_A L}{2EI}\right)}{2} \times L \times \frac{14\,l}{9} - \frac{w_0 l^2}{4\,EI} \times l \times \frac{1}{3} \times \frac{7l}{4} = 0$$

$$\Rightarrow \frac{R_A L^3}{3EI} + \frac{42 R_A L^3}{36EI} - \frac{7 w_0 l^4}{48EI} = 0$$

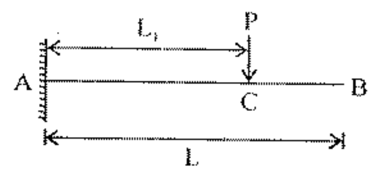
$$\Rightarrow \frac{54 R_A L^3}{36\,EI} = \frac{7 w_0 l^4}{48EI}$$

$$R_A = \frac{7 w_0 l \times 36}{48 \times 54}$$

$$\boxed{R_A = \frac{7 w_0 l}{72}}$$

OBJECTIVE QUESTIONS

1. A cantilever carries a load P at C as shown in the given figure.



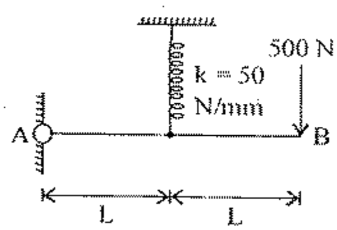
EI = Constant

The deflection at B is

(a) $\frac{PL_1^2}{2EI}(L - L_1)$ (b) $\frac{PL_1^2}{3EI}(L - L_1)$

(c) $\frac{PL_1^2}{2EI}\left(L - \frac{L_1}{3}\right)$ (d) $\frac{PL_1^2}{3EI}\left(L - \frac{L_1}{3}\right)$

2. A rigid bar is supported by a spring as shown in the given figure.



The deflection of the point B will be

(a) 10 mm upward

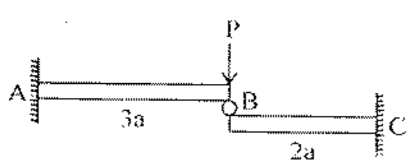
(b) 20 mm downward

(c) 5 mm upward

(d) 40 mm downward

6

3. The free end of a cantilever beam is supported by the free end of another cantilever beam using a roller as shown in the figure given below. What is the deflection at the roller support B?

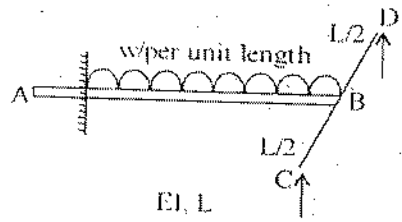


EI is constant

(a) $\frac{8Pa^3}{3EI}$ (b) $\frac{9Pa^3}{3EI}$

(c) $\frac{64Pa^3}{35EI}$ (d) $\frac{216Pa^3}{105EI}$

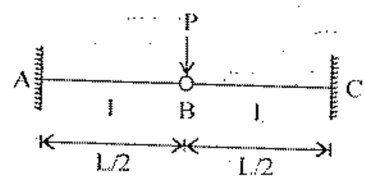
4. Which one of the following is the reaction of the cantilever at B as shown in the figure below? 7.



(a) $\frac{3}{8}$wL (b) $\frac{5}{8}$wL

(c) $\frac{6}{17}$wL (d) $\frac{3}{21}$wL

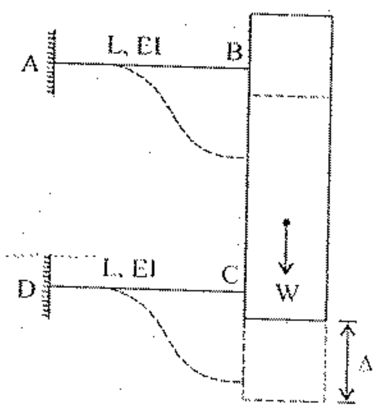
5. What is the deflection at the hinge for the beam shown below?



(a) 0 (b) $\frac{PL^3}{3\,EI}$

(c) $\frac{PL^3}{24\,EI}$ (d) $\frac{PL^3}{48\,EI}$

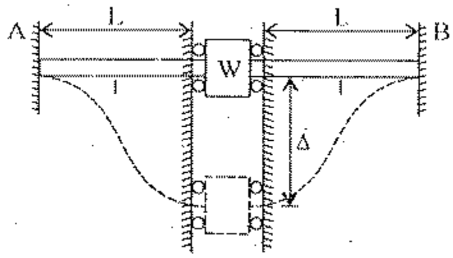
6. Assuming that the bar BC (see figure given below) is infinitely rigid and is of weight W and bars AB and CD have flexural rigidity EI and are of negligible weight, what would be the downward deflection (Δ) of the bar BC (Joints B and C are rigid)?



(a) $\frac{WL^3}{12EI}$ (b) $\frac{WL^3}{24EI}$

(c) $\frac{WL^3}{3EI}$ (d) $\frac{WL^3}{6EI}$

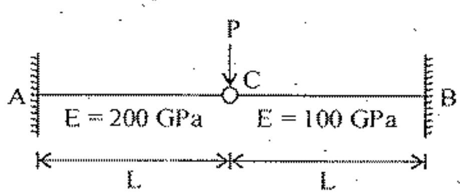
7. A heavy weight attached to a rod can slide in a grooved support as shown. What is the equilibrium sliding distance Δ?



(a) $\frac{WL^3}{3\,EI}$ (b) $\frac{WL^3}{12\,EI}$

(c) $\frac{WL^3}{24\,EI}$ (d) $\frac{WL^3}{48\,EI}$

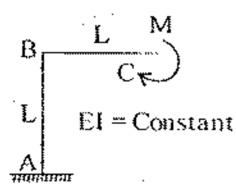
8. What is the bending moment at A for the beam shown below?



(a) $\frac{PL}{3}$ (b) $\frac{3\,PL}{2}$

(c) $\frac{PL}{2}$ (d) $\frac{2\,PL}{3}$

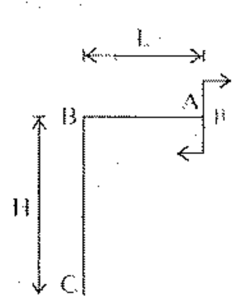
9. What is the horizontal deflection at free end C of the frame shown in the given figure?



(a) $\frac{ML^2}{2EI}$ (b) $\frac{ML^2}{EI}$

(c) $\frac{3ML^2}{2EI}$ (d) $\frac{2ML^2}{EI}$

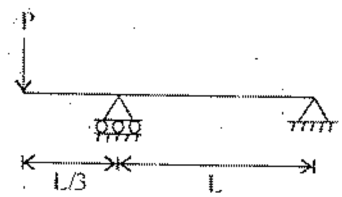
10. A rigid cantilever frame ABC is fixed at C and carries a couple μ at the free end A as shown in the given figure below. Neglecting axial deformation and assuming the flexural rigidity EI to be constant throughout the frame, the vertical deflection of A is



(a) $\frac{\mu L}{EI}\left(H+\frac{L}{2}\right)$ (b) $\frac{\mu L^2}{EI}\left(H+\frac{L}{2}\right)$

(c) $\frac{\mu H^2}{2\,EI}\left(\frac{H}{2}+L\right)$ (d) $\frac{\mu H}{EI}\left(\frac{H}{2}+L\right)$

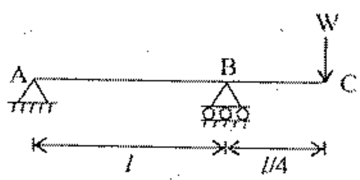
11. An overhang beam of uniform EI is loaded as shown in the figure below. The deflection at the free end will be



(a) $\frac{PL^3}{81EI}$ (b) $\frac{PL^3}{27EI}$

(c) $\frac{4PL^3}{81EI}$ (d) $\frac{2PL^3}{27EI}$

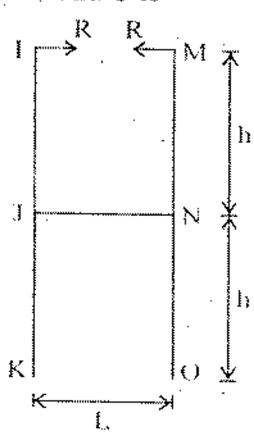
12. The slope at the support A of the overhanging beam shown in figure is



(a) $\frac{Wl^2}{12EI}$ (b) $\frac{Wl^2}{24EI}$

(c) $\frac{Wl}{3EI}$ (d) $\frac{Wl}{6EI}$

13. A "H" shaped frame of uniform flexural rigidity EI is loaded as shown in the figure. The relative outward displace-ment between points K and O is



(a) $\frac{RLh^2}{EI}$ (b) $\frac{RL^2h}{EI}$

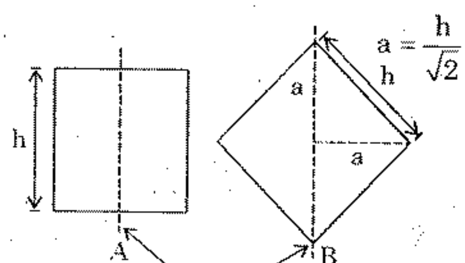
(c) $\frac{RLh^2}{3EI}$ (d) $\frac{RL^2h}{3EI}$

14. The maximum deflection of a fixed beam carrying a central load W is equal to

(a) $\frac{WL^3}{484EI}$ (b) $\frac{WL^3}{96EI}$

(c) $\frac{WL^3}{192EI}$ (d) $\frac{5}{384}\frac{WL^3}{EI}$

15. The span and the material of the two beams A and B are the same. The area of cross section of the two beams are equal. The cross-section is square. In the case of beam A, the plane of moment is parallel to the sides of the square and in the case of beam B, the plane of moment coincides with the diagonal as shown in the given figure.



Consider the following inferences from the above data:

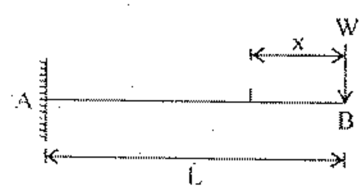
1. For the same loading, the deflection of the beam B is smaller than that of beam A.
2. If the load on the two beams is the same, then the maximum stress in beam B is greater than that in beam A.
3. Beam A can resist smaller load than beam B.
4. Flexural rigidities of both the beams are equal.

Which of these inferences are incorrect?

(a) 2 and 4 (b) 1 and 3

(c) 1 and 4 (d) 2, 3 and 4

16. For the cantilever beam shown in the given figure, which one of the following pairs is not correctly matched?



Theorem		(EI) times deflection at B
(a) Mohr's Theorem	:	Area of BMD × Distance of centroid of BMD from B
(b) Castigliano's Theorem	:	$\int_0^L Wx^2\,dx$

(c) Conjugate beam : Shear force at the fixed end of conjugate beam

(d) Successive integration : $\iint -Wx\,dx$

17. A fixed beam of uniform section is carrying a point load at its midspan. If the moment of inertia of the middle half length is now reduced to half its previous value, then the fixed end moments will
(a) increase
(b) decrease
(c) remain constant
(d) change their directions

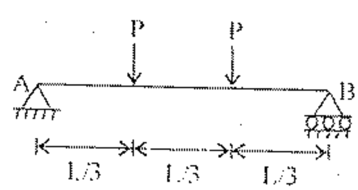
18. If the area under the shear curve for a beam between the two points X_1 and X_2 is k, then the difference between the moments at the two points X_1 and X_2 will be equal to
(a) k (b) 2k
(c) k/2 (d) k^2

19. Consider the following statements regarding a beam of uniform cross-section simply supported at its ends and carrying a concentrated load at one of its middle third points:
1. Its deflection under the load will be maximum.
2. The bending moment under the load will be maximum.
3. The deflection at the midpoint of the span will be maximum.
4. The slope at the nearest support will be maximum.

Which of these statements are correct?
(a) 1 and 3 (b) 2 and 4
(c) 1 and 2 (d) 3 and 4

20. **Assertion (A)**: Macaulay's method to determine the slope and deflection at a point in a beam is suitable for beams subjected to concentrated loads and can be extended to uniformly distributed loads.
Reason (R): Macaulay's method is based upon the modification of moment area method. This is applicable to a simple beam carrying a single concentrated load but by superposition, this method can be extended to cover any kind of loading.
Of these statements
(a) both A and R are true and R is the correct explanation of A
(b) both A and R are true but R is not a correct explanation of A
(c) A is true but R is false
(d) A is false but R is true

21. A simply supported beam of uniform flexural rigidity is loaded as shown in the given figure. The rotation of the end A is

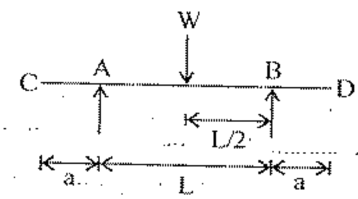


(a) $\frac{PL^2}{9EI}$ (b) $\frac{PL^2}{6EI}$

(c) $\frac{PL^2}{18\,EI}$ (d) $\frac{PL^2}{12\,EI}$

22. The maximum deflection of simply supported beam occurs at zero
(a) bending moment location
(b) shear force location
(c) slope location
(d) shear force location and also zero bending moment location

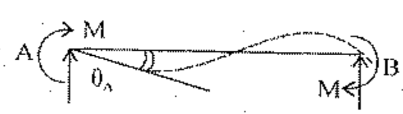
23. Consider the loaded beam shown in the given figure:



EI Constant

Assertion (A): The deflection at the free end C is 'a' times the slope at A.
Reason (R): The elastic curve for the overhang portion AC or BD is a straight line tangential to the elastic curve at A and B.
Of these statements
(a) both A and R are true and R is the correct explanation of A
(b) both A and R are true but R is not a correct explanation of A
(c) A is true but R is false
(d) A is false but R is true

24. A beam (EI = constant) of span L is subjected to clockwise moments M at both the ends A and B. The rotation of end A works out to be



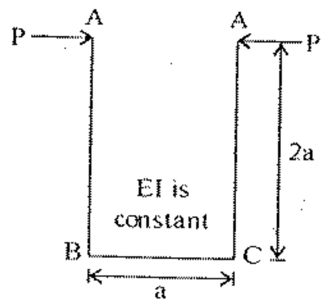
(a) $\frac{ML}{2\,EI}$ (b) $\frac{ML}{3\,EI}$
(c) $\frac{ML}{4\,EI}$ (d) $\frac{ML}{6\,EI}$

25. A propped cantilever of span 4 m is fixed at A and propped at B. The beam carries a UDL of 1 t/m over the entire span. The reaction at B is
(a) 2.5 t (b) 2 t
(c) 1 t (d) 1.5 t

26. In a system two weightless rigid bars AB and BC of length 'a' each having hinge supports at the ends A and C, respectively, are connected to each other at B by a frictionless hinge (internal hinge). The rotation at the hinge is restrained by a rotational spring of stiffness 'k' and system assumes a straight line configuration ABC. The rotation at the supports due to vertical load P acting at joint B is
(a) $\frac{Pa}{2k}$ (b) $\frac{Pa}{4k}$

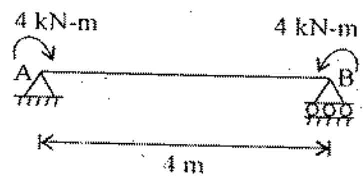
(c) $\frac{Pa}{k}$ (d) $\frac{2Pa}{k}$

27. In the structure shown in the figure below, what is the distance through which the points A move towards each other?



(a) $4Pa^3/EI$ (b) $16Pa^3/3EI$
(c) $28Pa^3/3EI$ (d) $6Pa^3/EI$

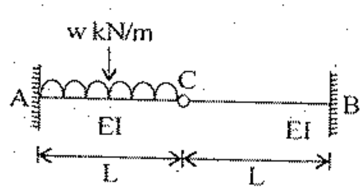
28. A simply supported beam AB of span 4 m is subjected to terminal couples as shown in the figure below.



If EI is in kN-m², what is the magnitude of the central deflection of the beam in metres ?

(a) $\frac{4}{EI}$ (b) $\frac{8}{EI}$
(c) $\frac{2}{EI}$ (d) $\frac{16}{EI}$

29. What is the reaction on the pin C for a beam as shown in the figure below?

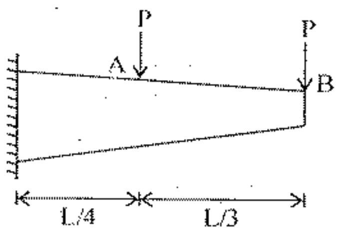


(a) $\frac{3}{8}$ wL kN (b) $\frac{1}{2}$ wL kN
(c) $\frac{1}{4}$ wL kN (d) $\frac{3}{16}$ wL kN

30. If the hinged end of a propped cantilever of span L settles by an amount δ, then the rotation of the hinged end will be

(a) $\frac{\delta}{L}$ (b) $\frac{2\delta}{L}$
(c) $\frac{3\delta}{2L}$ (d) $\frac{4\delta}{3L}$

31. The cantilever beam shown in the given figure has load P acting at points A and B. The deflection at B is Δ when the load at B is removed. Wh...



(a) Δ/4 (b) Δ/2
(c) Δ (d) 2Δ/3

32. Match **List-I** with **List-II** and select the correct answer using the codes given below the lists:

List-I

A Partial derivative of strain energy
B Derivative of deflection
C Derivative of slope
D Derivative of moment

List-II

1. Equation for shear force
2. Equation for slope
3. Equation for BM
4. Deflection under the load

Codes:

	A	B	C	D
(a)	4	3	2	1
(b)	2	1	3	4
(c)	4	2	3	1
(d)	2	3	1	4

33. Match **List-I** (Nature of beam) with **List-II** (Maximum deflection) and select the correct answer using the codes given below the lists:

List-I

A. Cantilever beam subjected to a concentrated load W at free end
B. Simply supported beam subjected to a point load W at the centre
C. Cantilever beam subjected to a hydrostatic load with zero intensity at the free end and W at the fixed end
D. Simply supported beam subjected to a triangularly distributed load with its apex of magnitude W at the mid-span

List-II

1. $\frac{WL^3}{30EI}$
2. $\frac{WL^3}{3EI}$
3. $\frac{WL^3}{120EI}$

4. $\frac{WL^3}{48EI}$

Codes:

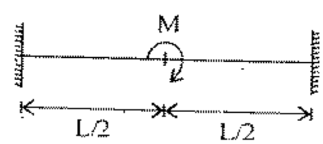
	A	B	C	D
(a)	2	1	4	3
(b)	3	1	2	4
(c)	2	4	1	3
(d)	1	4	2	3

34. A cantilever beam of span L and uniform flexural rigidity EI is loaded with an upward force W at the mid-point and downward force P at the free end. The deflection at the free end will be zero, if

(a) $W = \frac{3P}{2}$ (b) $W = 2P$

(c) $W = \frac{16P}{5}$ (d) $W = 5P$

35. A fixed beam as shown in the figure below has a span L and uniform flexural rigidity EI. It is subjected to a concentrated clockwise moment M at the centre.

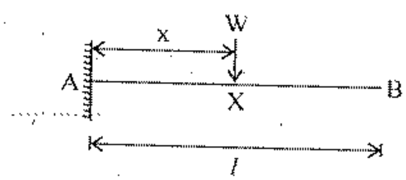


The deflection at the centre of the beam is

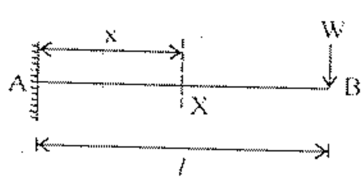
(a) $\frac{ML^2}{8EI}(\uparrow)$ (b) $\frac{ML^2}{8EI}(\downarrow)$

(c) zero (d) $\frac{ML^2}{384EI}(\downarrow)$

36. **Assertion (A):** For the cantilever shown below, the free end deflection for any position of load distance 'x' from the fixed end is as under $\delta_B = Wx^2(3l - x)/6EI$.



Reason (R): For the beam shown below, the deflection at any section distance 'x' from the fixed end is as under $\delta_x = Wx^2(3l - x)/6EI$.



Of these statements

(a) both A and R are true and R is the correct explanation of A

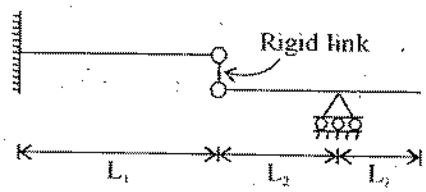
(b) both A and R are true but R is not a correct explanation of A

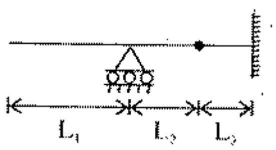
(c) A is true but R is false

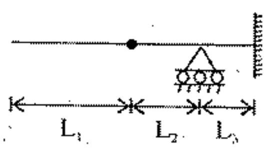
(d) A is false but R is true

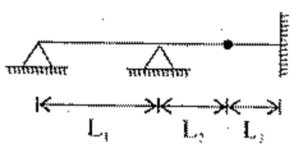
37. The principle of superposition is applicable to
(a) non-linear behaviour of material and small displacement theory
(b) non-linear behaviour of material and large displacement theory
(c) linear elastic behaviour of material and small displacement theory
(d) linear elastic behaviour of material and large displacement theory

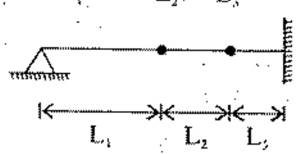
38. For the beam shown in the figure given below, which among the following is the conjugate beam?



(a)


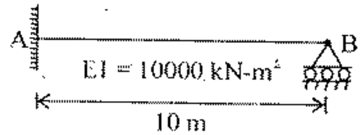
(b)


(c)


(d)


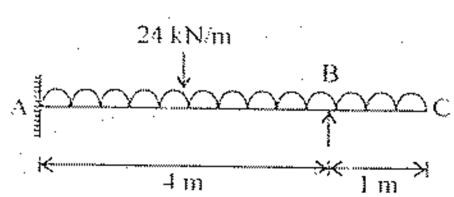
39. A simply supported beam is subjected to an eccentric concentrated load. Where does the maximum deflection of the beam due to the applied load occur?
(a) Directly under load
(b) At the centre of the beam
(c) Between the load point and nearest end support
(d) Between the load point and centre of the beam

40. The beam AB shown below, of span 10 m and having uniform EI = 10000 kN-m^2, is subjected to a rotation of 0.001 radian at end B. What is the fixed end moment at A?



(a) 1.5 kN-m (b) 2.0 kN-m
(c) 3.0 kN-m (d) 4.0 kN-m

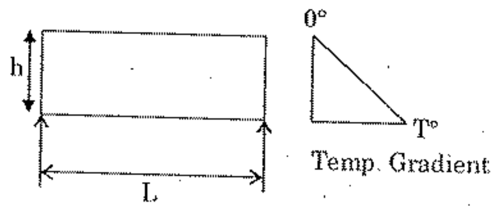
41. What is the moment at joint A for the beam shown below?



(a) – 32 kN-m (b) – 22 kN-m

(c) -42 kN-m (d) -52 kN-m

42. A simply supported beam of uniform rectangular cross-section of width b and depth h is subjected to linear temperature gradient, 0° at the top and T° at the bottom, as shown in the figure. The coefficient of linear expansion of the beam material is α. The resulting vertical deflection at the mid-span of the beam is



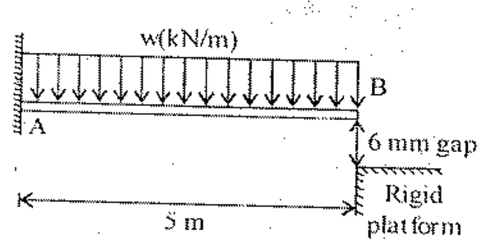
(a) $\frac{\alpha Th^2}{8L}$ upward

(b) $\frac{\alpha TL^2}{8h}$ upward

(c) $\frac{\alpha Th^2}{8L}$ downward

(d) $\frac{\alpha TL^2}{8h}$ downward

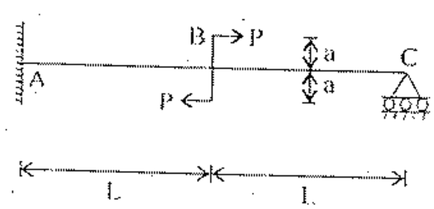
43. For the linear elastic beam shown in the figure, the flexural rigidity, EI is 781250 kN-m². When w = 10 kN/m, the vertical reaction R_A at A is 50 kN. The value of R_A for w = 100 kN/m is



(a) 500 kN (b) 425 kN
(c) 250 kN (d) 75 kN

Common Data for Questions 44 and 45:

Consider a propped cantilever beam ABC under two loads of magnitude P each as shown in the figure below. Flexural rigidity of the beam is EI.



44. The reaction at C is

(a) $\frac{9\,Pa}{16\,L}$ (upwards)

(b) $\frac{9\,Pa}{16\,L}$ (downwards)

(c) $\frac{9\,Pa}{8\,L}$ (upwards)

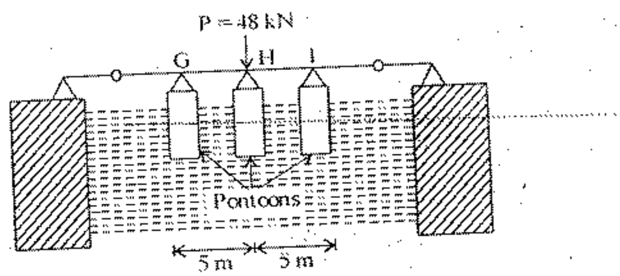
(d) $\frac{9\,Pa}{8\,L}$ (downwards)

45. The rotation at B is

(a) $\frac{5\,PLa}{16\,EI}$ (clockwise)
(b) $\frac{5\,PLa}{16\,EI}$ (anticlockwise)
(c) $\frac{59\,PLa}{16\,EI}$ (clockwise)
(d) $\frac{59\,PLa}{16\,EI}$ (anticlockwise)

Statement for Linked Answer Questions 46 and 47:

Beam GHI is supported by three pontoons as shown in the figure below. The horizontal cross sectional area of each pontoon is 8 m^2, the flexural rigidity of the beam is 10000 kN-m^2 and the unit weight of water is 10 kN/m^3.

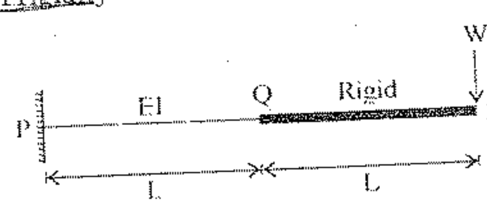


46. When the middle pontoon is removed, the deflection at H will be
(a) 0.2 m
(b) 0.4 m
(c) 0.6 m
(d) 0.8 m

47. When the middle pontoon is brought back to its position as shown in the figure above, the reaction at H will be
(a) 8.6 kN
(b) 15.7 kN
(c) 19.2 kN
(d) 24.2 kN

Statement for Linked Answer Questions 48and 49:

In the cantilever beam PQR shown in figure below, the segment PQ has flexural rigidity EI and the segment QR has infinite flexural rigidity



48. The deflection and slope of the beam at Q are respectively

(a) $\frac{5WL^3}{6EI}$ and $\frac{3WL^2}{2EI}$
(b) $\frac{WL^3}{3EI}$ and $\frac{WL^2}{2EI}$

(c) $\frac{WL^3}{2EI}$ and $\frac{WL^2}{EI}$ (d) $\frac{WL^3}{3EI}$ and $\frac{3WL^2}{2EI}$

49. The deflection of the beam at R is

(a) $\frac{8WL^3}{EI}$ (b) $\frac{5WL^3}{6EI}$

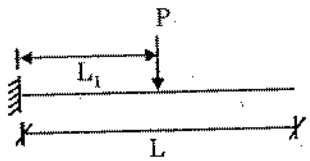
(c) $\frac{7WL^3}{3EI}$ (d) $\frac{8WL^3}{6EI}$

ANSWERS

1. (c)	13. (a)	25. (d)	37. (c)
2. (d)	14. (c)	26. (b)	38. (a)
3. (d)	15. (b)	27. (c)	39. (d)
4. (c)	16. (c)	28. (b)	40. (b)
5. (d)	17. (a)	29. (d)	41. (c)
6. (b)	18. (a)	30. (c)	42. (d)
7. (c)	19. (b)	31. (c)	43. (b)
8. (d)	20. (c)	32. (c)	44. (c)
9. (a)	21. (a)	33. (c)	45. (a)
10. (a)	22. (c)	34. (c)	46. (b)
11. (c)	23. (a)	35. (c)	47. (c)
12. (b)	24. (d)	36. (a)	48. (a)
			49. (c)

SOLUTION...

1. **(c)**

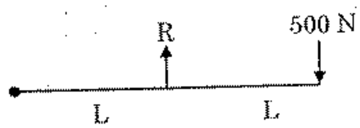


$$EI = \text{constant}$$

$$= \frac{PL_1^3}{3\,EI} + \frac{PL_1^2}{2\,EI} \times (L - L_1)$$

$$\frac{PL_1^2}{2\,EI}\left(L - L_1 + \frac{2}{3}L_1\right) = \frac{PL_1^2}{2\,EI}\left(L - \frac{L_1}{3}\right)$$

2. **(d)**



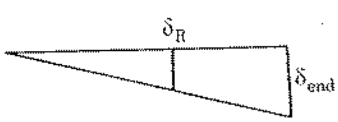
$$500 \times 2L = (R) \times L$$

$$R = 1000\text{ N}$$

$$\therefore \quad \delta_{atR} = \frac{1000\text{ N}}{50\text{ N/mm}} = 20\text{ mm}$$

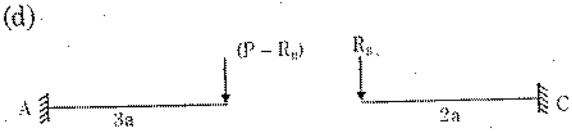
$$\left[K = \frac{R}{\delta}\right]$$

From Similar triangles



$$\delta_{end} = 40\text{ mm }(\downarrow)$$

3. **(d)**



$$\delta_{AB} = \frac{(P - R_B)(3a)^3}{3EI}$$

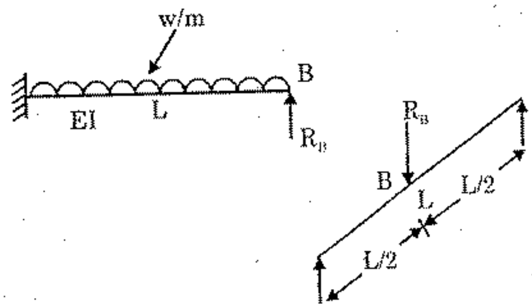
$$\delta_{CB} = \frac{R_B(2a)^3}{3EI}$$

$$(P - R_B)3^3 = R_B 2^3$$

$$27P = (27 + 8)R_B = 35\,R_B$$

$$\delta_{CB} = \frac{27}{35}P \times \frac{8a^3}{3EI} = \frac{72}{35}\frac{Pa^3}{EI} \times \frac{3}{3} = \frac{216\,Pa^3}{105\,EI}$$

4. **(c)**



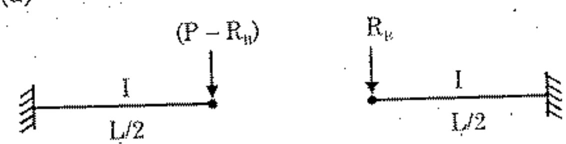
Net deflection at B is same

$$\frac{wL^4}{8EI}(\downarrow) - \frac{R_B L^3}{3EI}(\uparrow) = \frac{R_B L^3}{48EI}(\downarrow)$$

$$\frac{6}{17}\frac{wL}{EI} = R_B$$

+ve implies assumed direction is OK.

5. **(d)**



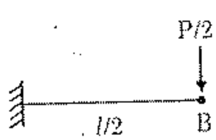
$$\frac{(P - R_B)\left(\frac{L}{2}\right)^3}{3EI} = \frac{R_B\left(\frac{L}{2}\right)^3}{3EI}$$

$$\Rightarrow \quad R_B = \frac{P}{2}$$

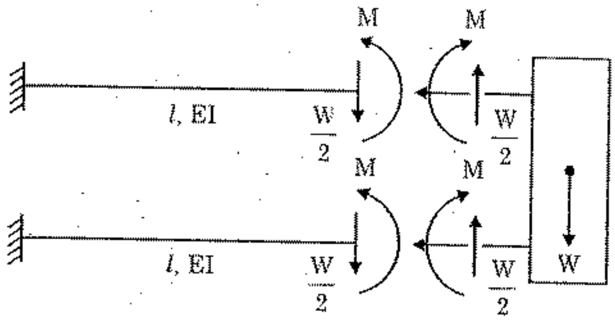
$$\therefore \quad \delta_B = \frac{\frac{P}{2}\left(\frac{L}{2}\right)^3}{3EI} = \frac{PL^3}{48EI}(\downarrow)$$

Alternatively as there is symmetry in loading the δ_B will be

$$= \frac{\left(\frac{P}{2}\right)\left(\frac{l}{2}\right)^3}{3EI} = \frac{Pl^3}{48\,EI}$$



6. (b) FBD



Both beams have same stiffness, hence moment equally distributed. For same deflection in both beam {equal stiffness},

transverse load should be same = $\frac{W}{2}$

Rigid joints hence slope = 0

$$\Rightarrow \frac{\left(\frac{W}{2}\right)l^2}{2EI} - \frac{Ml}{EI} = 0$$

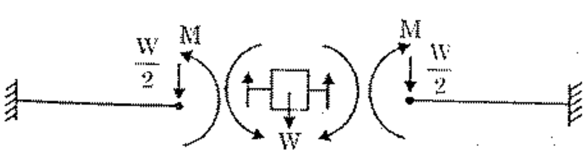
$$\Rightarrow \quad M = \frac{Wl}{4}$$

$$\Rightarrow \quad \delta = \frac{\left(\frac{W}{2}\right)l^3}{3EI} - \frac{Ml^2}{2EI}$$

$$= \frac{Wl^3}{6EI} - \frac{Wl^3}{8EI}$$

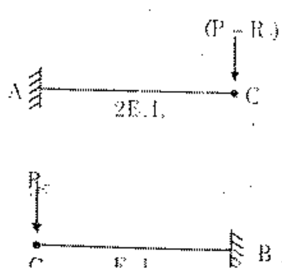
$$= \frac{Wl^3}{6EI} - \frac{Wl^3}{24EI} (\downarrow)$$

7. (c) FBD



Rest process same as previous question.

8. (d)



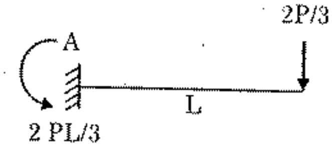
$$\delta_C = \frac{(P - R_C)L^3}{3(2E)I} (\downarrow)$$

$$\delta_C = \frac{R_C L^3}{3EI} (\downarrow)$$

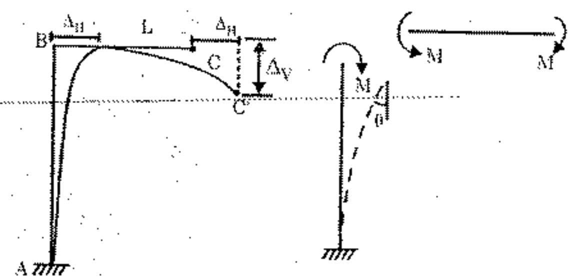
$$\frac{P - R_C}{2} = R_C$$

$$R_c = \frac{P}{3}$$

$$M_A = \frac{2PL}{3}$$



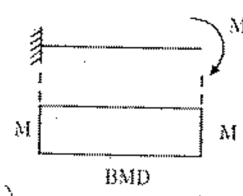
9. (a)



This can be simplified to a cantilever with moment.

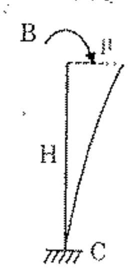
$$\Delta_{horizontal})_C = \frac{ML^2}{2EI}$$

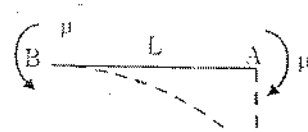
$$\theta = \frac{ML}{EI}$$



$$\Delta V)_C = \left(\frac{ML^2}{2EI}\right)_{BC} + \left(\frac{ML}{EI}\right)_{AB} \times L = \frac{3}{2}\frac{ML^2}{EI}$$

10. (a) Vertical deflection at A. (Procedure same as above)





$$= \frac{\mu L^2}{2EI} + \left(\frac{\mu H}{EI}\right) \times L = \frac{\mu L}{EI}\left(\frac{L}{2} + H\right)$$

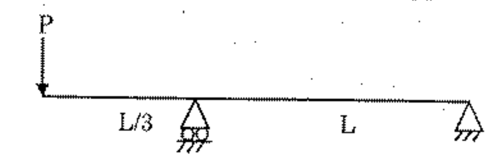
11. (c) **figure**

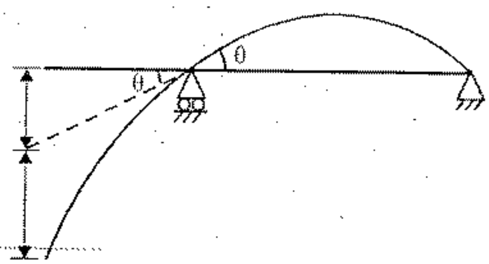
$$\theta = \frac{ML}{3EI} = \frac{PL/3.L}{3EI} = \frac{PL^2}{9EI}$$

$$\delta = \delta_I + \delta_{II}$$

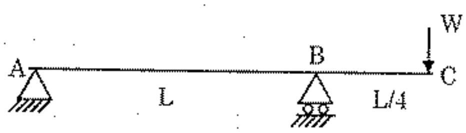
$$= \frac{PL}{9EI} \times \frac{L}{3} + \frac{P(L/3)^3}{3EI}$$

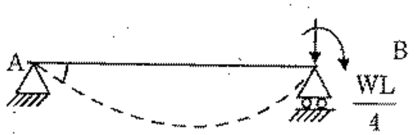
$$= \frac{PL^3}{27EI} + \frac{PL^3}{81EI} = \frac{4PL^3}{81EI}$$





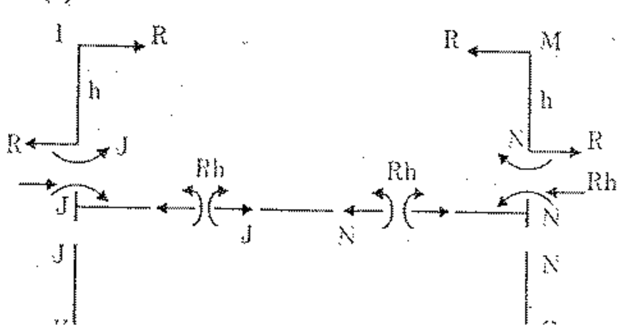
12 (b)

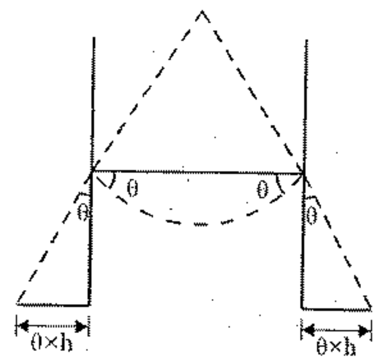


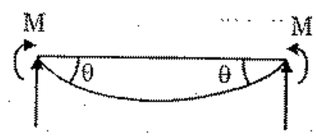


$$\theta = \frac{ML}{6EI} = \frac{(WL/u)L}{6EI} = \frac{WL^2}{24EI}$$

13. (a)







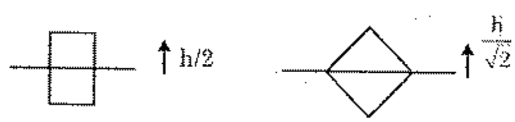
$$\theta = \frac{WL}{3EI} + \frac{Ml}{6EI} = \frac{RhL}{2EI}$$

$$\Rightarrow \qquad \delta = 2\theta h = \frac{2RhL}{2EI} \times h = \frac{RLh^2}{EI}$$

15. (b) $I_{\square} = \frac{a^4}{12}; \; I_{\diamond} = \frac{a^4}{12}$

$Z_{\square} < Z_{\diamond}$

Diagonally placed section is stronger by $\sqrt{2}$ times flexural rigidity = EI = same for both members

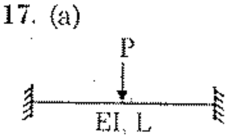


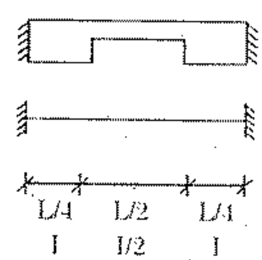
$\sigma \propto y$ extreme fibre distance

∴ Stress will be higher in ◇

⇒ it takes less load than ▫

17. (a)

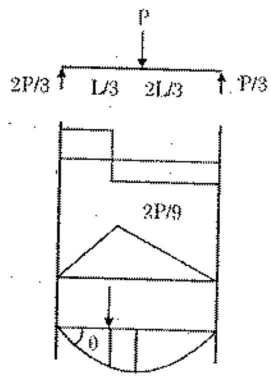




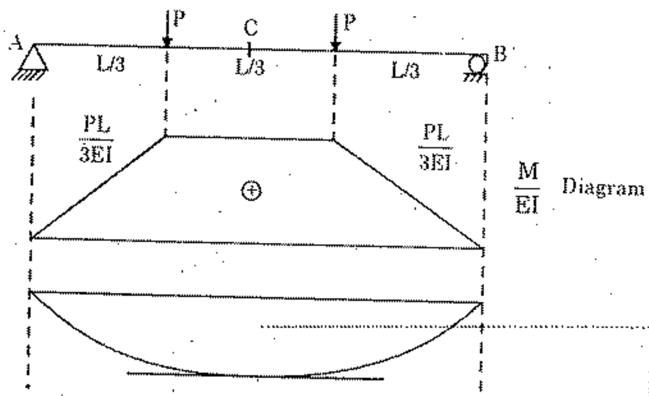
Reason : Staffer parts attracts greater moment (concept of MD method / stiffness method)

19. (b) Max θ will occur at near end
BM is max under load
Max deflection can occur any where not necessarily under load.

$y_{max} \rightarrow \frac{dy}{dx} = 0$



21. (a)

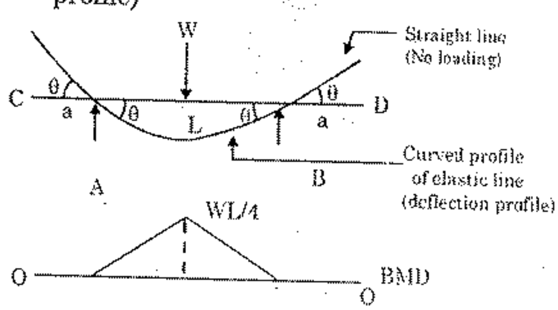


Slope at mid span = 0 ⇒ $\theta_c = 0$
⇒ Using moment area theorem b/w A and C;

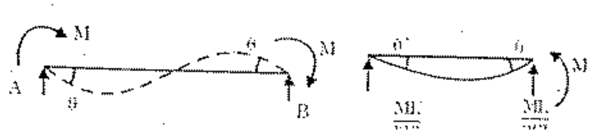
$$\theta_A - \theta_C = \frac{1}{2} \times \frac{PL}{3EI} \times \frac{L}{3} + \frac{PL}{3EI} \times \frac{L}{3} \times \frac{1}{2}$$

$$\theta_A = \frac{PL}{9EI}$$

23. (a) Curved profile of elastic line (deflection profile)

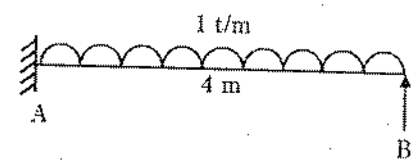


24. (d)



$$\theta_A = \left(\frac{ML}{3EI}\right)(↻) + \left(\frac{ML}{6EI}\right)(↺) = \frac{ML}{6EI}(↻)$$

25. (d)



Compatibility condition

$$\Delta_B = 0$$

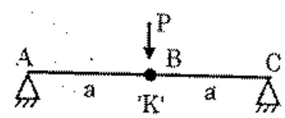
$$\frac{R_B L^3}{3EI} = \frac{wL^4}{8EI}$$

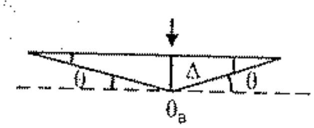
$$R_B = \frac{3wL}{8}$$

$$R_p = \frac{3wL}{8}$$

$$R_B = \frac{3 \times 1 \times 4}{8} = 1.5\ t$$

26. (b)





$$\theta_B = \frac{M_B}{K} = \frac{\frac{P}{2}.a}{K}$$

$$\theta_B = \frac{Pa}{2K} \quad \text{[total rotation at B]}$$

Rotational stiffness

$$K = \frac{P}{\Delta}$$

$$K = \frac{M}{\theta = 1}$$

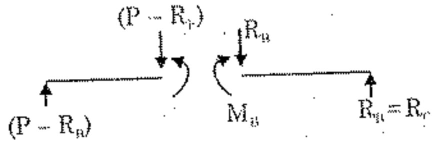
$$(P - R_B)a = M_B$$

$$M_B = R_B \times a$$

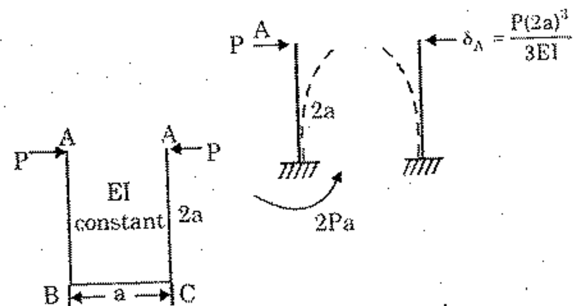
$$P - R_B = R_B$$

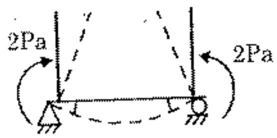
$$R_B = \frac{P}{2}$$

∴ Rotation at supports $\theta_A = \frac{\theta_B}{2} = \frac{Pa}{4K}$



27. (c) Since BC is not flexurally rigid, it bends and thus contributes to AA movements.





$$\theta = \frac{ML}{3EI} + \frac{ML}{6EI} = \frac{ML}{2EI} = \frac{Pa.a}{2EI} = \frac{Pa^2}{EI}$$

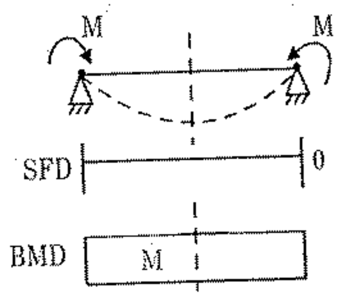
$$\therefore \quad \delta = \delta_A + (\theta \times 2a)$$

$$\Rightarrow \quad \delta = \frac{8Pa^3}{3EI} + \frac{Pa^2}{EI} \times 2a = \frac{Pa^3}{EI} \times \frac{14}{3}$$

Movement towards each other

$$= 2\left(\frac{14Pa^3}{3EI}\right) = \frac{28Pa^3}{3EI}$$

28. (b)

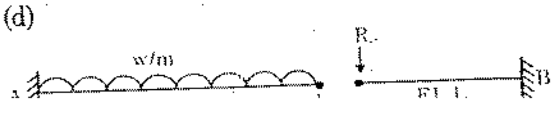


Max deflection it at mid span centre

By moment Area theorem

$$\delta = \frac{M\left(\frac{L}{2}\right) \times \frac{L}{4}}{EI} = \frac{ML^2}{8EI} = \frac{4 \times 4^2}{8EI} = \frac{8}{EI}$$

29. (d)



$$\delta_C = \frac{wL^4}{8EI}(\downarrow) - \frac{R_C L^3}{3EI}(\uparrow)$$

$$\delta_C = \frac{R_C L^3}{3EI}(\downarrow)$$

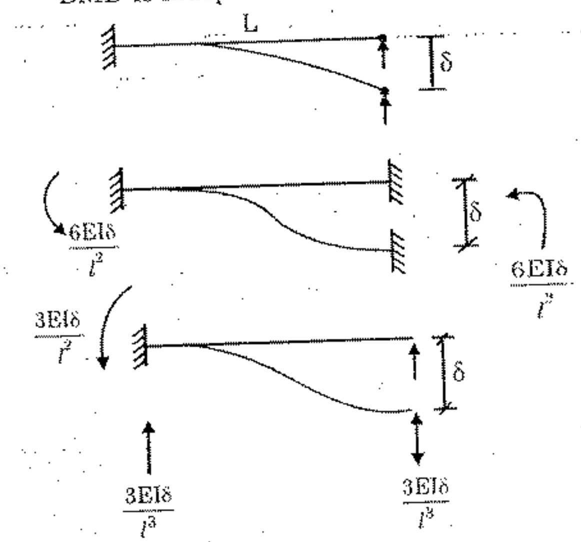
Net deflection is same

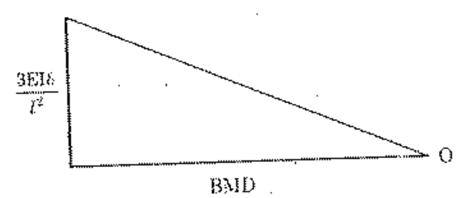
$$\frac{WL}{8} = 2\frac{R_C}{3}$$

$$R_C = \frac{3WL}{16}$$

30. (c)

BMD is independent of EI

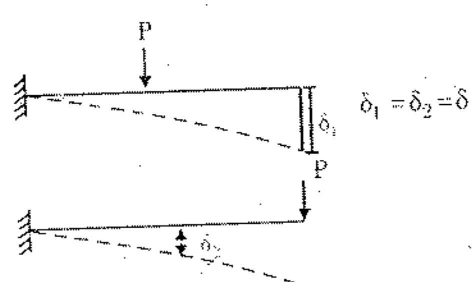




Moment Area theorem

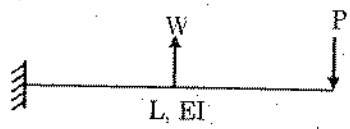
$$\theta = \frac{1}{2} \times \frac{3EI\delta}{l^2} \times \frac{l}{EI} = \frac{3EI}{2}\frac{\delta}{l} \times \frac{1}{EI} = \frac{3}{2}\frac{\delta}{l}$$

31. (c)



(Betti's & max wells theorems)

34. (c)

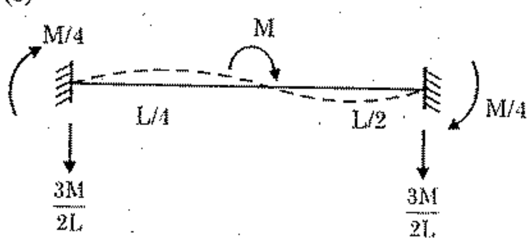


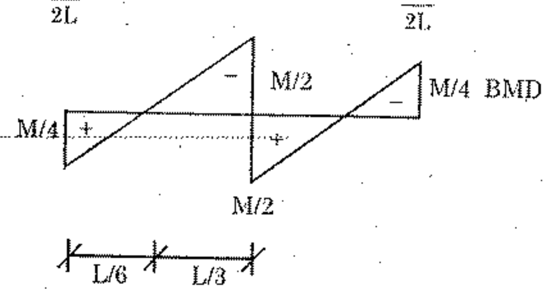
$$\frac{-PL^3}{3EI}(\downarrow) - \frac{W(L/2)^3}{3EI} - \frac{W\left(\frac{L}{2}\right)^2}{2EI} \times \frac{L}{2} = 0$$

$$\frac{P}{3} = \frac{W}{3} \times \frac{1}{8} + \frac{W}{8 \times 2}$$

$$\frac{6 \times 8P}{3 \times 5} = W = \frac{16P}{5}$$

35. (c)





$$= \left(\frac{M}{4} \times \frac{L}{6} \times \frac{1}{2}\right)\left(\frac{L}{3} + \frac{2}{3} \cdot \frac{L}{6}\right) - \left(\frac{M}{2} \times \frac{1}{2} \times \frac{L}{3}\right)\left(\frac{1}{3} \cdot \frac{L}{3}\right)$$

$$= \frac{ML^2}{4 \times 3}\left[\frac{1}{4 \times 3} \frac{4}{3} - \frac{1}{3 \times 3}\right] = 0$$

37. **(c)** Superposition principle is valid for small displacement only. All laws we study in engineering are valid for linear elastic members only.

[Except in plastic theory]

38. (a) Real Beam → Conjugate Beam

Fixed → Free

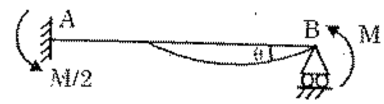
Hinge → Roller

Link releases M, AF but transfers SF

⇒ Replaced by roller support transfer M, AF.

Internal hinge → Internal Roller

40. (b)



$$\theta = \frac{ML}{4EI}$$

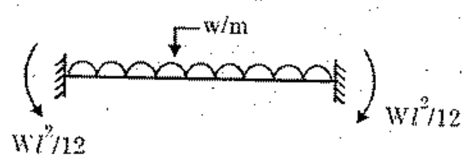
$$0.001 = \frac{M \times 10}{4 \times 10^4}$$

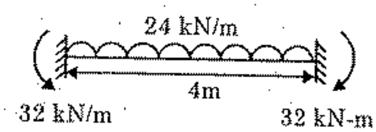
$$M = 4 \text{ kN-m}$$

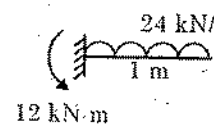
$$M_A = 2 \text{ kN-m}$$

41. (c) By moment distribution

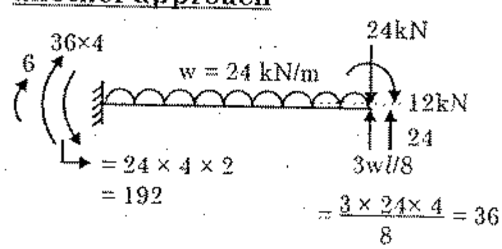
	L/1	1	0
FEM	–32	32	–12
		– 20	0
	– 10		
	–42	12	–12







another approach



$$= \frac{3 \times 24 \times 4}{8} = 36$$

$$M_A = 192 - 6 - 144$$

$$M_A = 42 \text{ kN-m}$$

42. (d)

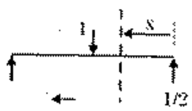
$$d\theta = \frac{\alpha \Delta T}{d} dx$$

$$d\theta \times R = dx$$

$$\frac{M}{EI} = \frac{1}{R} = \frac{\alpha \Delta T}{d}$$

By unit load method

$$1.\delta = \int M.m.\frac{dx}{EI}$$



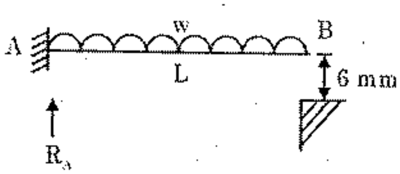
$$\delta = \int \frac{\alpha \Delta T}{d} m.dx \text{ [formula]}$$

$$m = \frac{1}{2}.x, \ x \to 0 \text{ to } \frac{L}{2}$$

$$\Rightarrow \quad \delta = 2\int_0^{L/2} \frac{\alpha \Delta T}{d}.\frac{x}{2}dx = \frac{\alpha \Delta T}{d}\left(\frac{x^2}{2}\right)_0^{L/2}$$

$$= \frac{\alpha \Delta T L^2}{8d} = \left(\frac{\alpha \Delta T}{d}\right)\left(\frac{L^2}{8}\right)(\downarrow)$$

43. (b)



$$\delta_B = \frac{wL^4}{8EI}(\downarrow)$$

$$\delta_B = \frac{wL^4}{8EI}(\downarrow) \ ; \ \delta_B = \frac{R_B L^3}{3EI}(\uparrow)$$

$$\frac{wL^4}{8EI} - \frac{R_B L^3}{3EI} = 6 \text{ [} \because \text{ Net deflection} = 6\text{mm]}$$

[This eqn. is valid when UDL deflection > gap]
[Here this is not valid]

For w = 10 kN/m

$R_A = 50$ kN

$R_B = 0$ [∵ deflection caused (1mm) << gap of 6 mm, hence above eqn. not valid in this case]

For w = 100 kN/m

$$\frac{wL^4}{8EI} - \frac{R_B L^3}{3EI} = 6 \text{ mm}$$

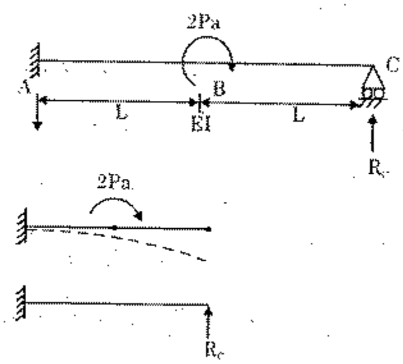
$$\frac{5^3}{781250}\left[\frac{100 \times 5}{8} - \frac{R_B}{3}\right] = \frac{6}{1000}$$

$$R_B = 75 \text{ kN}$$

$$R_A + R_B = 500 \text{ kN}$$

$$R_A = 425 \text{ kN}$$

44. (c)



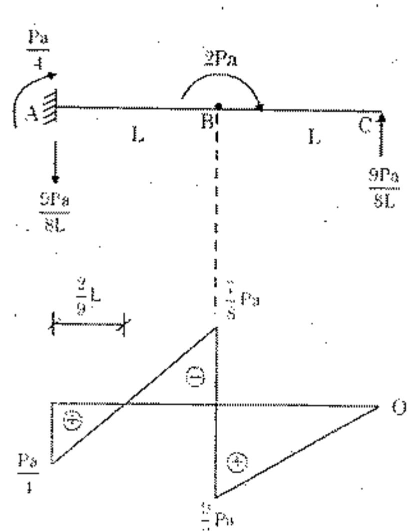
Net 'δ' at C = 0

$$\frac{2PaL^2}{2EI} + \frac{2PaL}{EI} \times L = \frac{R_C \times (2L)^3}{3EI}$$

$$2Pa \times \frac{3}{2} = R_C \times \frac{8}{3}L$$

$$R_C = \frac{9Pa}{8L} \text{ (upward)}$$

45. (a)



$$\frac{9}{8}\text{Pa}-\frac{2\text{Pa}}{4\times 2} = \frac{7}{8}\text{Pa}$$

$$2\text{Pa}-\frac{7\text{Pa}}{8} = \frac{9}{8}\text{Pa}$$

$$\frac{9}{8}\frac{\text{Pa}}{\text{L}}\times x = \frac{\text{Pa}}{4}$$

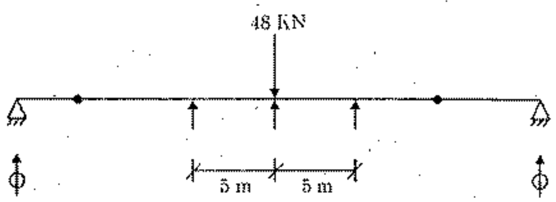
$$x = \frac{2\text{L}}{9}$$

Area between A & B

$$\theta_A-\theta_B=+\frac{1}{2}\frac{\text{Pa}}{4}\times\frac{2}{9}\frac{\text{L}}{\text{EI}}-\frac{1}{2}\frac{7}{8}\text{Pa}\times\frac{7}{9}\frac{\text{L}}{\text{EI}}=\frac{-5}{16}\frac{\text{Pa.L}}{\text{EI}}$$

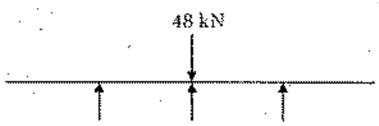
$$\text{As } \theta_A=0 \Rightarrow \theta_B=\frac{5}{16}\frac{\text{PaL}}{\text{EI}}\ (\curvearrowright)$$

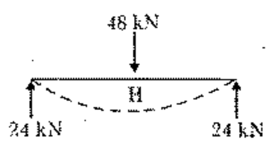
46. (b)



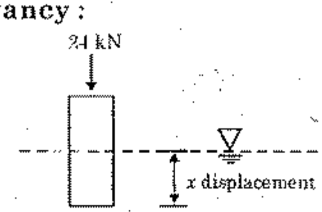
Since hinge supports cannot take moments, reactions = 0

Given Problem





Buoyancy :



$$24 = (x\times 8)\times 10 \text{ kN/m}^3$$

$$x = 0.3 \text{ m}$$

This is same for all points (displacement) on the system.

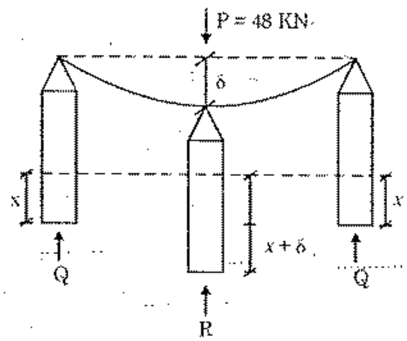
$$\delta = \frac{\text{PL}^3}{48\text{EI}}(\downarrow) \text{ [due to loading]}$$

[Since the beam is not rigid]

$$\therefore \text{ Total deflection at H} = 0.3+\frac{48\times 10^3}{48\times 10^4}$$

$$= 0.3+0.1 = 0.4 \text{ m } (\downarrow)$$

47. (c)



We have

$$\text{P} = 2\text{Q}+\text{R} \quad \dots(1)$$

$$\delta = \frac{(\text{P}-\text{R})\text{L}^3}{48\text{EI}} \quad \dots(2)$$

$$(x+\delta)\,8\times 10 = \text{R} \quad \dots(3)$$

$$\Rightarrow \quad x+\delta = \frac{\text{R}}{80}$$

$$x\times 80 = \text{Q} \quad \dots(4)$$

$$x = \frac{\text{Q}}{80}$$

$$\frac{\text{Q}}{80}+\delta = \frac{\text{R}}{80}$$

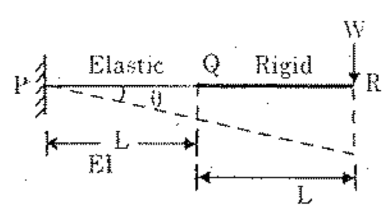
$$\frac{48-\text{R}}{2\times 80}+\delta = \frac{\text{R}}{80} \quad \left[\because \text{Q}=\frac{\text{P}-\text{R}}{2}\right]$$

By solving these equation (4)

$$\delta = \frac{3\text{R}-48}{160} = \frac{(48-\text{R})10^3}{48\times 10^4}$$

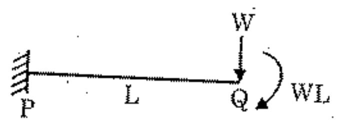
$$\text{R} = 19.2 \text{ kN}$$

48. (a) Rigid → No elastic deflection



This is equivalent to $\delta_Q = \frac{WL^3}{3EI} + \frac{(WL)L^2}{2EI} = \frac{5WL^3}{6EI}$

$$\theta_Q = \frac{WL^2}{2EI} + \frac{(WL)L}{EI} = \frac{3WL^2}{2EI}$$



49. (c)

$$\delta_R = \delta_Q + (\theta_Q \times L)$$

$$= \frac{5WL^3}{6EI} + \frac{3WL^3}{2EI} = \frac{7}{3}\frac{WL^3}{EI}$$

4

Transformation of Stress and Strain

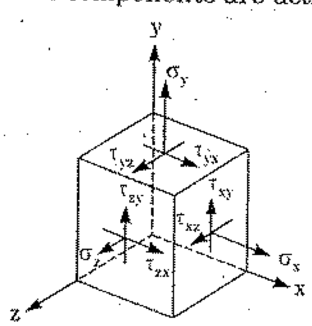
INTRODUCTION

To ensure the safety of a structural component, we not only have to ensure that the structural component is under equilibrium due to external forces, but also, each and every point inside the volume of the structural component must be in equilibrium and must have stress less than the max permissible stress.

Thus we need to know on which plane max normal stress will act, on which plane max shear stress will act, what is the magnitude of max normal and shear stress?

As the magnitude of normal and shear stress varies with the inclination of planes, stress on a particular plane can be calculated from stress on other planes using method of transformation of stress.

At any point, most general state of stress is represented by six components. (σ_x, σ_y, σ_z, τ_{xy}, τ_{yz}, τ_{zx}) as shown below. But in a particular case not all of these six components act simultaneously. In this chapter we will consider the case in which only three stress components are acting.



PLANE STRESS

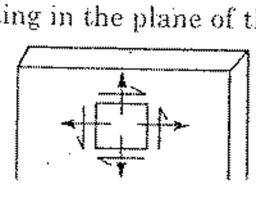
When two faces of cubic elements are free of any stress, the stress condition is called plane stress condition. For example, if z-axis is chosen perpendicular to the face on which no stress is acting then

$$\sigma_z = \tau_{zx} = \tau_{zy} = 0$$

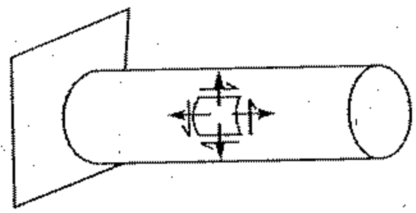
Hence remaining stress components are:

$$\boxed{\sigma_x, \sigma_y \text{ and } \tau_{xy} \rightarrow \text{Plane Stress Components}}$$

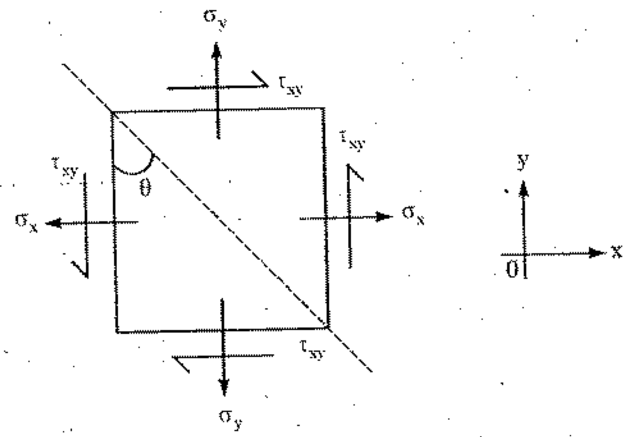
Examples of plane stress conditions are: (a) Bar in tension/Compression. (b) Shaft in torsion. (c) Beam in bending. (d) Plates subjected to forces acting in the plane of the plate.

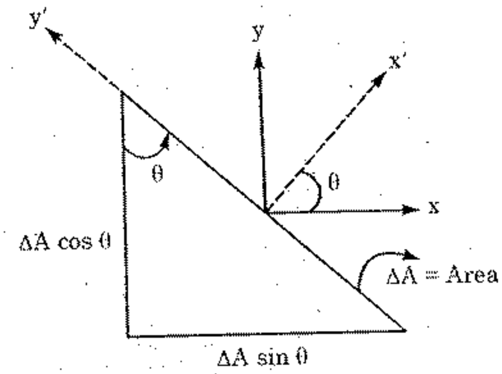


(e) Stress on the surface of structural element that is not subjected to external forces etc.



TRANSFORMATION OF PLANE STRESS

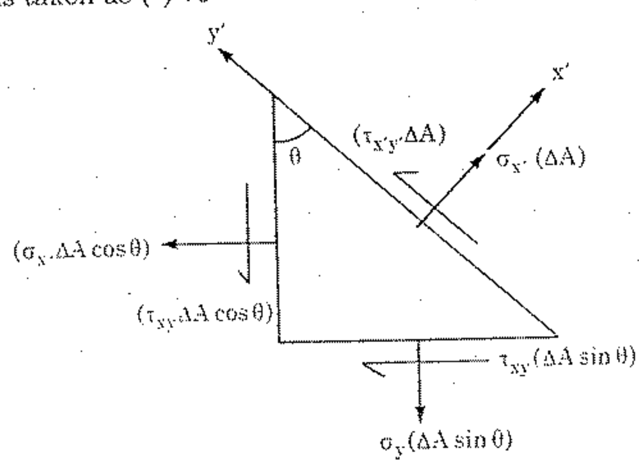




Transformation of stress means we have the stresses σ_x, τ_{xy}, σ_y on x and y faces and we have to find out stresses on plane, the normal to which are in x′ and y′ directions.

Sign Convention:

1. (+) → Tension
2. (–) → Compression
3. Shear on (+) ve face in (+) ve direction = (+) ve
   Shear on (–) ve face in (–) ve direction = (+) ve
   Shear on (–) ve face in (+) ve direction = (–) ve
   Shear on (+) ve face in (–) ve direction = (–) ve
4. Anticlockwise rotation (θ) is taken as (+) ve
5. Clockwise rotation (θ) is taken as (–) ve



Let $\sigma_{x'}$ and $\tau_{x'y'}$ be the normal and shear stresses acting on plane AC the normal to which is (+) ve x′

Let the area of plane AC be ΔA

$\Rightarrow$ Area of plane AB = $\Delta A \sin\theta$

From equilibrium equations

$$\sum F_{x'} = 0$$

$$\sigma_{x'} \times \Delta A - (\sigma_x \Delta A \cos\theta)\cos\theta - (\tau_{xy}\Delta A\cos\theta)\sin\theta$$

$$-(\sigma_y \times \Delta A \sin\theta)(\sin\theta) - (\tau_{xy}\Delta A \sin\theta)(\cos\theta) = 0 \quad \text{------ (i)}$$

$$\sum F_{y}' = 0$$

$$\tau_{x'y'}\Delta A + (\sigma_x \cdot \Delta A\cos\theta)\cdot\sin\theta - (\tau_{xy}\Delta A\cos\theta)\cdot\cos\theta$$

$$-(\sigma_y \Delta A \sin\theta)\cdot\cos\theta + (\tau_{xy}\Delta A\sin\theta)\sin\theta = 0 \quad \text{------ (ii)}$$

From (i) and (ii)

$$\Rightarrow \quad \sigma'_x = \sigma_x \cdot \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta$$

$$\tau_{x'y'} = -(\sigma_x - \sigma_y)\sin\theta\cdot\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)$$

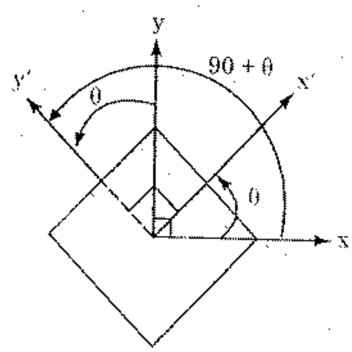
$$\Rightarrow \quad \sigma'_x = \frac{\sigma_x(1+\cos 2\theta)}{2} + \sigma_y\frac{(1-\cos 2\theta)}{2} + \tau_{xy}\sin 2\theta$$

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

Thus,

$$\boxed{\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta} \quad \text{------ (A)}$$

$$\boxed{\tau_{x'y'} = -\frac{(\sigma_x - \sigma_y)}{2}\sin 2\theta + \tau_{xy}\cos 2\theta} \quad \text{------ (B)}$$



As $\sigma_{y'}$ is obtained by replacing θ with $\theta + 90°$ in the above equation (A)

and $\cos 2(\theta + 90°) = \cos(180° + 2\theta) = -\cos 2\theta$

$\sin 2(\theta + 90°) = \cos(180° + 2\theta) = -\sin 2\theta$

Hence,

$$\boxed{\sigma_{y'} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta} \quad \text{------ (C)}$$

From (A) and (B)

$$\boxed{\sigma_x + \sigma_y = \sigma_{x'} + \sigma_{y'}} \quad \text{------ (D)}$$

i.e In case of plane stress, the sum of normal stresses exerted on a cubic element of material is independent of the orientation of element.

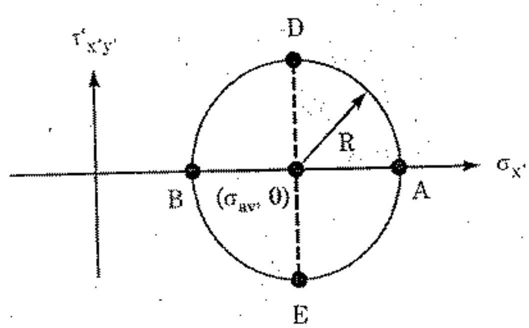
PRINCIPAL STRESS AND MAXIMUM SHEAR STRESS

From (A) and (B) above

$$\left(\sigma_{x'} - \frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau_{x'y'}^2 = \left[\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \left(\tau_{xy}\right)^2}\right]^2$$

If $\sigma_{av} = \dfrac{\sigma_x + \sigma_y}{2}$ and $R = \sqrt{\left(\dfrac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$ then $\boxed{\left(\sigma_{x'} - \sigma_{av}\right)^2 + \tau_{x'y'}^2 = R^2}$

This eq. represents equation of a circle with co-ordinate system $\sigma_{x'} - \tau_{x'y'}$ with centre at $(\sigma_{av}, 0)$ and radius R.



- Thus every point on the circle represents a state of stress. At point A and B, shear stress are zero and normal stress are max. and min. These max and min normal stresses are called principal stresses. The points where shear stresses are zero is given by equating $\tau_{x'y'} = 0$ in equation (B).

  Thus from (B)

  $$\Rightarrow \quad \left[\tan 2\theta_p = \frac{2\tau_{xy}}{(\sigma_x - \sigma_y)}\right]\left[\text{This relation can also be obtained by } \frac{d\sigma_{x'}}{d\theta} = 0\right] \text{ --- (E)}$$

  From this we get two values of $2\theta_p$ which are separated by 180° because $\boxed{\tan(180° + \alpha) = \tan\alpha}$

- Thus, the two values of θ_p are separated by 90° i.e. θ_p and $90 + \theta_p$. One of these orientations gives max normal stress and other gives min normal stress. Thus we have two planes on which shear stresses are zero. On one of them normal stress is max and on other it is min.

  Major principal stress (σ_{max}) = Max value of normal stress

  Minor principal stress (σ_{min}) = Min value of normal stress

  Major principal plane → Plane on which σ_{max} acts

- Thus from the circular representation of state of stress, $\sigma_{max/min} = \sigma_{av} \pm R$

i.e.
$$\boxed{\sigma_{max/min} = \frac{\sigma_y + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \left(\tau_{xy}\right)^2}} \quad \text{------------------- (F)}$$

Normally in 3-D case: We have three principal stresses, acting on three mutually perpendicular planes. But in case of plane-stress the third principal stress = 0. [The third principal stress is parallel to z-axis].

- To find out which θ_p (as discussed above) corresponds to max/min principal stress, we put the value of θ_p in eq. (A)

$$\text{i.e. in } \sigma'_x = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

and check whether we get σ_{max} or σ_{min}.

- Point D and E corresponds to max. shear stress

Magnitude of max shear stress = Radius of circle = $\frac{1}{2}$ × difference of principal stresses

$$\boxed{\tau_{max} = \frac{\sigma_{max} - \sigma_{min}}{2} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}} \quad \text{---------------------- (G)}$$

Cordinate of point D is $\left(\frac{\sigma_x + \sigma_y}{2}, \tau_{xy\,max}\right)$

⇒ **The point where shear stress is max, Normal stress = $\frac{\sigma_x + \sigma_y}{2}$**

- By putting $\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2}$ in equation (A), we get

$$\frac{\sigma_x + \sigma_y}{2} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

⇒ $\boxed{\tan 2\theta_S = \frac{-(\sigma_x - \sigma_y)}{2\tau_{xy}}}$ [This relation can also be obtained from $\frac{d(\tau_{x'y'})}{d\theta} = 0$] -- [H]

Thus we get θ_s and $90 + \theta_S$ as two-orientation of plane on which shear stress is max

- Let $m_1 = \tan 2\theta_P$

$m_2 = \tan 2\theta_S$

⇒ $\tan 2\theta_P \times \tan 2\theta_S = -1$ [From E and H]

⇒ $m_1 \cdot m_2 = -1$

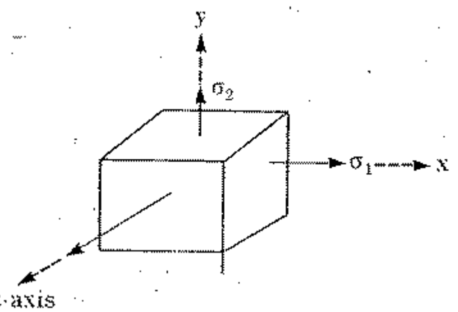
⇒ $2\theta_P$ and $2\theta_S$ are 90° apart

⇒ **θ_P and θ_S are 45° apart**

⇒ **Planes of max. shear stress are 45° to the principal planes**

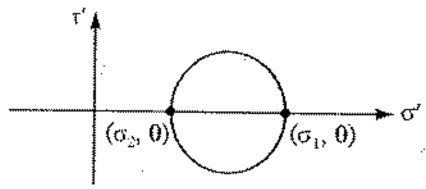
Note that in the above cases we have considered rotation of elements about z-axis. If we consider rotation about other axis, the max. shear stress will be different than the value for max in-plane shear stress discussed above i.e. (τ_{xy}).

ABSOLUTE MAX SHEAR STRESS



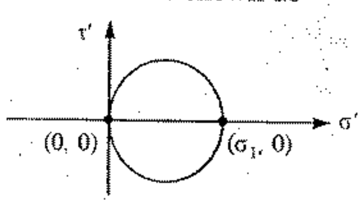
In case of plane stress condition if σ_1 and σ_2 are the principal stresses then

1. For rotation about z-axis the stress condition is shown as



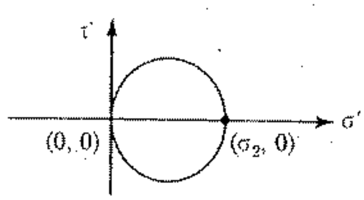
$$\Rightarrow \tau_{max} = \frac{\sigma_1 - \sigma_2}{2}$$

2. For rotation about y-axis the stress condition is shown as



$$\Rightarrow \tau_{max} = \frac{\sigma_1}{2}$$

3. For rotation about x-axis the stress condition is shown as



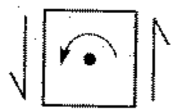
$$\Rightarrow \tau_{max} = \frac{\sigma_2}{2}$$

Hence absolute maximum shear stress is max of $\left\{\frac{|\sigma_1 - \sigma_2|}{2}, \frac{|\sigma_1|}{2}, \frac{|\sigma_2|}{2}\right\}$

MOHR CIRCLE FOR PLANE STRESS

Mohr circle is used for transformation of stresses. We already have shown that state of stress at a point in the body can be expressed as a points on the circumference of a circle. This concept is used for transformation of stress (i.e. by using known stresses on two ⊥r planes calculation of stress on any other planes]. To use Mohr circle concept following sign convention is adopted.

Sign Convention

1. Tension → (+) ve
2. Compression → (–) ve
3.  Shear producing clockwise moment about centre of the element is taken as (+) ve.
4.  Shear producing anticlockwise moment about the centre of element is taken as (–) ve.

In the transformation of stress we know stresses on plane (1) and (2) and desire to find out stress on plane (3) and (4) i.e. the planes, the normal to which are in x′ and y′ direction (as shown in figure below). We know that state of stress at a point can be represented by a point on the circumference of a circle. Hence, since, as per the sign convention just discussed, stress on plane (1) is $(\sigma_x, -\tau_{xy})$ and that on plane (2) is (σ_y, τ_{xy}), the points are represented by point A and B respectively on a circle as shown below. Similarly, stress on plane (3) and (4) i.e. $(\sigma_{x'}, -\tau_{x'y'})$ and $(\sigma_{y'}, \tau_{x'y'})$ are represented by point (C) and (D) on the Mohr circle.

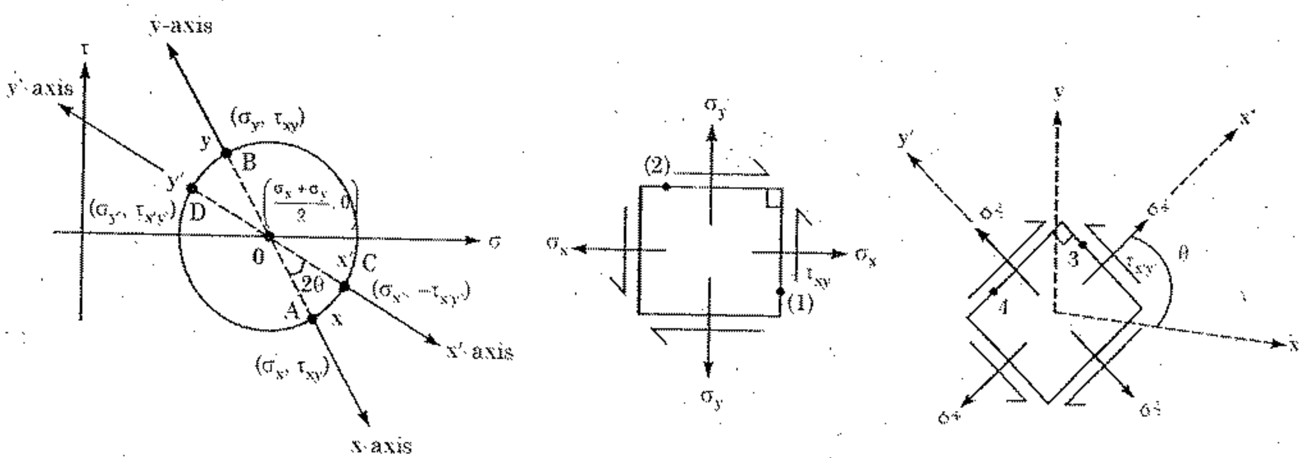


Fig. A

RULES FOR APPLYING MOHR'S CIRCLE FOR TRANSFORMATION OF STRESS

1. On rectangular σ-τ axis, plot points having coordinates corresponding to face 1 and face 2 adopting the sign convention of Mohr circle. In this case face 1 has co-ordinates $(\sigma_x, -\tau_{xy})$ and face 2 has co-ordinate (σ_y, τ_{xy}).
2. Join the points just plotted by straight line. This line is the diameter of a circle whose centre is on the σ-axis.

   [Note that the line joining the two point will be a diameter only when the two points correspond to two faces which are ⊥r to each other].

   If the two planes, the stresses of which have been plotted, are not ⊥r to each other, then to get the centre of circle, a perpendicular bisector is drawn on the line joining the two point. Intersection of this perpendicular bisector with the σ-axis is the centre of the circle.

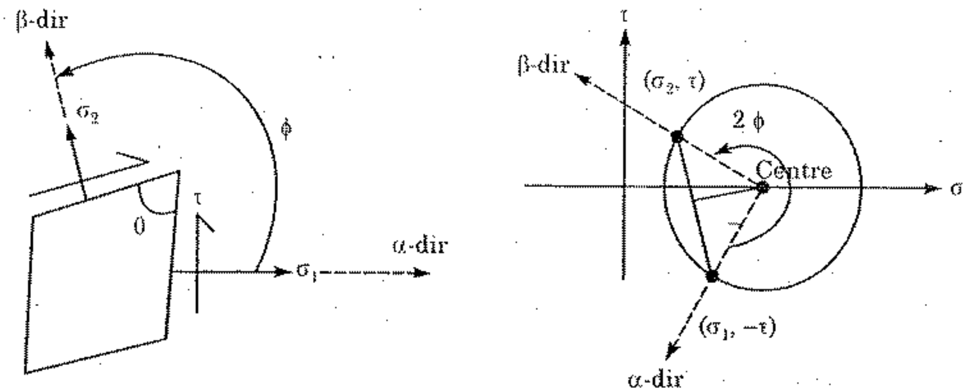


Fig. B

3. As different planes are passed through the selected points in a stressed body, the normal and shearing stress components on these planes are represented by the co-ordinates of points whose position shifts around the circumference of Mohr's circle.
4. The line joining centre of the circle to any point on its circumference represents the axis directed normal to the plane whose stress components are given by the co-ordinate of that point. Thus normal to the plane on which $(\sigma_x, -\tau_{xy})$ is acting i.e. normal to plane (1) in Fig. A is represented by line joining O and A.
5. The angle between the radii to selected points on Mohr circle is twice the angle between the normals to the actual planes represented by these point. The rotational sense of this angle corresponds to the rotational sense of actual angle between the normals to the planes. i.e. if x′-axis is at an angle θ in anticlockwise direction from x-axis, then on Mohr circle x′-radius is laid off at on anticlockwise angle 2θ from the x-radius.

HOOKE'S LAW FOR PLANE-STRESS

If plane-stress components are $(\sigma_x, \sigma_y, \tau_{xy})$ and

ε_x = Normal strain in x-direction

ε_y = Normal strain in y-direction

ε_z = Normal strain in z-direction

τ_{xy} = Shear strain

Then from Hooke's law

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \mu\sigma_y)$$
$$\varepsilon_y = \frac{1}{E}(\sigma_y - \mu\sigma_x)$$
$$\varepsilon_z = \frac{1}{E}(-\mu)(\sigma_x + \sigma_y) \qquad \text{(I)}$$
$$\gamma_{xy} = \frac{\tau_{xy}}{G}$$

Note: *Normal and shear stress are independent. One is not affected by other.*

By simplifying equation (I), we get

$$\sigma_x = \frac{E}{1-\mu^2}(\varepsilon_x + \mu\varepsilon_y)$$
$$\sigma_y = \frac{E}{1-\mu^2}(\varepsilon_y + \mu\varepsilon_x) \qquad \text{(J)}$$
$$\tau_{xy} = G \cdot \gamma_{xy}$$

SPECIAL CASES

(i) **Bi-axial stress condition** (i.e. when only σ_x and σ_y are acting and $\tau_{xy} = 0$.)

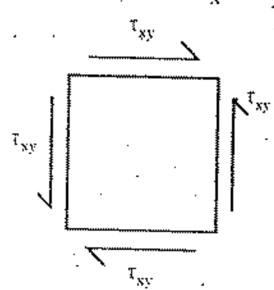
Same equation as derived above in (A) and (B) are applicable because effects of normal and shear stresses are independent of each other.

(ii) **Uniaxial stress** (i.e. when only σ_x is acting and $\sigma_y = 0$ and $\tau_{xy} = 0$.)

$$\boxed{\varepsilon_x = \frac{\sigma_x}{E}}$$

$$\boxed{\varepsilon_y = \varepsilon_z = -\frac{\mu\sigma_x}{E}}$$

(iii) **Pure shear case** (i.e. when only τ_{xy} is acting and $\sigma_x = \sigma_y = 0$.)



$$\boxed{\varepsilon_x = \varepsilon_y = \varepsilon_z = 0}$$ i.e., Normal strain in x, y, and z direction are zero.

$$\boxed{\gamma_{xy} = \frac{\tau_{xy}}{G}}$$

Volume Changes: (Plane Stress Condition)

- We know that volumetric strain is given by

$$\varepsilon_v = \frac{\Delta V}{V_0} = \frac{(\sigma_x + \sigma_y + \sigma_z)}{E}(1-2\mu)$$

Hence in <u>plane stress condition</u>

$$\boxed{\varepsilon_v = \frac{(\sigma_x + \sigma_y)}{E}(1-2\mu)}$$

- In bi-axial stress condition (i.e. $\tau_{xy} = 0$) the same formula as above is valid
- In uniaxial stress condition (i.e. $\sigma_y = 0$, $\tau_{xy} = 0$):

$$\Rightarrow \boxed{\varepsilon_v = \frac{\sigma_x(1-2\mu)}{E}}$$

- In pure shear case (i.e. $\sigma_x = 0 = \sigma_y$), $\varepsilon_v = 0$.

Note: *Shear strain only leads to distortion of element. It doesnot lead to change in volume. Normal stresses on the other hand leads to change in volume.*

Strain Energy: (Plane Stress Condition)

- Strain energy per unit volume in most general state of stress is given by

$$U = \frac{1}{2}\left[\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_z\varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{xz}\gamma_{xz} + \tau_{zy}\gamma_{zy}\right]$$

- Hence under plain stress condition when only σ_x, σ_y, τ_{xy} exist.

  Strain energy due to normal stress (per unit volume), $U_1 = \frac{1}{2}(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y)$

- Strain energy due to shear stress, $U_2 = \frac{\tau_{xy}\cdot\gamma_{xy}}{2}$ (per unit volume)
- Total strain energy per unit volume
  $U = U_1 + U_2$

$$\Rightarrow \boxed{U = \frac{1}{2}(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \tau_{xy}\gamma_{xy})}$$

- If σ_1 and σ_2 are principal stresses then strain energy per unit volume is

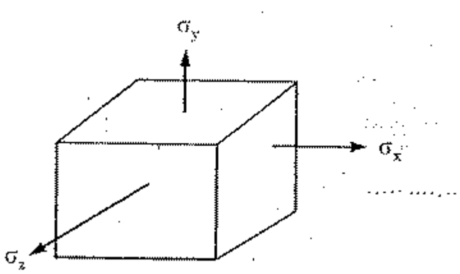
$$U = \frac{1}{2}\left[\sigma_1(\varepsilon_1) + \sigma_2(\varepsilon_2)\right]$$

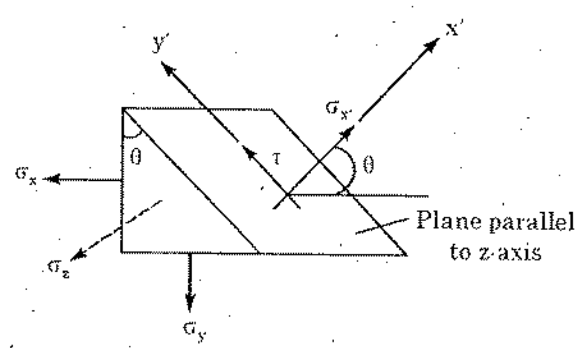
$$= \frac{1}{2}\left[\sigma_1\left(\frac{\sigma_1}{E} - \frac{\mu\sigma_2}{E}\right) + \sigma_2\left(\frac{\sigma_2}{E} - \frac{\mu\sigma_1}{E}\right)\right]$$

$$\boxed{U = \frac{1}{2E}\left[\sigma_1^2 + \sigma_2^2 - 2\mu\sigma_1\sigma_2\right]} \text{ (Strain energy per unit volume)}$$

Triaxial Stresses

- An element subjected to σ_x, σ_y and σ_z only [Shear = zero on x, y, z faces] is said to be in triaxial stress.
  $\Rightarrow$ σ_x, σ_y and σ_z are principal stresses





$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

- In triaxial stress condition also if transformation of stress is done to find out stress on planes obtained by rotation about z-axis, the transformation eq. will be same as that under plane stress condition (i.e. eq. A).
- This is because transformation eq. (A) has been derived from eq. of statics $\Sigma F_{x'} = 0$ and $\Sigma F_{y'} = 0$. Hence even if σ_z is existing it will not appear in the equation, as σ_z will have no component along x' or y' axis. Note that if τ_{xz} had been acting, it would have appeared in the transformation eq. because τ_{xz} will have a component along x' and y' axis.

- Also as discussed earlier Max shear stress = $\max\left[\frac{|\sigma_x - \sigma_y|}{2}, \frac{|\sigma_y - \sigma_z|}{2}, \frac{|\sigma_z - \sigma_x|}{2}\right]$

HOOKE'S LAW FOR TRIAXIAL STRESS

$$\varepsilon_x = \frac{\sigma_x}{E} - \frac{\mu(\sigma_y + \sigma_z)}{E}$$

$$\varepsilon_y = \frac{\sigma_y - \mu(\sigma_x + \sigma_z)}{E} \quad \text{------------ (K)}$$

$$\varepsilon_z = \frac{\sigma_z - \mu(\sigma_x + \sigma_y)}{E}$$

or by rearranging

$$\sigma_x = \frac{E}{(1+\mu)(1-2\mu)}\left[(1-\mu)\,\varepsilon_x + \mu(\varepsilon_y + \varepsilon_z)\right]$$

$$\sigma_y = \frac{E}{(1+\mu)(1-2\mu)}\left[(1-\mu)\,\varepsilon_y + \mu(\varepsilon_z + \varepsilon_x)\right] \quad \text{------- (L)}$$

$$\sigma_z = \frac{E}{(1+\mu)(1-2\mu)}\left[(1-\mu)\,\varepsilon_z + \mu(\varepsilon_y + \varepsilon_x)\right]$$

Strain Energy Per Unit Volume (under triaxial stress)

General formula for strain energy per unit volume is

$$U = \frac{1}{2}\left[\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_z\varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{xz}\gamma_{xz} + \tau_{zx}\gamma_{zx}\right]$$

but for triaxial stress condition, $\tau_{xy} = \tau_{yz} = \tau_{zx} = 0$

$$\Rightarrow \ U = \frac{1}{2}\left[\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_z\varepsilon_z\right]$$

$$\Rightarrow \ U = \frac{1}{2}\left[\sigma_x\left(\frac{\sigma_x}{E} - \frac{\mu\sigma_y}{E} - \frac{\mu\sigma_z}{E}\right) + \sigma_y\left(\frac{\sigma_y}{E} - \frac{\mu\sigma_x}{E} - \frac{\mu\sigma_z}{E}\right) + \sigma_z\left(\frac{\sigma_z}{E} - \frac{\mu\sigma_x}{E} - \frac{\mu\sigma_y}{E}\right)\right]$$

$$\Rightarrow \ U = \frac{1}{2E}\left(\sigma_x^2 + \sigma_y^2 + \sigma_z^2 - 2\mu\sigma_x\sigma_y - 2\mu\sigma_y\sigma_z - 2\mu\sigma_z\sigma_x\right)$$

$$\boxed{U = \frac{1}{2E}\left(\sigma_x^2 + \sigma_y^2 + \sigma_z^2 - 2\mu(\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x)\right)} \ \text{(under triaxial stress condition)}$$

Spherical Stress:

[It is a special case of triaxial stress when $\sigma_x = \sigma_y = \sigma_z = \sigma_0$]

- Under this condition any plane cut through the element will be subjected to same normal stress = σ_0.
- Shear stress on all planes = 0.
- Every plane is a principal plane.
- Mohr circle reduces to a point.

- Normal strain = $\boxed{\varepsilon_0 = \frac{\sigma_0}{E}(1-2\mu)}$

Volumetric Strain (under spherical stress condition)

$$\varepsilon_v = \varepsilon_x + \varepsilon_y + \varepsilon_z = 3\varepsilon_0$$

$$\boxed{\varepsilon_v = \frac{3\sigma_0(1-2\mu)}{E}}$$

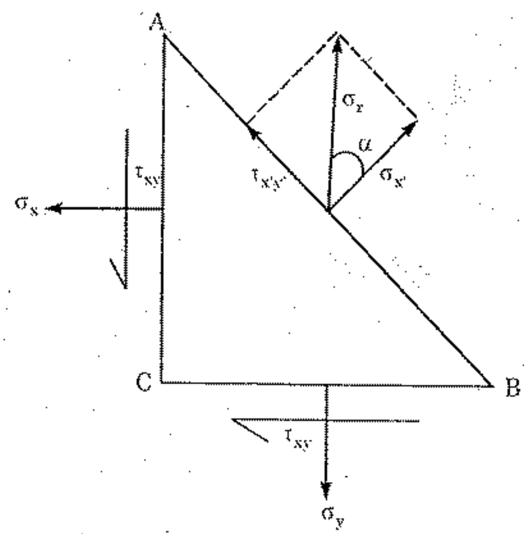
$$\Rightarrow \boxed{\varepsilon_v = \frac{\sigma_0}{K}}$$

where K = Bulk modules.

Spherical stress is also called hydrostatic stress.

ANGLE OF OBLIQUITY

Angle that line of action of resultant stress on a plane makes with the normal to the plane is called angle of obliquity.



Resultant stress on plane AB = σ_r

$$\sigma_r = \sqrt{\sigma_{x'}^2 + \tau_{x'y'}^2}$$

angle α is called angle of obliquity of plane AB.

Example 1

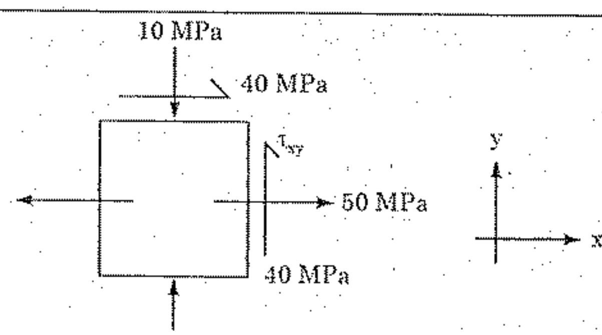
For the state of stress shown in figure below, determine:

(a) The principle planes

(b) The principal stresses

(c) The max shearing stress and its plane

(d) Normal stress corresponding to max shearing stress



Sol: $\sigma_x = 50$ MPa, $\sigma_y = -10$ MPa, $\tau_{xy} = 40$ MPa

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{2\times(40)}{50-(-10)}$$

$$\tan 2\theta_p = \frac{80}{60}$$

$$2\theta_p = 53.13° \quad \Rightarrow \quad \theta_p = 26.56°$$

$$2\theta_p = 180 + 53.13° \quad \Rightarrow \quad \theta_p = 116.56°$$

These are corresponding to two principle planes.

$$\sigma_{max/min} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$= \frac{50+(-10)}{2} \pm \sqrt{\left[\frac{50-(-10)}{2}\right]^2 + (40)^2}$$

$$= 20 \pm 50$$

$\sigma_{max} = 70$ MPa **Ans.**

$\sigma_{min} = -30$ MPa **Ans.**

Let us check out of the two values of θ_p, which one corresponds to σ_{max}.
By putting $\theta = 26.56°$

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

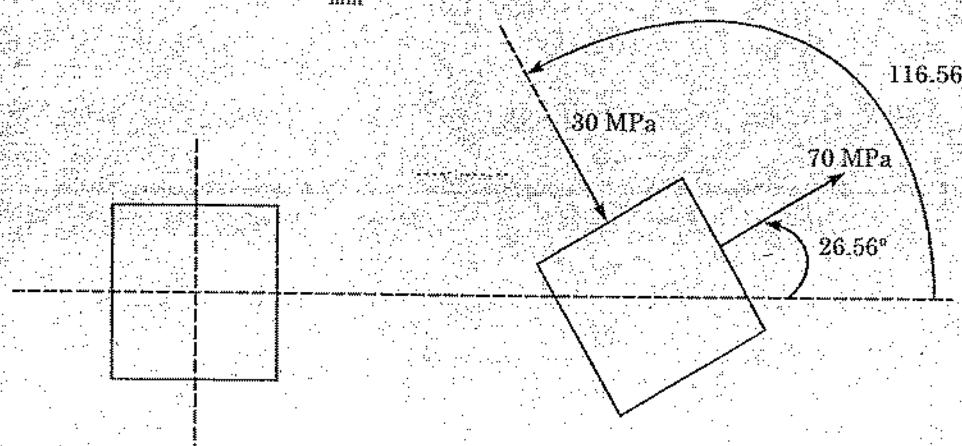
$$= \frac{50-10}{2} + \frac{50-(-10)}{2}\cos(53.13°) + 40\sin 53.13°$$

$$= 20 + 30\cos 53.13° + 40\sin 53.13°$$

$$= 20 + 18 + 32$$

$$= 70$$

$\Rightarrow$ $\theta_p = 26.56°$ corresponds to σ_{max} **Ans.**
hence, $\theta_p = 116.56°$ corresponds to σ_{min} **Ans.**



$$\text{Max. shearing stress} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \left(\tau_{xy}\right)^2}$$

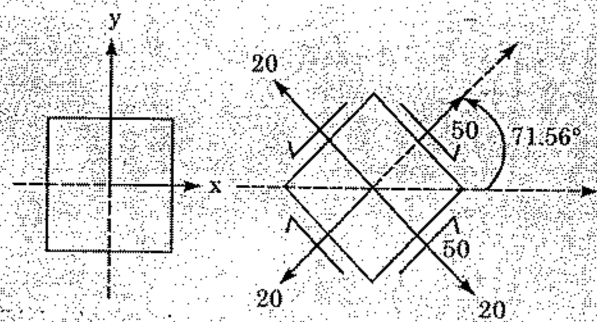
$$= \sqrt{\left(\frac{50-(-10)}{2}\right)^2 + (40)^2}$$

$$= \sqrt{(30)^2 + (40)^2} = 50 = \text{Max shear stress} \quad \textbf{Ans.}$$

Max shearing stress occur on planes which are at 45° with principal planes

Hence $\theta_s = 26.56° + 45° = 71.56°$ **Ans.**

and $26.56° - 45° = -18.42°$ **Ans.**



By putting $\theta_s = 71.56°$ in $\tau_{x'y'} = -\dfrac{(\sigma_x - \sigma_y)}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$, we get $\tau_{x'y'} = -50$

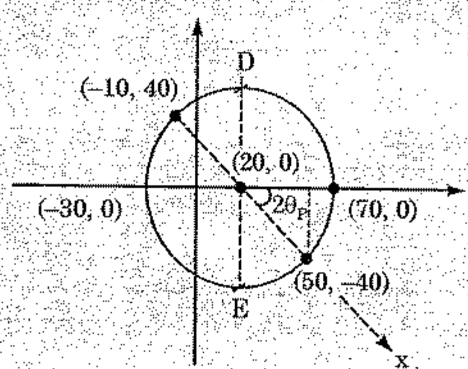
$\Rightarrow$ Shear on x'-plane is in (–) ve y'-direction

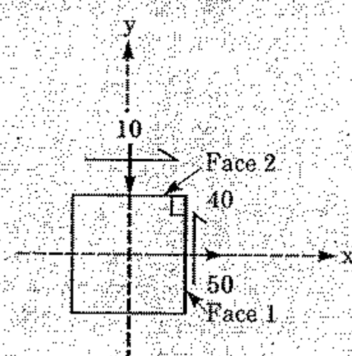
Normal stress on the plane of max shear stress is

$$\frac{\sigma_x + \sigma_y}{2} = \frac{50 - 10}{2} = 20 \text{ MPa}$$

Solution by Mohr Circle:

Stress co-ordinate for face 1 is (50, –40) and that for face 2 is (–10, 40). These two points are plotted. The line joining these two points represent the diameter of Mohr circle because face 1 and face 2 are ⊥r to each other. Taking this line as diameter Mohr circle is completed.



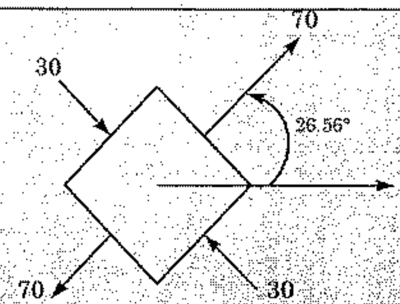


$$\tan 2\theta_P = \frac{40}{(50-20)} = \frac{40}{30}$$

$$\Rightarrow \quad 2\theta_P = 53.13°$$

$\theta_P = 26.56°$ = Major principal plane [from the Mohr circle it is clear that $\theta_P = 26.56°$ corresponds to major principal stress, because out of two values of principal stresses 70 and –30, 70 is max

θ_P for minor principle stress is given by $\dfrac{(180+53.12)}{2} = 116.56°$



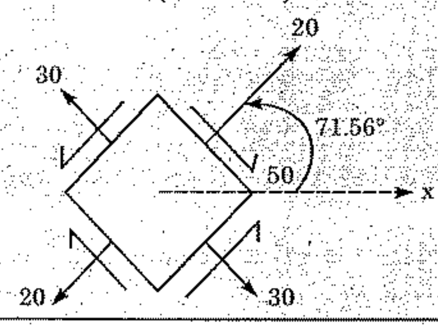
Major principal stress = 70 MPa (Tensile)

Minor principal stress = 30 MPa (Compressive)

Max shear stress = radius = 50 MPa

The orientation of plane on which max shear stress acts is such that normal to this plane makes an angle (45° + 26.56°) = 71.56° anticlockwise with the normal to the plane on which stress (50, –40) is acting (i.e. x-axis).

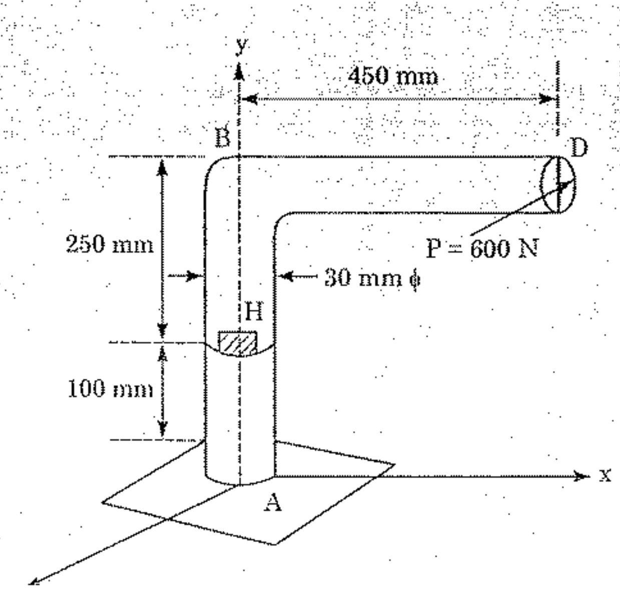
or $-45° + 26.56 = -18.42°$ (clockwise)



Note: *At D shear stress is (+) ve ⇒ rotation produced by it is clockwise about the centre of the element hence the figure.*

Normal stress on the plane on which max shear stress is acting (represented by point D & E) is 20 N/mm².

Example 2



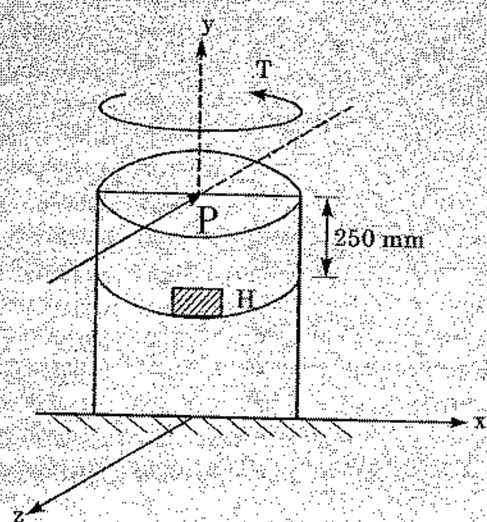
A single force P of megnitude 600 N is applied to end D of lever ABD. Knowing that portion AB of the lever has a diameter of 30 mm, determine:

(a) The normal and shearing stresses on an element located at point H and having sides parallel to the x and y-axis.

(b) The principal planes and principal stresses at point H.

Torque produced at H

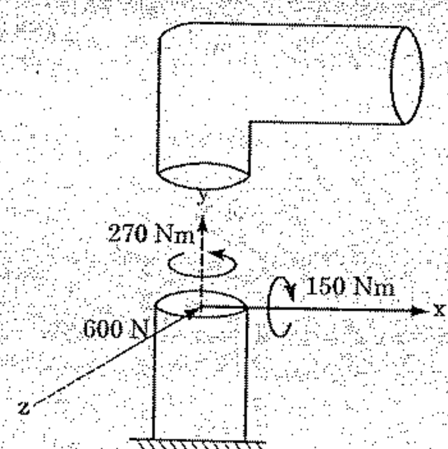
T = 600 N × 0.45 m = 270 Nm



Moment produced at H

= (600 × 0.25) = 150 Nm

Shear force at section passing through H = 600 N



On the element at H

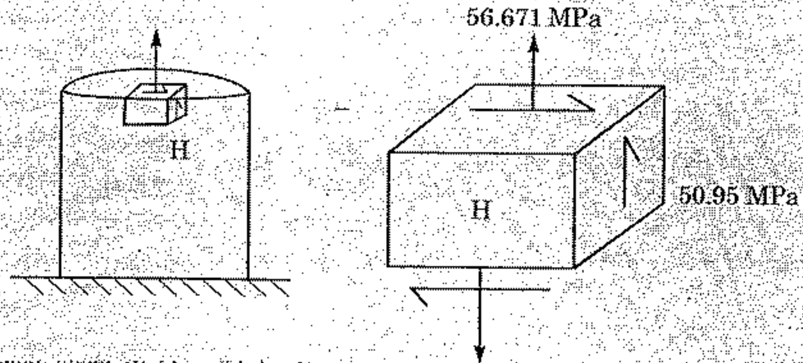
- Shear stress due to direct load = 0 because variation of this shear stress over the section passing through H is parabolic with megnitude at ends = zero.

- Normal stress due to bending = $\frac{MZ}{I_{xx}}$

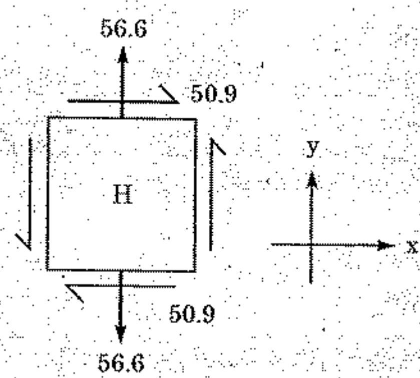
$$= \frac{150 \times 0.015}{\frac{\pi(0.030)^4}{64}}$$

= 56.617 MPa

- Shear stress due to torque $= \frac{Tr}{J} = \frac{270 \times 0.015}{\frac{\pi}{32}(0.030)^4} = 50.95$ MPa



Hence stress condition on element at H having faces parallel to x and y axis is as shown in the figure below.



$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

$$= \frac{2 \times 50.9}{0 - 56.6}$$

$$\Rightarrow \quad 2\theta_p = -61°$$

and $\quad 2\theta_p = 180 - 61°$

$$\Rightarrow \quad \theta_p = -30.5°$$

or $\quad \theta_p = 59.5°$

$$\sigma_{max/min} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \left(\tau_{xy}\right)^2}$$

$$= \frac{0 + 56.6}{2} \pm \sqrt{\left(\frac{0.56.6}{2}\right)^2 + (50.9)^2}$$

$$= 28.3 \pm \sqrt{(28.3)^2 + (50.9)^2}$$

$$= 28.3 \pm 58.24$$

$$\sigma_{max} = 86.54 \text{ MPa}$$

$$\sigma_{min} = -29.94 \text{ MPa}$$

Check (By putting $\theta_p = -30.5°$)

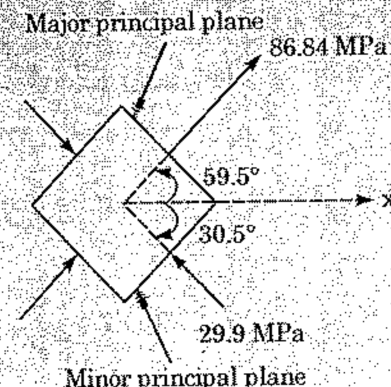
$$\sigma'_x = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$= 28.3\cos(-61°) + 58.9(\sin(-61°))$$

$$\sigma'_x = -29.93$$

$\Rightarrow$ $\theta_p = -30.5°$ corresponds to minor principal plane

and $\theta_p = 59.5°$ corresponds to major principal plane and



PLAIN STRAIN

If the only deformations are those in x-y plane then three strain components may exist

ε_x = Normal strain in x-direction

ε_y = Normal strain in y-direction

γ_{xy} = Shear strain associated with x-y plane.

- An element of material subjected to these strain (and only these strains) is said to be in a state of plane strain.
- For plain strain other three components of strain ε_z, γ_{xz}, γ_{yz} are zero.

COMPARISION OF PLANE STRESS AND PLANE STRAIN

	Plane stress	Plane strain
Stress	$\sigma_z = 0, \tau_{xz} = 0, \tau_{yz} = 0$ σ_x, σ_y and τ_{xy} may be non-zero	$\tau_{xz} = 0, \tau_{yz} = 0$ $\sigma_x, \sigma_y, \sigma_z$ and τ_{xy} may be non-zero
Strain	$\gamma_{xz} = 0, \gamma_{yz} = 0$ $\varepsilon_x, \varepsilon_y, \varepsilon_z$ and γ_{xy} may be non-zero	$\varepsilon_z = 0, \gamma_{xz} = 0, \gamma_{yz} = 0$ $\varepsilon_x, \varepsilon_y, \gamma_{xy}$ may be non-zero

Note:

$$\varepsilon_x = \frac{\sigma_x}{E} - \frac{\mu\sigma_y}{E} - \frac{\mu\sigma_z}{E} \qquad \gamma_{xy} = \frac{\tau_{xy}}{G}$$

$$\varepsilon_y = \frac{\sigma_y}{E} - \frac{\mu\sigma_z}{E} - \frac{\mu\sigma_x}{E} \qquad \gamma_{yz} = \frac{\tau_{yz}}{G}$$

$$\varepsilon_z = \frac{\sigma_z}{E} - \frac{\mu\sigma_x}{E} - \frac{\mu\sigma_y}{E} \qquad \gamma_{zx} = \frac{\tau_{zx}}{G}$$

- If $\sigma_z = 0$, that doesnot mean $\varepsilon_z = 0$ [except, in case when $\mu = 0$ i.e. ideal material or when $\sigma_x = -\sigma_y$]
  Similarly,
- If $\varepsilon_z = 0$, that doesnot mean $\sigma_z = 0$ [except in case when $\mu = 0$ i.e. ideal material or when $\sigma_x = -\sigma_y$]
- Thus note that plain stress and plain strain components are not same.
- In term of stress, $\sigma_z = 0$ in plane stress but σ_z may not be zero in plane strain.
  Strain condition under plane strain condition is presented by strain tensor as shown below.

$$\begin{bmatrix} \varepsilon_x & \gamma_{xy} & 0 \\ \gamma_{yx} & \varepsilon_y & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{Strain Tensor}$$

TRANSFORMATION OF PLANE STRAIN

- $\varepsilon_{x'} = \dfrac{\varepsilon_x + \varepsilon_y}{2} + \dfrac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \dfrac{\gamma_{xy}}{2}\sin 2\theta$
- $\varepsilon_{y'} = \dfrac{\varepsilon_x + \varepsilon_y}{2} - \dfrac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta - \dfrac{\gamma_{xy}}{2}\sin 2\theta$
- $\boxed{\varepsilon'_x + \varepsilon'_y = \varepsilon_x + \varepsilon_y}$
- $\boxed{\dfrac{\gamma_{x'y'}}{2} = -\dfrac{\varepsilon_x - \varepsilon_y}{2}\sin 2\theta + \dfrac{\gamma_{xy}}{2}\cos 2\theta}$

Above equations are analogous to that derived for plane stress. Note that these equations are obtained by replacing, normal stress by corresponding normal strain and shear stress by $\frac{1}{2}$ the corresponding shearing strain in the stress transformation equation.

Sign Convention

Normal tensile strain is taken as (+) ve

Normal compressive strain is taken as (–) ve

γ_{xy}(+) ve if angle between (+) ve faces reduces otherwise (–) ve

$\theta \rightarrow$ anti clockwise (+) ve

MOHR CIRCLE FOR PLANE STRAIN

Sign Convention:

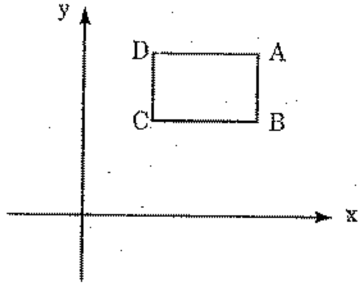
Normal Strain

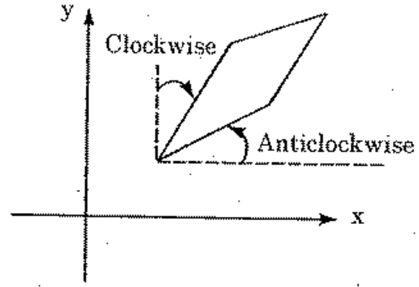
(+) ve if strain tries to elongate the element.

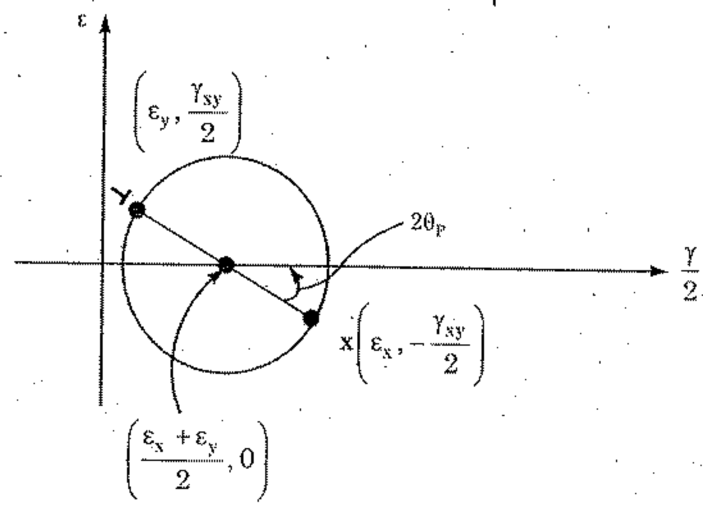
(–) ve if strain tries to contract the element.

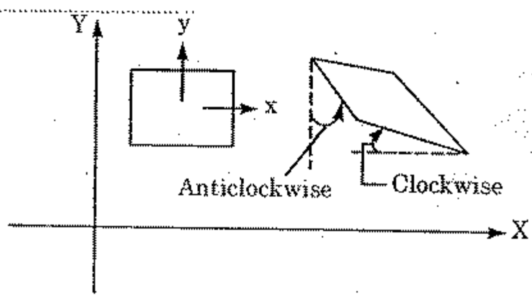
Shear Strain

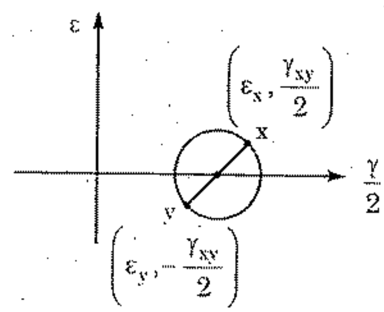
- If shear deformation causes a given side to rotate clockwise, corresponding point on Mohr's circle for plane strain is plotted above horizontal line i.e. (+) ve.
- If shear deformation causes a given side to rotate anti clockwise corresponding point on Mohr's circle for plane strain is plotted below horizontal line i.e. (–) ve.











Note: Strain associated with BC or AD line is taken as ε_x and that associated with AB line is taken as ε_y.

Radius of circle is $= \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$

Principal strains are

$$\varepsilon_{max/min} = \frac{\varepsilon_x + \varepsilon_y}{2} \pm \sqrt{\frac{(\varepsilon_x - \varepsilon_y)^2}{2} + \left(\frac{\tau_{xy}}{2}\right)^2}$$

$$\tan 2\theta_P = \frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y}$$

Note: $\gamma_{xy} \cdot G = \tau_{xy}$

$\Rightarrow$ when $\tau_{xy} = 0$, $\gamma_{xy} = 0$

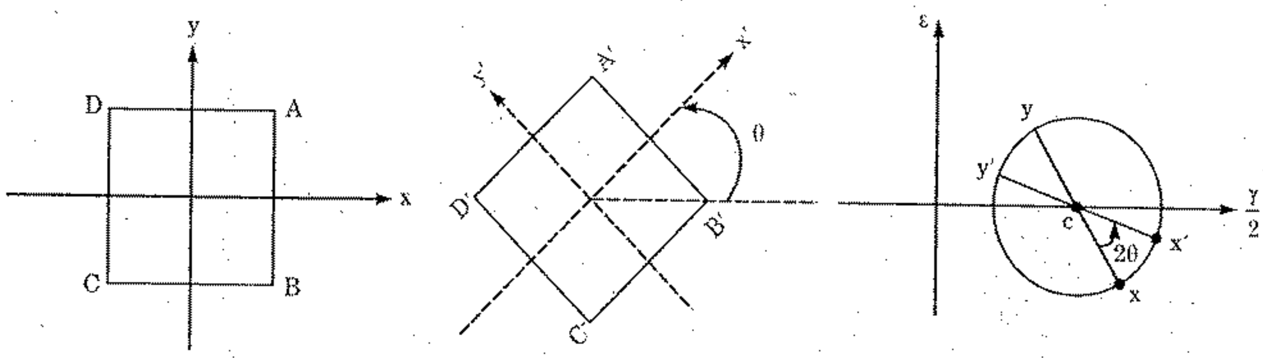
⇒ Principal axes of strain coincides with principal axis of stress

Max. in-plane shearing strain $\left(\frac{\gamma_{max}}{2}\right)_{in\text{-}plain} = R$

⇒ $\boxed{(\gamma_{max})_{in\,plane} = \sqrt{(\varepsilon_x - \varepsilon_y)^2 + \gamma_{xy}^2}}$

It occurs at 45° to the axis of principal strain

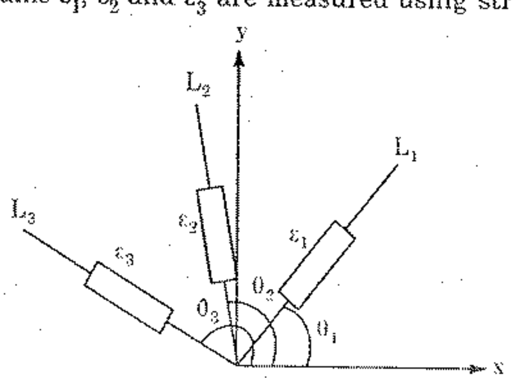
- Transformation of plane strain is done in the similar manner as discussed for plane stress.



- Thus, from strain associated with face AB and AD, strain associated with face A'B' and A'D' can be obtained using transformation equation of plane strain or using Mohr circle for plane strain.

STRAIN ROSETTE

- **Normal strain** on the surface of structural element can be measured in any given direction using strain guages. A strain rosette is a group of three gauges arranged is a particular pattern such that it can measure normal strain in three different directions on the surface of a structural element.
- If the strain rosette is mounted on the surface of a body where the material is in plane stress condition then the information derived (i.e. normal strain in three different directions) can be used to calculate strain/stress in various directions using strain transformation equations.
- The transformation equation requires ε_x, ε_y, γ_{xy} to find out state of strain in various direction. To find out these, three normal strains ε_1, ε_2 and ε_3 are measured using strain rosette.



Thus,

$$\varepsilon_1 = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta_1 + \frac{\gamma_{xy}}{2}\sin 2\theta_1$$

$$\varepsilon_2 = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta_2 + \frac{\gamma_{xy}}{2}\sin 2\theta_2$$

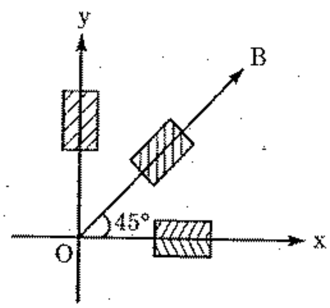
$$\varepsilon_3 = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta_3 + \frac{\gamma_{xy}}{2}\sin 2\theta_3$$

or

$$\boxed{\begin{aligned} \varepsilon_1 &= \varepsilon_x \cos^2\theta_1 + \varepsilon_y \sin^2\theta_1 + \gamma_{xy}\sin\theta_1\cos\theta_1 \\ \varepsilon_2 &= \varepsilon_x \cos^2\theta_2 + \varepsilon_y \sin^2\theta_2 + \gamma_{xy}\sin\theta_2\cos\theta_2 \\ \varepsilon_3 &= \varepsilon_x \cos^2\theta_3 + \varepsilon_y \sin^2\theta_3 + \gamma_{xy}\sin\theta_3\cos\theta_3 \end{aligned}}$$

By solving these three equation simultaneously, ε_x, ε_y and γ_{xy} can be calculated.

Note:



ε_{OB} = Normal strain along the bi sector of x and y-axis

then $\varepsilon_{OB} = \frac{\varepsilon_x + \varepsilon_y}{2} + 0 + \frac{\gamma_{xy}}{2}$

$\Rightarrow$ $\boxed{\gamma_{xy} = 2\varepsilon_{OB} - (\varepsilon_x + \varepsilon_y)}$

Thus, ε_x, ε_y, and ε_{OB} are measured by strain guage as shown above and $\boxed{\gamma_{xy} = [2\varepsilon_{OB} - (\varepsilon_x + \varepsilon_y)]}$ can be calculated.

THEORIES OF FAILURE

- We know that under uniaxial tension or compression practically, yielding begins at the yield strength at which plastic deformation is significant.
- But when several components of stress exist, the yielding depends on some combination of these components.
- Although, no theoritical method has been deviced that correlates yielding in the uniaxial tensile test with yielding in more complex loading, several theories of failure, based on predicting the onset of yielding has been proposed.
- The purpose of these theories of failure is to establish, from the behaviour of a material subjected to simple tension or compression, the point at which failure will occur under any type of combined loading.
- By failure we mean here either yielding (resulting in excessive permanent deformation) or actual ruputr (i.e. failure due to material failure).
- Structure or machine component may, on the other hand, fail due to local buckling or elastic instabilities also. However the theory of failure takes into account material failure only.
- Each theory of failure has been proposed to apply to a particular situations only.
- Thus, theory of failure for ductile material will be different from the theory of failure for brittle material. Similarly, some theory may be applicable under general loading conditions but may not be applicable for hydrostatic loading conditions.

- The theories of failures proposed in this chapter will be applicable to static loadings only.

The various theories of failure are:

(i) Max Principal Stress Theory

(Rankine Theory, Lame's theory or Max Stress theory)

For no failure, max principal stress should be less than yield stress under uniaxial loading.

i.e. $\boxed{\sigma_{max} \le f_y}$

- For design purpose $\boxed{\sigma \le \dfrac{f_y}{F.o.s.}}$ ---------- (A)
- This theory is applicable for Brittle material because brittle material fail under tension leading to fracture.
- Not suitable for ductile material in which strength is limited by shear.
- Not suitable for pure shear case because under pure shear, max principal stress = τ (shear stress) hence as per principal stress theory

$$\tau = f_y$$

but in ductile material under pure shear, strength should not exceed $\dfrac{f_y}{\sqrt{3}}$

(ii) Max Principal Strain Theory (St. Venant Theory)

- *For no failure, max principal strain should be less then or equal to the yield strain in uniaxial loading.*

$$\text{i.e.} \quad \varepsilon_1 \le \frac{f_y}{E}$$

$$\text{or} \quad \frac{\sigma_1 - \mu(\sigma_2 + \sigma_3)}{E} \le \frac{f_y}{E} \quad \text{.........(B)}$$

where σ_1, σ_2 and σ_3 are principal stresses.

- For design purpose

$$\sigma_1 - \mu(\sigma_2 + \sigma_3) \le \frac{f_y}{F.o.s.}$$

- This theory is satisfactory for brittle material.
- It over estimates the elastic strength of ductile material.
- Not suitable for hydrostatic stress condition i.e. when

$$\sigma_1 = \sigma_2 = \sigma_3 = p$$

because as per this theory

$$p - 2p.\mu \le f_y$$

$$\Rightarrow \quad p \le \frac{f_y}{1 - 2\mu}$$

$\Rightarrow$ As per this theory $p \le 2f_y$ [for steel, $\mu = 0.25$] which is not acceptable.

- Not suitable for pure shear case because in pure shear $\sigma_1 = \tau$, $\sigma_2 = -\tau$

$\Rightarrow$ $t - \mu(-\tau) \le f_y$ $\Rightarrow$ $\tau \le \frac{f_y}{1+\mu}$

for steel $\mu = .25$

$\Rightarrow$ $\tau \le 0.8\ f_y$, which is not acceptable because τ should be less then $\frac{f_y}{\sqrt{3}}$ i.e. $0.57\ f_y$

(iii) Max Shear Stress Theory (Tresca, Guest, Coulomb Theory)

Max shear stress should be less than or equal to max shear stress under uniaxial loading.

- Max Shear Stress under uniaxial loading like $f_y \leftarrow \square \rightarrow f_y$ is $\frac{f_y}{2}$.

$$\Rightarrow \text{max of}\left[\frac{|\sigma_{max} - \sigma_{min}|}{2}, \frac{|\sigma_{max}|}{2}, \frac{|\sigma_{min}|}{2}\right] \le \frac{f_y}{2} \quad \text{------------------ (C)}$$

- This theory is applicable for ductile material.
- This method gives the most conservative design out of various other theories of failure.
- Not suitable for hydrostatic loading because under hydrostatic loading when $\sigma_{max} = \sigma_{min} = p$ $\Rightarrow \tau_{max} = 0$.

(iv) Max Strain Energy Theory (Beltrami–Haigh Theory)

- *Total strain energy per unit volume absorbed at a point should be less than or equal to total strain energy per unit volume under uniaxial loading, when the material is subjected to stress upto elastic limit.*
- Total strain energy per unit volume is given by

$U = \frac{1}{2}\sigma_1\varepsilon_1 + \frac{1}{2}\sigma_2\varepsilon_2 + \frac{1}{2}\sigma_3\varepsilon_3$ (σ_1, σ_2, σ_3 are principal stresses and ε_1, ε_2 and ε_3 are principal strains)

$$= \frac{1}{2}\sigma_1\frac{[\sigma_1 - \mu(\sigma_2+\sigma_3)]}{E} + \frac{1}{2}\sigma_2\frac{[\sigma_2 - \mu(\sigma_3+\sigma_1)]}{E} + \frac{1}{2}\sigma_3\frac{[\sigma_3 - \mu(\sigma_1+\sigma_2)]}{E}$$

$$= \frac{1}{2E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\mu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)\right]$$

- Max strain energy per unit volume under uniaxial loading is $\frac{f_y^2}{2E}$

$$\Rightarrow \quad \frac{1}{2E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\mu\left(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1\right)\right] \le \frac{f_y^2}{2E}$$

$$\Rightarrow \quad \boxed{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\mu\ (\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) \le f_y^2} \quad \text{------------------ (D)}$$

$$\boxed{\text{For design purpose, } \left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\mu\ (\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)\right] \le \left(\frac{f_y}{\text{F.o.s.}}\right)^2}$$

For 2D: $\boxed{\sigma_1^2 + \sigma_2^2 - 2\mu\sigma_1\sigma_2 \le f_y^2}$

For design $\boxed{\sigma_1^2 + \sigma_2^2 - 2\mu\sigma_1\sigma_2 \le \left(\frac{f_y}{\text{F.o.s.}}\right)^2}$

- This theory is applicable for ductile material.
- Not suitable for Brittle material.
- For pure shear $\sigma_1 = \tau$, $\sigma_2 = -\tau$

$\Rightarrow \tau^2 + \tau^2 + 2\mu\tau^2 \le f_y^2$

$\Rightarrow 2\tau^2 (1 + \mu) \le f_y^2$

$\Rightarrow \tau \le \dfrac{f_y}{\sqrt{2(1+\mu)}}$

for steel $\mu = 0.25$

$\Rightarrow \tau \le 0.632\ f_y$

However, we know that τ should be less then equal to $\dfrac{f_y}{\sqrt{3}}$ i.e. $0.577\ f_y$

(v) Max Shear Strain Energy Theory (Distortion Energy Theory) (Huber – Hencky – Von Mises Theory)

Max shear strain energy in a body should be less than or equal to max shear strain energy under uni-axial loading

$$\boxed{\frac{1}{2}\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right] \le f_y^2} \quad \text{------------ (E)}$$

$$\boxed{\text{for design } \frac{1}{2}\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right] \le \left(\frac{f_y}{\text{F.o.s.}}\right)^2}$$

- This theory is applicable for ductile material.
- For 2D case

$$\boxed{\sigma_1^2+\sigma_2^2-\sigma_1\sigma_2 \le f_y^2}$$

For pure shear $\sigma_1 = \tau$, $\sigma_2 = -\tau$

$\Rightarrow 3\tau^2 \le f_y^2$

$$\boxed{\tau \le \frac{f_y}{\sqrt{3}}}$$

Hence this theory is in perfect aggrement with the case of pure shear.

Note:

- We know that normal stress causes change in volume, but shear stress causes no change in volume, it only causes distortion. Hence out of total strain energy, if strain energy due to volumetric strain is subtracted, we get strain energy due to distortion.

- Volumetric strain energy = $\dfrac{1}{2}$ × Volumetric stress × Volumetric strain

$$= \frac{1}{2}\left(\frac{\sigma_1+\sigma_2+\sigma_3}{3}\right)\left(\frac{(\sigma_1+\sigma_2+\sigma_3)(1-2\mu)}{E}\right)$$

$$= \frac{(\sigma_1+\sigma_2+\sigma_3)^2(1-2\mu)}{}$$

$\Rightarrow$ Volumetric strain energy $= \dfrac{(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + 2\sigma_1\sigma_2 + 2\sigma_2\sigma_3 + 2\sigma_3\sigma_1)(1-2\mu)}{6E}$

Distortion Energy = Total strain energy – Volumetric strain energy

$$= \frac{1}{2E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\mu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)\right] - \frac{1-2\mu}{6E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + 2\sigma_1\sigma_2 + 2\sigma_2\sigma_3 + 2\sigma_3\sigma_1\right]$$

$$= \frac{1}{6E}\begin{bmatrix} 2\sigma_1^2 + 2\sigma_2^2 + 2\sigma_3^2 - 2\sigma_1\sigma_2 - 2\sigma_2\sigma_3 - 2\sigma_3\sigma_1 - 6\mu\sigma_1\sigma_2 - 6\mu\sigma_2\sigma_3 - 6\mu\sigma_3\sigma_1 \\ + 2\mu\sigma_1^2 + 2\mu\sigma_2^2 + 2\mu\sigma_3^2 + 4\mu\sigma_1\sigma_2 + 4\mu\sigma_2\sigma_3 + 4\mu\sigma_3\sigma_1 \end{bmatrix}$$

$$= \frac{1}{6E}\left[2(1+\mu)(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) - 2(1+\mu)(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)\right]$$

$$= \frac{(1+\mu)}{3E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_3\sigma_1\right]$$

$$= \left(\frac{1+\mu}{6E}\right)\left[(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2\right]$$

$$\boxed{\text{Distortion energy} = \frac{1}{12G}\left[(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2\right]}$$

For unaxial loading $\sigma_1 = f_y$, $\sigma_2 = \sigma_3 = 0$

Distortion energy under uniaxial loading $= \dfrac{1}{12G}\left[2f_y^2\right] = \dfrac{f_y^2}{6G}$

$$\boxed{\text{Distortion energy for uniaxial loading} = \frac{f_y^2}{6G}}$$

Thus as per max distortion energy theory

$$\Rightarrow \frac{1}{12\,G}\left[(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2\right] \le \frac{f_y^2}{6G}$$

$$\Rightarrow \boxed{\frac{1}{2}\left[(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2\right] \le f_y^2}$$

(vi) Octahedral Shear Stress Theory (in 2D) is given by

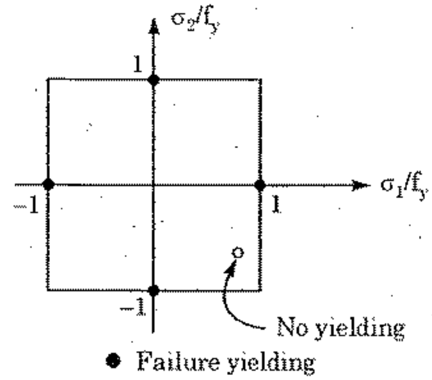
- $\boxed{\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2 \le f_y^2}$
- $\boxed{\text{for design } \sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2 \le \left(\dfrac{f_y}{\text{F.o.s.}}\right)^2}$

Thus Octahedral shear stress theory is same as max distortion energy theorem in 2D condition.

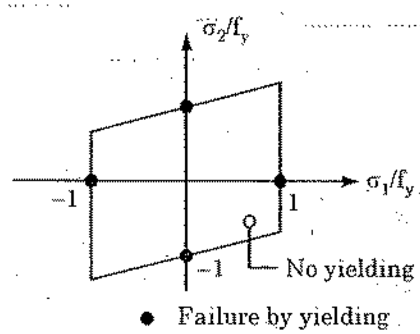
- Suitable for Ductile Material in pure shear case.

GRAPHICAL REPRESENTATION OF THEORIES OF FAILURE

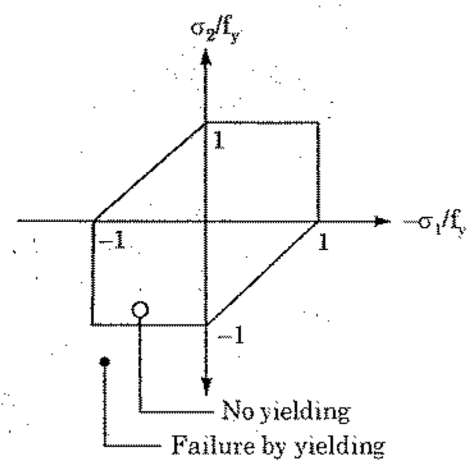
1. Max Principal Stress Theory



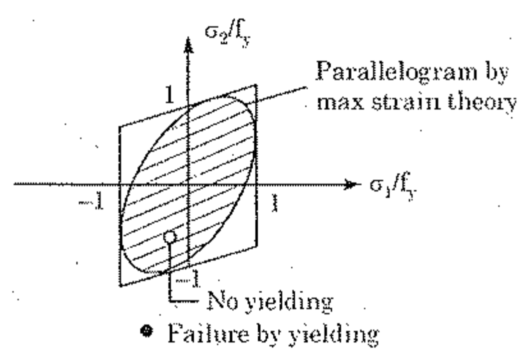
2. Max Strain Theory



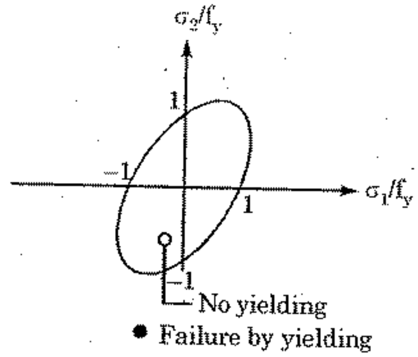
3. Max Shear Stress Theory



4. Max Strain Energy Theory



5. Max Distortion Energy Theory



CONCLUSION:

(a) Max shear stress theory → most conservative.

(b) Max distortion energy theorem → most appropriate for ductile material.

(c) Max principal stress theory → most appropriate for brittle material.

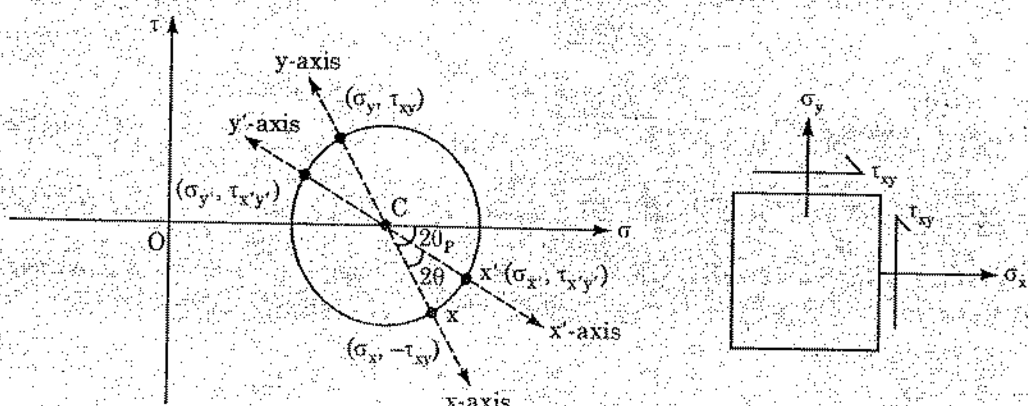
- All the theories compares the value under general state of stress with that under uniaxial loading. Hence all the theories will give similar result under uniaxial loading (or when one principal stress is large compared to other).

Example 3

Prove the validity of transformation equation

$\sigma_{x'} = \dfrac{\sigma_x+\sigma_y}{2}+\dfrac{\sigma_x-\sigma_y}{2}\cos 2\theta+\tau_{xy}\sin 2\theta$ and $\tau_{x'y'} = -\dfrac{\sigma_x-\sigma_y}{2}\sin 2\theta+\tau_{xy}\cos 2\theta$, using Stress circle approach.

Sol:



$$OC = \frac{\sigma_x+\sigma_y}{2}$$

$$CX = CX'$$

$$CX\cos 2\theta_p = \sigma_x - \frac{\sigma_x+\sigma_y}{2} = \frac{\sigma_x-\sigma_y}{2} = CX'\cos 2\theta_p$$

$$CX\sin 2\theta_p = \tau_{xy} = CX'\sin 2\theta_p$$

$$\sigma_{x'} = OC + CX'\cos(2\theta_p - 2\theta)$$

$$= OC + CX'\cos 2\theta_p\cos 2\theta + CX'\sin 2\theta_p\sin 2\theta$$

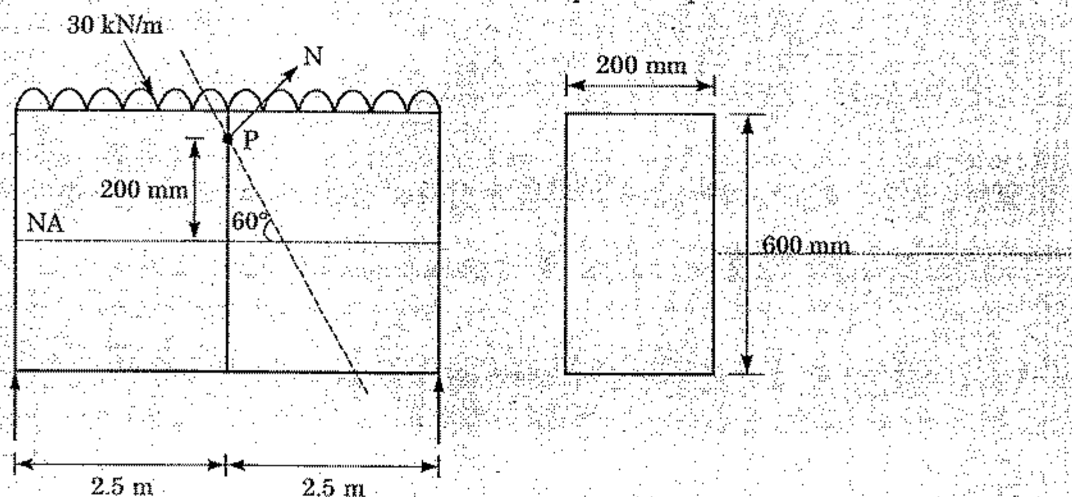
$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\tau_{x'y'} = CX' \times \sin(2\theta_p - 2\theta)$$
$$= CX'(\sin 2\theta_p \,.\, \cos 2\theta - \cos 2\theta_p \,.\, \sin 2\theta)$$

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

Example 4

A simply supported beam of span 5 m and cross-section 200 mm × 600 mm is subjected to a uniformly distributed load of 30 kN/m including self-weight. A plane inclined at 60° to the axis of the beam is passing through a point P located on the central cross-section of the beam and 200 mm above the neutral axis. Find the normal stress and shear stress on the inclined plane at point P.



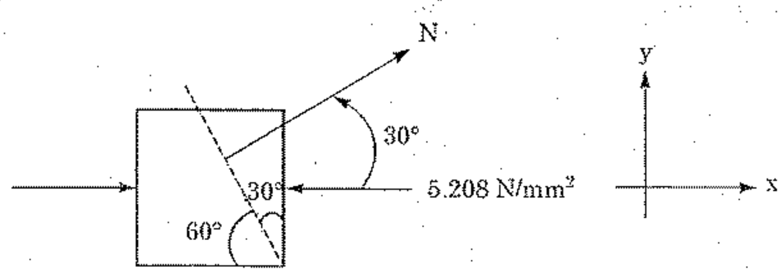
Sol:

$$\text{BM at section at mid span} = \frac{wl^2}{8} = \frac{30\,(5)^2}{8} = 93.75\text{ kNm}$$

$$\text{Bending stress at point P} = \frac{My}{I} = \frac{93.75 \times 10^6 \times 200}{\frac{200(600)^3}{12}}$$
$$= 5.208\text{ N/mm}^2$$

Shear stress at point P = $\frac{VA\bar{y}}{Ib}$ = 0 (Since shear force at mid span = 0)

The element at P =

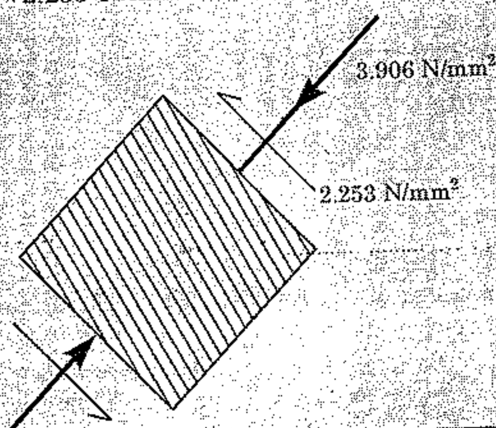


$$= \frac{-5.208}{2}\left(1+\frac{1}{2}\right)$$

$$= -3.906 \text{ N/mm}^2$$

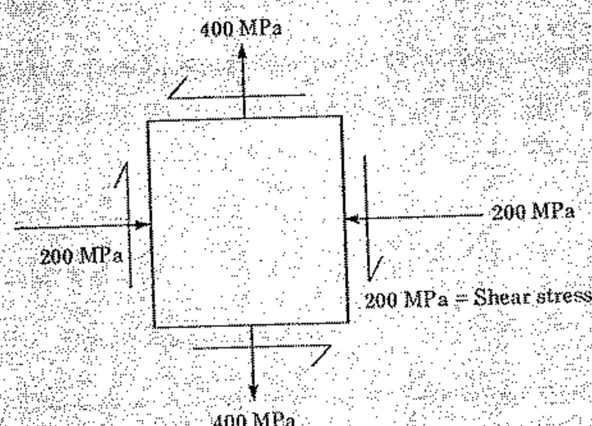
$$\text{Shear stress} = \frac{-\sigma_x}{2}\sin 60° = \frac{+5.208}{2} \times \frac{\sqrt{3}}{2}$$

$$= +2.255 \text{ N/mm}^2$$

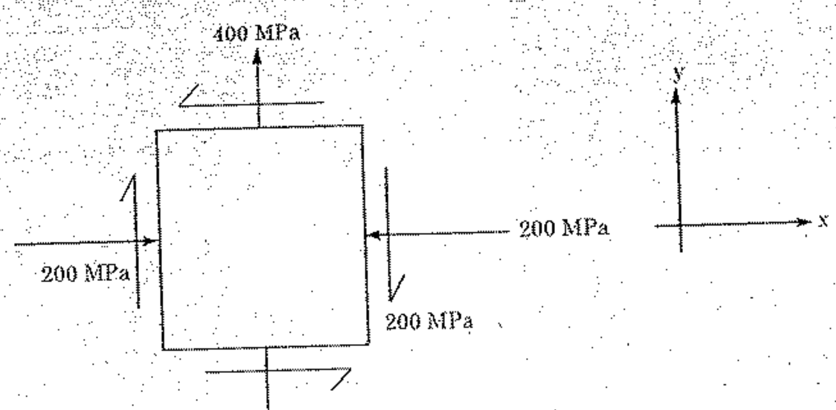


Example 5

Given the state of stress shown in the figure below, find the (i) principal stresses, and (ii) maximum shearing stresses and the associated normal stress. Calculate the principal planes and maximum shear stress plane.



Sol:



Principal stresses $\sigma_{1/2} = \frac{-200+400}{2} \pm \sqrt{\left[\frac{(400+200)}{2}\right]^2 + (200)^2}$

$$= 100 \pm \sqrt{(300)^2 + (200)^2}$$

$$= 100 \pm 360.555$$

$$= 460.555, -260.55$$

(i) σ_{max} = Max principal stress = 460.555 N/mm^2

σ_{min} = min principal stress = –260.555 N/mm^2

(ii) Max shearing stress = $\frac{\sigma_{max} - \sigma_{min}}{2}$ = 360.555 N/mm^2

(iii) Associated normal stress = $\frac{\sigma_{max} + \sigma_{min}}{2}$ = 100 N/mm^2

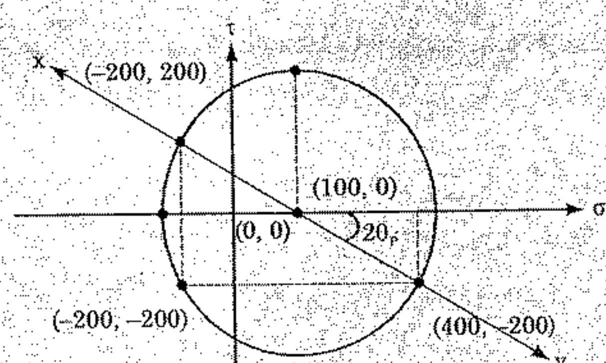
(iv) $\tan 2\theta_P = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{2 \times (-200)}{-200-400}$

$$= \frac{-400}{-600} = \frac{2}{3}$$

$\theta_P = 16.84°, \; 106.84°$ from x-axis.

Max shear stress plane = 16.84 + 45 = 61.84° from x-axis in anticlockwise direction.

Mohr Circle Approach



Radius = $\sqrt{(400-100)^2 + (200)^2}$

= 360.555

(i) σ_{max} = 460.555 N/mm^2

σ_{min} = 100 – 360.555 = –260.555 N/mm^2

(ii) Max shear stress = radius = 360.555 N/mm^2

(iii) Associated normal stress = 100 N/mm^2

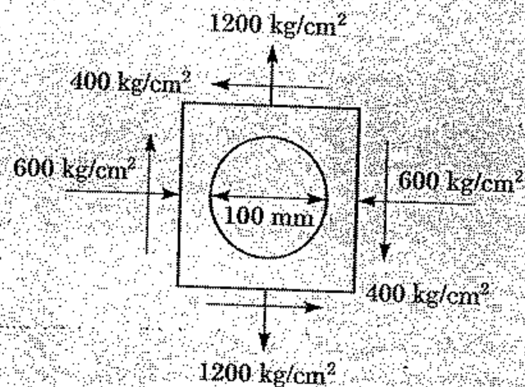
(iv) $\tan 2\theta_P = \frac{200}{300}$

θ_P = 16.84° anticlockwise from y-axis for max principal plane

Normal on Minor principal plane will be 90 + 16.84 = 106.84° anticlockwise from y-axis

Example 6

A circle of 100 mm diameter is inscribed on a steel plate before it is stressed and then the plate is loaded as shown in figure. Then the circle is deformed into an ellipse. Determine the major and minor axes of the ellipse and their directions.

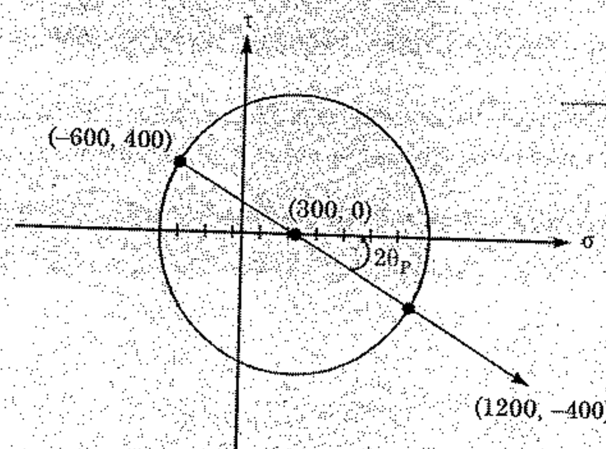


$E = 2.1 \times 10^6$ kg/cm², $\frac{1}{m} = \mu = 0.28$

Sol:

$E = 2.1 \times 10^6$ kg/cm²

$\mu = 0.28$



Radius $= \sqrt{(1200-300)^2+(400)^2}$

$= 984.886$

$\Rightarrow \sigma_{max} = 300 + 984.885 = 1284.886$ kg/cm²

$\sigma_{min} = 300 - 984.885 = -684.885$ kg/cm²

$$\varepsilon_{max} = \frac{\sigma_{max} - \mu\sigma_{min}}{E} = \frac{1284.886 + 0.28 \times 684.886}{2.1 \times 10^6}$$

$= 7.0317 \times 10^{-4}$

$$\varepsilon_{min} = \frac{\sigma_{min} - \mu\sigma_{max}}{E} = \frac{-684.886 - 0.28 \times 1284.886}{2.1 \times 10^6}$$

$= -4.9745 \times 10^{-4}$

$\Rightarrow$ Major axis $= 100 \times 7.0317 \times 10^{-4} + 100$

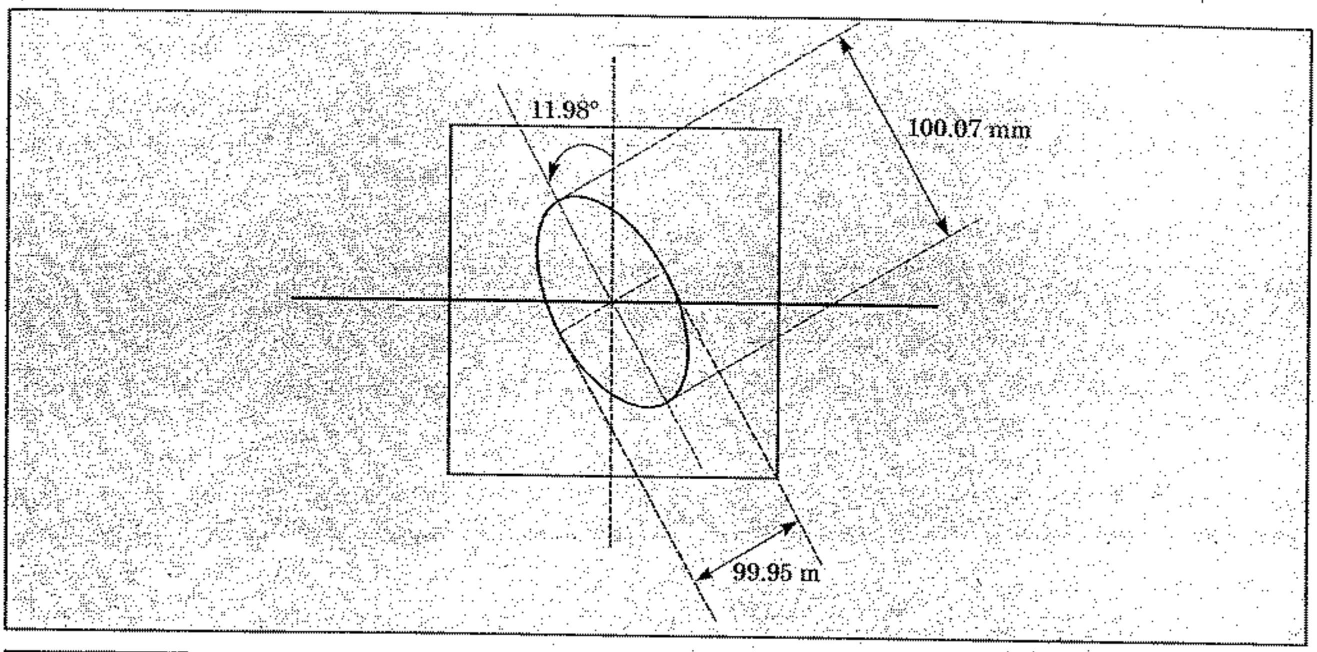
$= 100.0703$ mm

Minor axis $= -100 \times 4.9745 \times 10^{-4} + 100$

$= 99.95$ mm

$$\tan 2\theta_p = \frac{400}{900}$$

$\theta_p = 11.98$ anticlockwise from plane on which 1200 kg/cm² stress is acting.



Example 7

The stresses in a flat steel plate in a condition of plane stress are:

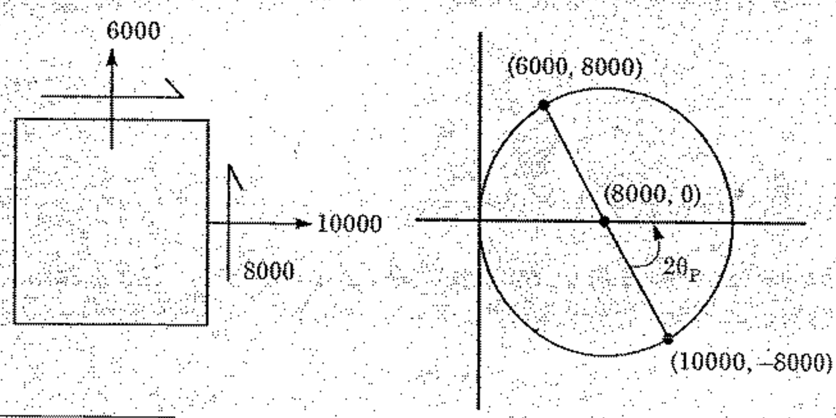
$\sigma_x = 10000$ N/mm²

$\sigma_y = 6000$ N/mm²

$\tau_{xy} = 8000$ N/mm²

Find the magnitude and orientation of the principal stresses in the plane of the plate.

Sol:



$$\text{Radius} = \sqrt{(2000)^2 + (8000)^2}$$

$$= 8246.21 \text{ N/mm}^2$$

$$\Rightarrow \sigma_{max} = 8000 + 8246.21 = 16246.21 \text{ N/mm}^2$$

$$\sigma_{min} = -246.21 \text{ N/mm}^2$$

$$\tan 2\theta_P = \frac{8000}{2000}$$

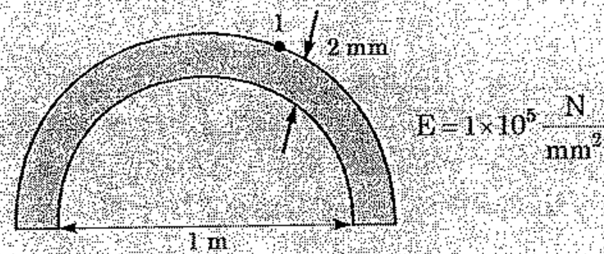
$$\theta_P = 37.98°$$

Major principal stress plane is such that normal to it makes 37.98 angle in anticlockwise direction from the plane on which 10000 N/mm² normal stress acts.

Example 8

A wire of 2 mm diameter is wound round a circular shaft of 1 m diameter. What are the maximum normal and shear stresses in the wire? Specify the planes of their occurrence. $E = 1 \times 10^5$ N/mm^2. Assume no applied tensile pull on the wire.

Sol:

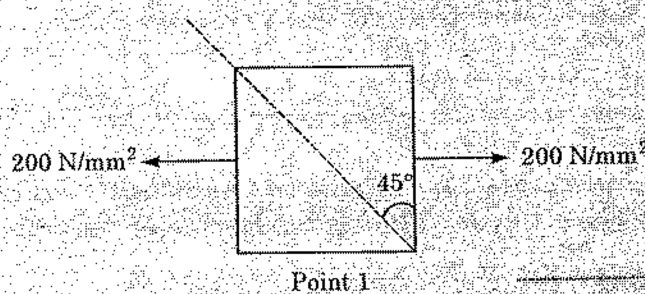


$$\frac{M}{I} = \frac{f}{y} = \frac{E}{R}$$

$$f = \frac{Ey}{R} = \frac{1 \times 10^5 \times 1mm}{0.5 \times 1000\ mm}\ N/mm^2$$

$$f = 200\ N/mm^2$$

Bending stress will give max direct stress of 200 N/mm^2.



$$\text{Max normal stress} = 200\ N/mm^2$$

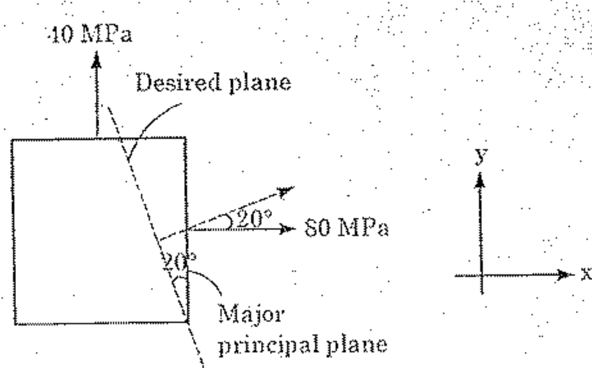
$$\text{Max shear stress} = \frac{200-0}{2} = 100\ N/mm^2$$

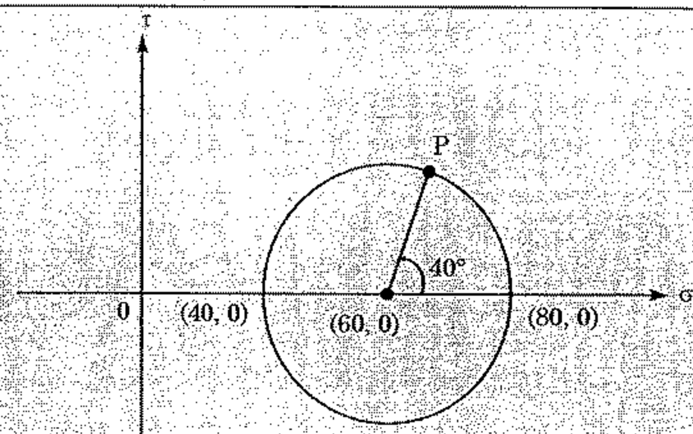
Max normal stress will occurs an shown on the element. Max shear stress will occur at 45° to max principal plane

Example 9

The principal tensile stresses at a point are: $\sigma_x = 80$ MPa and $\sigma_y = 40$ MPa. Find the magnitudes of normal, tangential and the resultant stress on a plane at 20° with the major principal plane. What is the angle of obliquity of the resultant stress with the major principal plane? Show by means of a sketch how normal stresses (σ_n) and tangential stress (σ_t) act on the triangular element.

Sol:





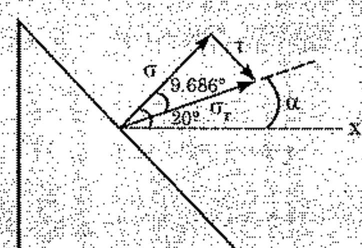
Radius = 20

Normal stress on P = 20 cos 40° + 60 = 75.32 MPa

Stress stress on P = 20 sin 40° = 12.856 MPa

Resultant stress = $\sqrt{(75.32)^2 + (12.856)^2}$ = 76.41 N/mm²

$\tan\phi = \dfrac{12.856}{75.32}$

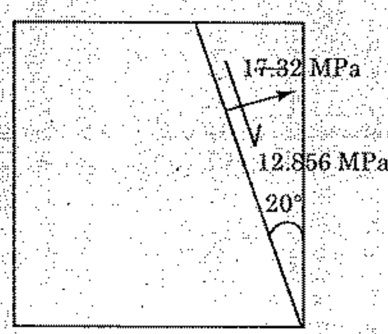


⇒ ϕ = 9.686° = angle of obliquity

α = angle of obliquity with major principal stress

= α = 20° – 9.686° = 10.314°

⇒ Angle that resultant stress makes with major principal stress = 10.314°.



Example 10

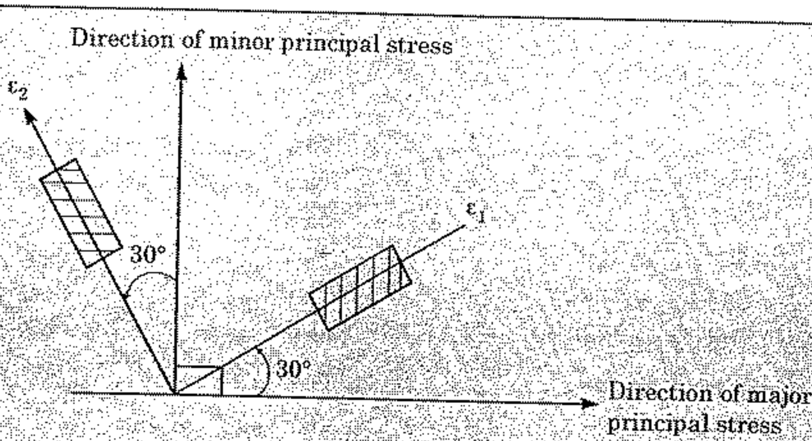
$\varepsilon_1 = 445 \times 10^{-6}$

$\varepsilon_2 = -32 \times 10^{-6}$

$E = 2.1 \times 10^5$ N/mm²

$\mu = 0.3$

Determine the magnitude of principal st...



Sol: Let ε_{max} and ε_{min} are the principal strains

$$\Rightarrow \quad \varepsilon_1 = \frac{\varepsilon_{max}+\varepsilon_{min}}{2} + \frac{\varepsilon_{max}-\varepsilon_{min}}{2}\cos 2(30°)$$

$$= \frac{\varepsilon_{max}+\varepsilon_{min}}{2} + \frac{\varepsilon_{max}-\varepsilon_{min}}{4}$$

$$\varepsilon_1 = \frac{3}{4}\varepsilon_{max} + \frac{\varepsilon_{min}}{4}$$

$$\boxed{3\varepsilon_{max}+\varepsilon_{min} = 4\times 445\times 10^{-6} = 1780\times 10^{-6}} \quad \text{------ (A)}$$

$$\varepsilon_2 = \frac{\varepsilon_{max}+\varepsilon_{min}}{2} + \frac{\varepsilon_{max}-\varepsilon_{min}}{2}\cos(240°)$$

$$= \frac{\varepsilon_{max}+\varepsilon_{min}}{2} + \frac{\varepsilon_{max}-\varepsilon_{min}}{2}\left(-\frac{1}{2}\right)$$

$$\varepsilon_2 = \frac{\varepsilon_{max}}{4} + \frac{3\varepsilon_{min}}{4}$$

$$\Rightarrow \quad \boxed{\varepsilon_{max}+3\varepsilon_{min} = -128\times 10^{-6}} \quad \text{------ (B)}$$

From (A) and (B)

$$\varepsilon_{max} = 683.5\times 10^{-6} = \frac{\sigma_{max}-\mu\sigma_{min}}{E}$$

$$\varepsilon_{min} = -270.5\times 10^{-6} = \frac{\sigma_{min}-\mu\sigma_{max}}{E}$$

$$\sigma_{max} - 0.3\,\sigma_{min} = 143.535$$

$$\Rightarrow \quad \sigma_{min} - 0.3\,\sigma_{max} = -56.805$$

$$\Rightarrow \quad \boxed{\sigma_{min} = -15.104\ \text{N/mm}^2}$$

$$\boxed{\sigma_{max} = 139\ \text{N/mm}^2}$$

Example 11

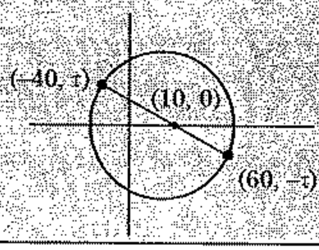
At a point in an elastic material, a direct tensile stress of 60 kN/mm^2 and a direct compressive stress of 40 kN/mm^2 are applied on planes at right angles to each other. If the maximum principal stresses are limited to 75 N/mm^2 (tensile). Find the shear stress that may be allowed on the planes. Also determine the minimum principal stress and the maximum shear stress.

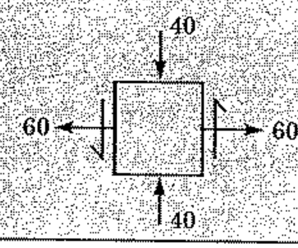
Sol:

Max principal stress $= 10+\sqrt{(50)^2+\tau^2} \le 75$

$\Rightarrow \tau \le 41.53$ N/mm^2

$\Rightarrow$ Max τ that is allowed $= 41.53$ N/mm^2.





Example 12

The principal stresses at a point in an elastic material are 1.5σ (tensile), σ (tensile) and 0.5σ (compressive). The elastic limit in tension is 210 MPa and μ = 0.3. What would be the value of σ at failure when computed by different theories of failure.

Sol: Principal stresses are 1.5 σ (tensile)

σ (tensile)

0.5 σ (compressive)

Elastic limit in tension = 210 MPa

$\mu = 0.3$

(1) Max principal stress theory

$1.5\sigma \le 210$

$\sigma \le 140$ N/mm^2

(2) Max principal strain theory

$$\frac{\sigma_1 - 0.3\,(\sigma_2+\sigma_3)}{E} \le \frac{f_y}{E}$$

$1.5\sigma - 0.3\,(\sigma - 0.5\,\sigma) \le 210$

$\boxed{\sigma \le 155.55}$

(3) Maximum shear stress theory

$$\text{Max shear stress} \le \frac{f_y}{2}$$

$$\frac{1.5\sigma-(-0.5\sigma)}{2} \le \frac{210}{2}$$

$2\sigma \le 210$

$\sigma \le 105$ N/mm^2

(4) Max strain energy theory

$$\sigma_1^2+\sigma_2^2+\sigma_3^2-2\mu\,(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1) \le f_y^2$$

$(1.5\sigma)^2 + \sigma^2 + (0.5\sigma)^2 - 0.6\,(1.5\,\sigma^2 - 0.5\sigma^2 - 0.75\sigma^2) \le f_y^2$

$3.35\,\sigma^2 \le f_y^2$

$$\sigma \le \sqrt{\frac{(210)^2}{3.35}}$$

$$\sigma \le 114.7 \text{ N/mm}^2$$

(5) Max distortion energy theorem

$$\frac{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2}{2} \le f_y^2$$

$$(0.5\sigma)^2 + (1.5\sigma)^2 + (-2.0\sigma)^2 \le 2 \times (210)^2$$

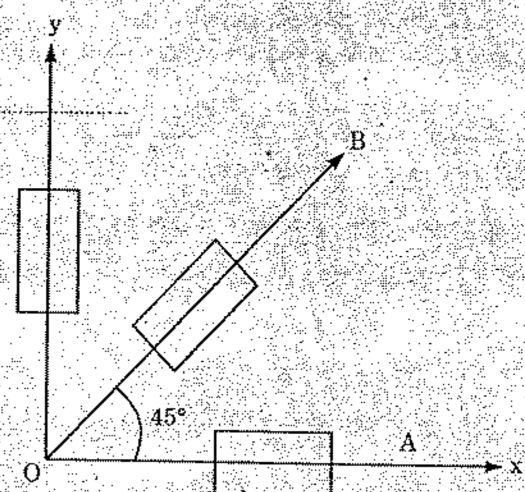
$$0.25\sigma^2 + 2.25\sigma^2 + 4\sigma^2 \le 2 \times (210)^2$$

$$6.5\sigma^2 \le 2 \times (210)^2$$

$$\boxed{\sigma \le 116.487 \text{ N/mm}^2}$$

Example 13

The strain measurements from a rectangular strain Rosette were $e_0 = 600 \times 10^{-6}$, $e_{45} = 500 \times 10^{-6}$ and $e_{90} = 200 \times 10^{-6}$. Find the magnitude and direction of principal strains. If $E = 2 \times 10^5$ N/mm² and $\mu = 0.3$. Find the principal stresses.



Sol:

$$e_0 = 600 \times 10^{-6}$$

$$e_{45} = 500 \times 10^{-6}$$

$$e_{90} = 200 \times 10^{-6}$$

$$E = 2 \times 10^5 \text{ N/mm}^2$$

$$\mu = 0.3$$

$$e_{OB} = \frac{e_x+e_y}{2} + \frac{e_x-e_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

$$e_{45} = \frac{e_x+e_y}{2} + 0 + \frac{\gamma_{xy}}{2}$$

$$500 \times 10^{-6} = \frac{600+200}{2} \times 10^{-6} + \frac{\gamma_{xy}}{2}$$

$$100 \times 10^{-6} = \frac{\gamma_{xy}}{2}$$

$$\gamma_{xy} = 200 \times 10^{-6}$$

$$\varepsilon_{1/2} = \frac{\varepsilon_x + \varepsilon_y}{2} \pm \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$$\varepsilon_{1/2} = 400 \times 10^{-6} \pm \sqrt{(200 \times 10^{-6})^2 + (100 \times 10^{-6})^2}$$

$$\varepsilon_{1/2} = (400 \pm 223.607) \times 10^{-6}$$

$$\boxed{\varepsilon_1 = 623.607 \times 10^{-6}} \quad \text{Major principal strain}$$

$$\boxed{\varepsilon_2 = 176.393 \times 10^{-6}} \quad \text{Minor principal strain}$$

$$\tan 2\theta_p = \frac{\gamma_{xy/2}}{\frac{\varepsilon_x - \varepsilon_y}{2}} = \frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y} = \frac{200}{400}$$

$$\boxed{\theta_p = 13.282°} \qquad \boxed{103.282°}$$

Putting $\theta_p = 13.282°$

$$e = \frac{e_x + e_y}{2} + \frac{e_x - e_y}{2} \cos(2 \times 13.282°) + \frac{\gamma_{xy}}{2} \sin(2 \times 13.282°)$$

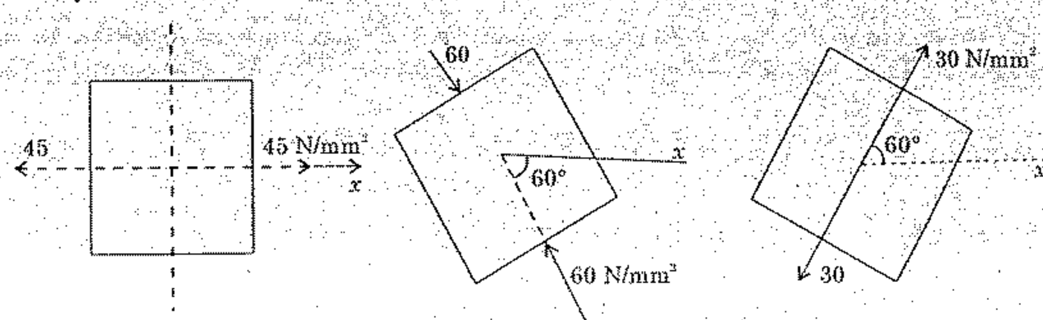
$$= (400 \times 10^{-6}) + (200 \times 10^{-6} \times 0.894) + (100 \times 0.4472 \times 10^{-6})$$

$$e = 623.607 \times 10^{-6}$$

Major principal strain plane is inclined at 13.282° to x-axis.

Example 14

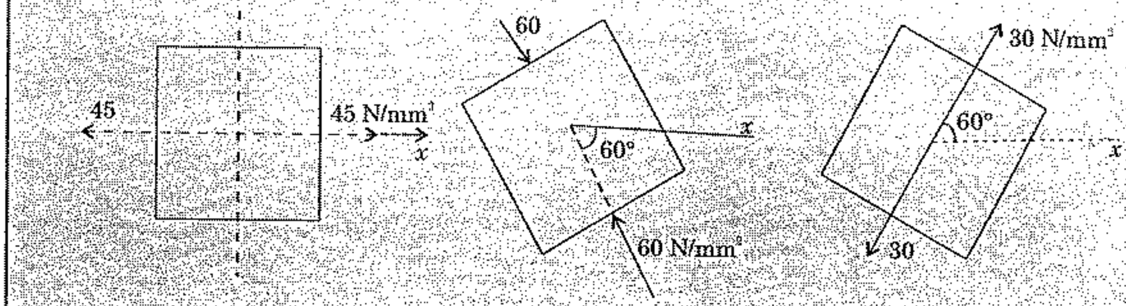
The state of stress at a point is the result of the three separate actions that produce the three states of stresses shown in Fig. Determine the principal stresses and principal planes caused by the superposition of these three stress states.



(a) (b) (c)

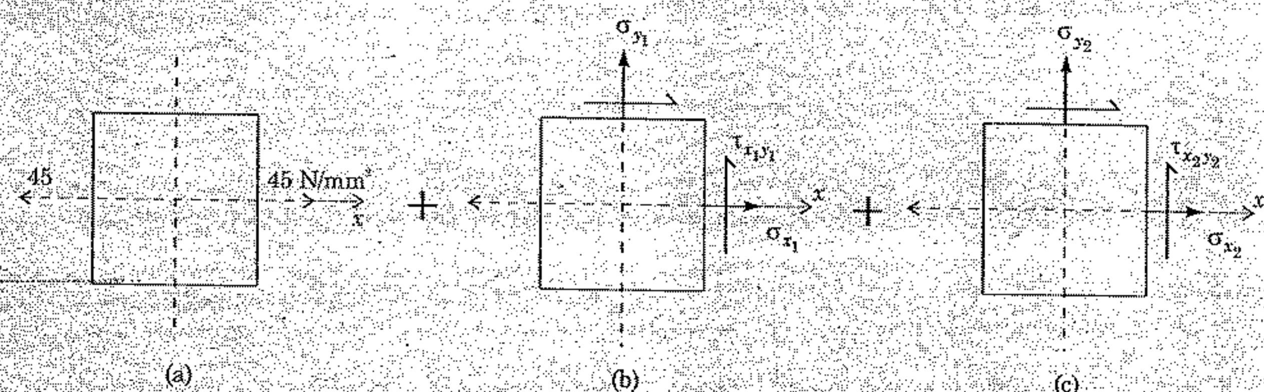
Sol. All the stress element can be transformed in such way that the transformed elements becomes parallel to that shown in figure (a) above.

Original

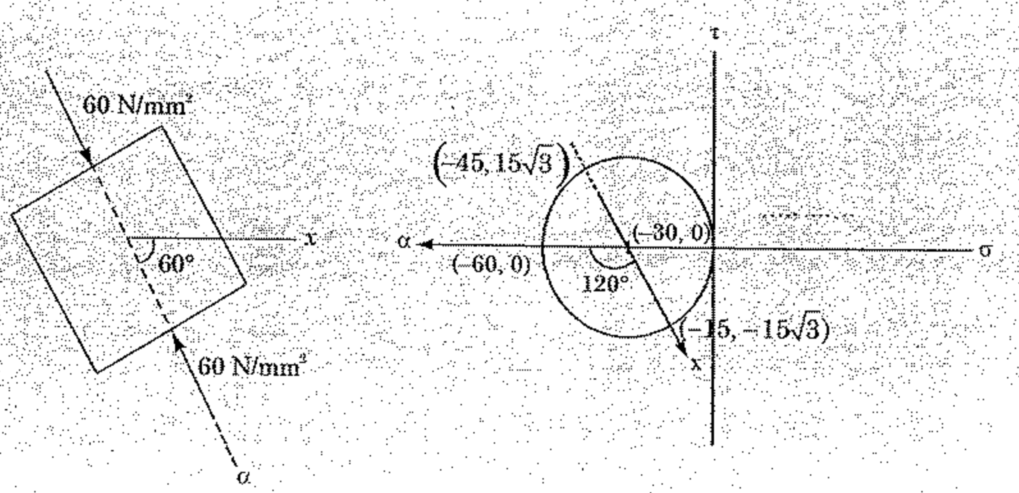


The net combination will be like as shown below.

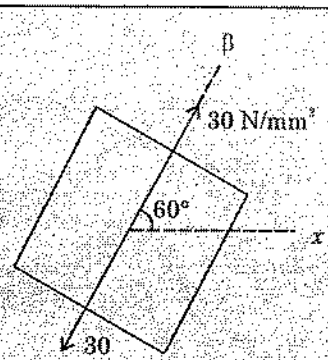
Transformed

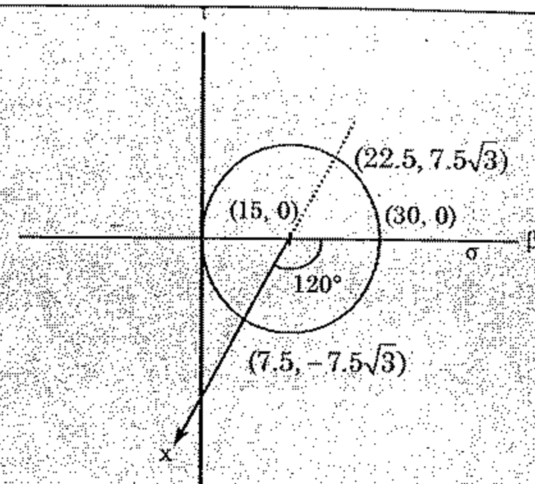


Let us now calculate $\sigma_{x_1}, \tau_{x_1y_1}, \sigma_{y_1}, \sigma_{x_2}, \tau_{x_2y_2}$ and σ_{y_2}.



Hence $\sigma_{x_1} = -15$ N/mm² and $\tau_{x_1y_1} = -15\sqrt{3}$ N/mm² and $\sigma_{y_1} = -45$ N/mm²





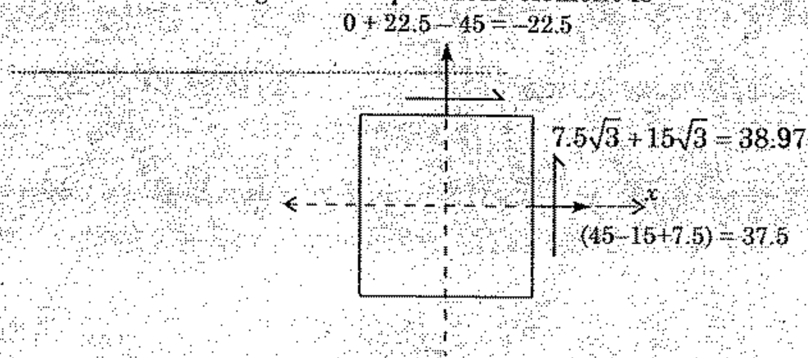
Hence

$$\sigma_{x_2} = 7.5 \text{ N/mm}^2$$

$$\sigma_{y_2} = 22.5 \text{ N/mm}^2$$

$$\tau_{x_2y_2} = -7.5\sqrt{3} \text{ N/mm}^2$$

Thus, the combined loading on the equivalent element is



$$\sigma_{1/3} = \frac{37.5 - 22.5}{2} \pm \sqrt{\left(\frac{37.5 + 22.5}{2}\right)^2 + (38.97)^2}$$

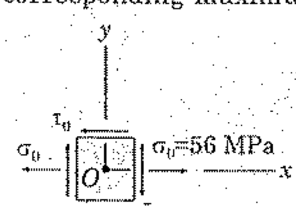
i.e. $\sigma_1 = 56.7 \text{ N/mm}^2$, $\sigma_3 = -41.7 \text{ N/mm}^2$

$$\tan 2\theta_p = \frac{38.97 \times 2}{37.5 + 22.5}$$

i.e. $\theta_p = 26.1°$, $116.205°$ ⇒ Principal planes are inclined at 26.1° and 116.205° with x-axis.

Example 15

A state of plane stress consists of a tensile stress $\sigma_0 = 56$ MPa exerted on vertical surfaces and of unknown shearing stresses. Determine (a) the magnitude of the shearing stress τ_0 for which the largest normal stress is 70 MPa, (b) the corresponding maximum shearing stress.



Sol. Construction of Mohr's Circle: We assume that the shearing stresses act in the senses shown. Thus, the shearing stress τ_0 on a face perpendicular to the x axis tends to rotate the element clockwise and we plot the point X of coordinates 56 MPa and τ_0 above the horizontal axis. Considering a horizontal face of the element, we observe that $\sigma_y = 0$ and that τ_0 tends to rotate the element counterclockwise; thus, we plot point Y at a distance τ_0 below O.

We note that the abscissa of the center C of Mohr's circle is

$$\sigma_{ave} = \frac{1}{2}(\sigma_x + \sigma_y) = \frac{1}{2}(56+0) = 28 \text{ MPa}$$

The radius R of the circle is determined by observing that the maximum normal stress, $\sigma_{max} = 70$ MPa, is represented by the abscissa of point A and writing

$$\sigma_{max} = \sigma_{ave} + R$$

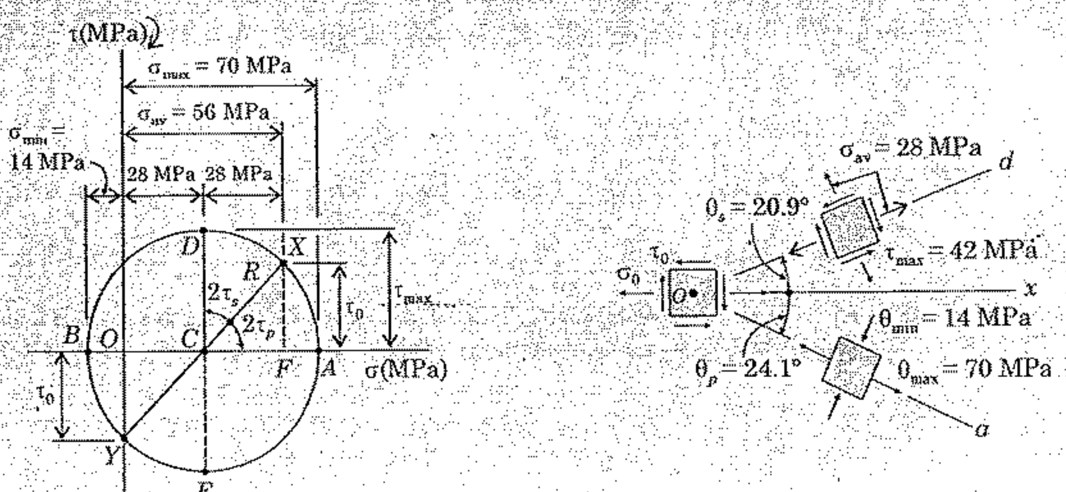
70 MPa = 28 MPa + R R = 42 MPa

(a) Shearing Stress τ_0: Considering the right triangle CFX, we find

$$\cos 2\theta_p = \frac{CF}{CX} = \frac{CF}{R} = \frac{28 \text{ MPa}}{42 \text{ MPa}}$$

$$2\theta_p = 48.2^\circ \qquad \theta_p = 24.1^\circ$$

$$\tau_0 = FX = R\sin 2\theta_p = (42 \text{ MPa})\sin 48.2^\circ \qquad \tau_0 = 31.3 \text{ MPa}$$



(b) Maximum Shearing Stress: The coordinates of point D of Mohr's circle represent the maximum shearing stress and the corresponding normal stress.

$$2\theta_s = 90^\circ - 2\theta_p = 90^\circ - 48.2^\circ = 41.8^\circ \quad \tau_{max} = R = 42 \text{ MPa} \quad \tau_{max} = 42 \text{ MPa}$$

$$2\theta_s = 90^\circ - 2\theta_p = 90^\circ - 48.2^\circ = 41.8^\circ \qquad \theta_x = 20.9^\circ$$

The maximum shearing stress is exerted on an element that is oriented as shown in Fig. a.

Example 16

A piece of material is subjected to two perpendicular stresses p_1 (tensile) and p_2 (compressive). Find an expression for the strain energy per unit volume.

If a stress of 128 N/mm² acting alone gives the same value of strain energy as the expression already found, find the value of p_2 when p_1 = 112 N/mm². Take μ = 0.3

Sol. We have

$$U=\frac{1}{2E}\left(p_1^2+p_2^2-2\mu p_1 p_2\right) \text{ per unit volume.}$$

Here since p_2 is compressive; hence reversing the sign of p_2 we get

$$U=\frac{1}{2E}\left(p_1^2+p_2^2+2\mu p_1 p_2\right) \text{ per unit volume} \qquad \text{(where } f_2 \text{ is compressive)}$$

For a single stress p acting alone, $U=\frac{p^2}{2E}$ per unit volume

$$\therefore \frac{p^2}{2E}=\frac{1}{2E}\left(p_1^2+p_2^2+2\mu p_1 p_2\right)$$

$$\text{or } (128)^2=\left\{(112)^2+p_2^2+2\times 112 p_2\times 0.3\right\}$$

$$\text{or } p_2^2+67.2p-3840=0$$

$$\text{From which } p=\left[-67.2\pm\sqrt{4515.84+4\times 3840}\right]\frac{1}{2}$$

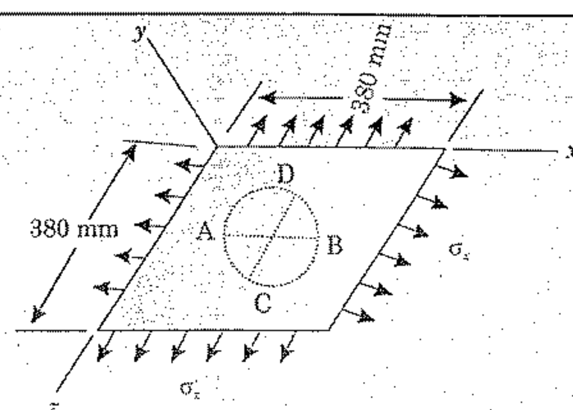
$$\text{or } p=\left[-67.2\pm 141\right]\frac{1}{2}=-33.6\pm 70.5$$

Since p_2 has already been assumed negative, *i.e.* compressive, the positive sign must be taken

$$p_2=-33.6+70.5=36.9 \text{ N/mm}^2$$

Example 17

A circle of diameter d = 225 mm is scribed on an unstressed aluminum plate of thickness t = 18 mm. Forces acting in the plane of the plate later cause normal stresses $\sigma_x = 84$ MPa and $\sigma_z = 140$ MPa. For E = 70 GPa and $\upsilon=\frac{1}{3}$ (υ = poisson's ratio), determine the change in



(i) the length of diameter AB,

(ii) the length of diameter CD,

(iii) the thickness of the plate,

Sol. Hooke's Law: We note that $\sigma_y = 0$, Using equations(2.28) we find the strain in each of the coordinate directions.

$$\epsilon_x = +\frac{\sigma_x}{E} - \frac{\upsilon\sigma_y}{E} - \frac{\upsilon\sigma_z}{E}$$

$$= \frac{1}{70\text{ GPa}}\left[(84\text{ MPa}) - 0 - \frac{1}{3}(140\text{ MPa})\right] = +0.533 \times 10^{-3}\text{ mm/mm}$$

$$\epsilon_y = -\frac{\upsilon\sigma_x}{E} + \frac{\sigma_y}{E} - \frac{\upsilon\sigma_z}{E}$$

$$= \frac{1}{70\text{ GPa}}\left[-\frac{1}{3}(84\text{ MPa}) + 0 - \frac{1}{3}(140\text{ MPa})\right] = -1.067 \times 10^{-3}\text{ mm/mm}$$

$$\epsilon_z = -\frac{\upsilon\sigma_x}{E} - \frac{\upsilon\sigma_y}{E} + \frac{\sigma_z}{E}$$

$$= \frac{1}{70\text{ GPa}}\left[-\frac{1}{3}(84\text{ MPa}) - 0 + (140\text{ MPa})\right] = +1.600 \times 10^{-3}\text{ mm/mm}$$

(i) Diameter AB: The change in length is $\delta_{B/A} = \epsilon_x d$

$\delta_{B/A} = \epsilon_x d = (+0.533 \times 10^{-3}$ mm/mm) (225 mm)

$\delta_{B/A} = +0.12$ mm

(ii) Diameter CD

$\delta_{C/D} = \epsilon_z d = (+1.600 \times 10^{-3}$ mm/mm) (225 mm)

$\delta_{C/D} = +0.36$ mm

(iii) Thickness: Recalling that $t = 18$ mm, we have

$\delta_t = \epsilon_y t = (-1.067 \times 10^{-3}$ mm/mm) (18 mm)

$\delta_t = -0.0192$ mm

(iv) Volume of the Plate:

$$e = \varepsilon_x + \varepsilon_y + \varepsilon_z = (+0.533 - 1.067 + 1.600)10^{-3} = +1.067 \times 10^{-3}$$

$$\Delta V = eV = +1.067 \times 10^{-3}\,[(380\text{ mm})(380\text{ mm})(18\text{ mm})] = 2733\text{ mm}^3$$

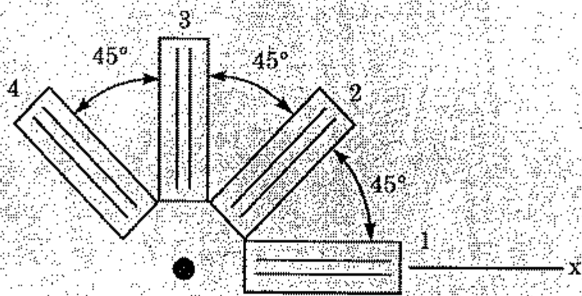
Example 18

The rosette shown has been used to determine the following strains at a point on the surface of a crane book:

$\epsilon_1 = +420\mu$ $\epsilon_2 = -45\mu$ $\epsilon_3 = +165\mu$

(a) What should be the reading of gage 3?

(b) Determine the principal strains and the maximum in-plane shearing strain.



Sol:

(a) Gases 2 and 4 are 90° apart $\varepsilon_{ave} = \frac{1}{2}(\varepsilon_2 + \varepsilon_4)$

$$\varepsilon_{ave} = \frac{1}{2}(-45\,\mu + 165\,\mu) = 60\text{ m}$$

Gases 1 and 3 are also 90° apart $\varepsilon_{ave} = \frac{1}{2}(\varepsilon_1 + \varepsilon_3)$

$$\varepsilon_3 = 2\varepsilon_{ave} - \varepsilon_1 = (2)(60\,\mu) - 420\,\mu = -300\,\mu$$

(b) $\varepsilon_x = 420\,\mu \qquad \varepsilon_y = \varepsilon_3 = -300\,\mu$

$$\gamma_{xy} = 2\varepsilon_2 - \varepsilon_x - \varepsilon_1 = (2)(-45\,\mu) - 420\,\mu + 300\,\mu$$

$$= -210\,\mu$$

$$R = \sqrt{\frac{(\varepsilon_x - \varepsilon_y)^2}{2} + \left(\frac{\gamma_{xy}}{2}\right)^2} = \sqrt{\left(\frac{420\mu + 300\mu}{2}\right)^2 + \left(-\frac{210\mu}{2}\right)^2}$$

$$= 375\,\mu$$

$$\varepsilon_a = \varepsilon_{ave} + R = 60\,\mu + 375\,\mu = 435\,\mu$$

$$\varepsilon_b = \varepsilon_{ave} - R = 60\mu - 375\,\mu = -315\,\mu$$

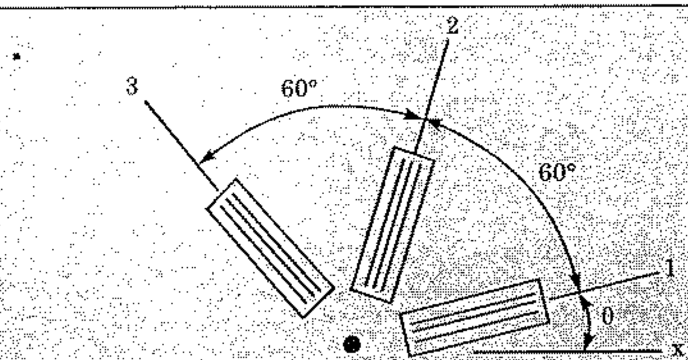
$$\gamma_{max\ (inplane)} = 2R = 750\,\mu$$

Example 19

Show that the sum of the three strain measurements made with a 60° rosette is independent of the orientation of the rosette and equal to

$$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 3\varepsilon_{ave}$$

where ε_{ave} is the abscissa of the center of the corresponding Mohr's circle for strain.



Sol: $\varepsilon_1 = \varepsilon_{ave} + \dfrac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \dfrac{\gamma_{xy}}{2}\sin 2\theta$ ---------- (i)

$$\varepsilon_2 = \varepsilon_{ave} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos(2\theta + 120°) + \frac{\gamma_{xy}}{2}\sin(2\theta + 120°)$$

$$= \varepsilon_{ave} + \frac{\varepsilon_x - \varepsilon_y}{2}(\cos 120°\cos 2\theta - \sin 120°\sin 2\theta) + \frac{\gamma_{xy}}{2}(\cos 120°\sin 2\theta + \sin 120°\cos 2\theta)$$

$$= \varepsilon_{ave} + \frac{\varepsilon_x - \varepsilon_y}{2}\left(-\frac{1}{2}\cos 2\theta - \frac{\sqrt{3}}{2}\sin 2\theta\right) + \frac{\gamma_{xy}}{2}\left(-\frac{1}{2}\sin 2\theta + \frac{\sqrt{3}}{2}\cos 2\theta\right) \text{ ---------- (ii)}$$

$$\varepsilon_3 = \varepsilon_{ave} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos(2\theta + 240°) + \frac{\gamma_{xy}}{2}\sin(2\theta + 240°)$$

$$= \varepsilon_{ave} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos(2\theta + 240°) + \frac{\gamma_{xy}}{2}\sin(2\theta + 240°)$$

$$= \varepsilon_{ave} + \frac{\varepsilon_x - \varepsilon_y}{2}(\cos 240°\cos 2\theta - \sin 240°\sin 2\theta) + \frac{\gamma_{xy}}{2}(\cos 240°\sin 2\theta + \sin 240°\cos 2\theta)$$

$$= \varepsilon_{ave} + \frac{\varepsilon_x - \varepsilon_y}{2}\left(-\frac{1}{2}\cos 2\theta + \frac{\sqrt{3}}{2}\sin 2\theta\right) + \frac{\gamma_{xy}}{2}\left(-\frac{1}{2}\sin 2\theta - \frac{\sqrt{3}}{2}\cos 2\theta\right) \text{ ---------- (iii)}$$

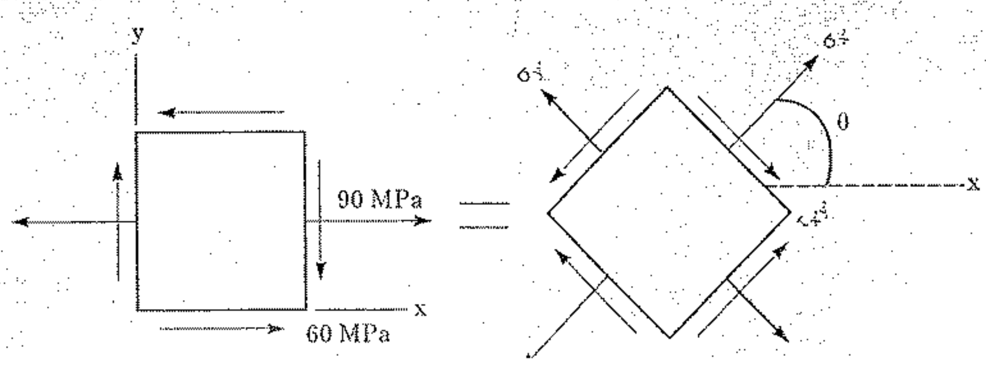
Adding (1), (2), and (3)

$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 3\varepsilon_{ave} + 0 + 0$

$3\varepsilon_{ave} = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$

Example 20

For the state of stress shown, determine the range of values of θ for which the normal stress $\sigma_{x'}$ is equal to or less than 100 MPa.



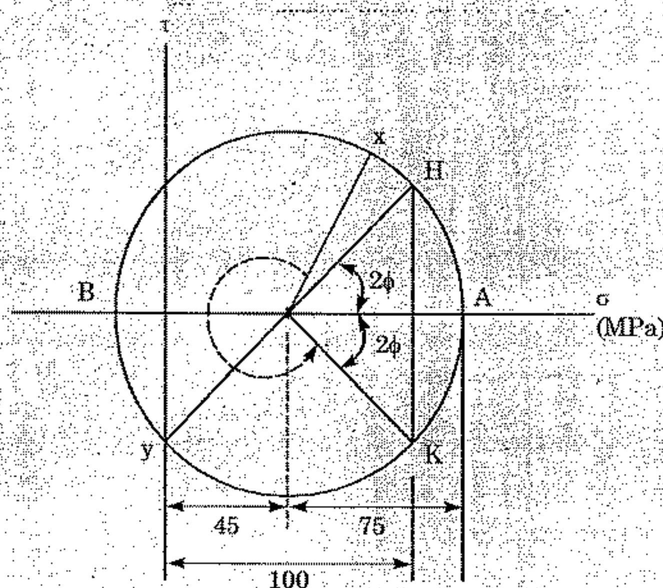
Sol: $\sigma_x = 90$ MPa, $\sigma_y = 0$, $\gamma_{xy} = -60$ MPa

$$\sigma_{ave} = \frac{1}{2}(\sigma_x + \sigma_y) = 45 \text{ MPa}$$

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \gamma_{xy}^2}$$

$$= \sqrt{45^2 + 60^2} = 75 \text{ MPa}$$

$$\tan 2\theta_p = \frac{2\gamma_{xy}}{\sigma_x - \sigma_y} = \frac{(2)(-60)}{90} = -\frac{4}{3}$$



$2\theta_p = -53.13°$

$\theta_p = -26.565°$

$\sigma_x \le 100$ MPa for states of stress corresponding to arc HBK of Mohr's circle.

From the circle

$R \cos 2\phi = 100 - 45 = 55$ MPa

$$\cos 2\phi = \frac{55}{75} = 0.73333$$

$2\phi = 42.833°$ $\phi = 21.417°$

$\theta_H = \theta_p + \phi = -26.565° + 21.417° = -5.15°$

$2\theta_K = 2\theta_H + 360° - 4\phi = -10.297° + 360° - 85.666° = 264.037°$

$\theta_K = 132.02°$

Permissible range of ϕ is $\theta_H \le \theta \le \theta_K$

$-5.15° \le \theta \le 132.02°$

Also $174.85° \le \theta \le 312.02°$

OBJECTIVE QUESTIONS

1. Consider the following statements:
   1. On planes having maximum and minimum principal stresses, there will be no tangential stress.
   2. Shear stresses on mutually perpendi-cular planes are numerically equal.
   3. Maximum shear stress is equal to half the sum of the maximum and minimum principal stresses.

   Which of these statements is/are correct?

   (a) Only 1 is (b) 1 and 2
   (c) 2 and 3 (d) 1 and 3

2. In a stressed body, an elementary cube of material is taken at a point with its faces perpendicular to X and Y reference axes. Tensile stresses equal to 15 kN/cm^2 and 9 kN/cm^2 are observed on these respective faces. They are also accompanied by shear equal to 4 kN/cm^2. The magnitude of the principal stresses at the point are

   (a) 12 kN/cm^2 tensile and 3 kN/cm^2 tensile
   (b) 17 kN/cm^2 tensile and 7 kN/cm^2 tensile
   (c) 9.5 kN/cm^2 compressive and 6.5 kN/cm^2 compressive
   (d) 12 kN/cm^2 tensile and 13 kN/cm^2 tensile

3. In a rectangular element subjected to like principal tensile stresses 'p_1' and 'p_2' in two mutually perpendicular directions X and Y, the maximum shear would occur along the

   (a) plane normal to X-axis
   (b) plane normal to Y-axis
   (c) plane at 45° to Y-direction
   (d) planes at 45° and 135° to the Y-direction

4. The lists given below refer to a bar of length L cross sectional area A, Young's modulus E, Poisson's ratio μ and subjected to axial stress 'p'. Match **List-I** with **List-II** and select the correct answer using the codes given below the lists:

   List-I

   A. Volumetric strain
   B. Strain energy per unit volume
   C. Ratio of Young's modulus to bulk modulus
   D. Ratio of Young's modulus to modulus of rigidity

   List-II

   1. $2(1+\mu)$
   2. $3(1-2\mu)$
   3. $\frac{p}{E}(1-2\mu)$
   4. $\frac{p^2}{2E}$
   5. $2(1-\mu)$

   Codes:

	A	B	C	D
(a)	3	4	2	1
(b)	5	4	1	2
(c)	5	4	2	[illegible]

(d) 2 3 1 5

5. Consider the following statements:
   1. In a member subjected to uniaxial tensile force the maximum normal stress is the external load divided by the maximum cross-sectional area.
   2. When the structural member is subjected to uniaxial loading, the shear stress is zero on a plane where the normal stress is maximum.
   3. In a member subjected to uniaxial loading, the normal stress on the planes of maximum shear stress is less than the maximum.

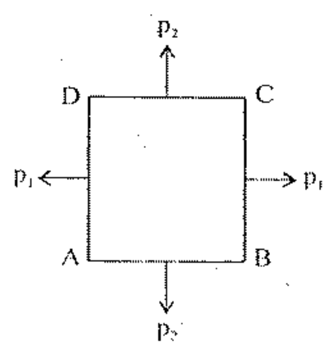
   Which of these statements are correct?
   (a) 1 and 2 (b) 1 and 3
   (c) 2 and 3 (d) 1, 2 and 3

6. A certain steel has proportionality limit of 3000 kg/cm^2 in simple tension. It is subjected to principal stresses of 1200 kg/cm^2 (tensile), 600 kg/cm^2 (tensile) and 300 kg/cm^2 (compressive). The factor of safety according to maximum shear stress theory is
   (a) 1.50 (b) 1.75
   (c) 1.80 (d) 2.00

7. At a point in a strained material, if two mutually perpendicular tensile stresses of 2000 kg/cm^2 and 1000 kg/cm^2 are acting, then the intensity of tangential stress on a plane inclined at 15° to the axis of the minor stress will be
   (a) 125 kg/cm^2 (b) 250 kg/cm^2
   (c) 500 kg/cm^2 (d) 1000 kg/cm^2

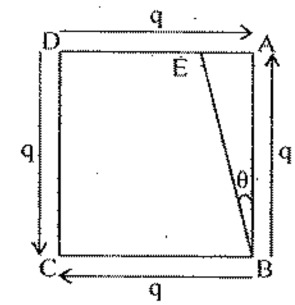
8. In a plane stress problem there are normal tensile stresses σ_x and σ_y accompanied by shear stress τ_{xy} at a point along orthogonal Cartesian co-ordinates x and y respectively. If it is observed that the minimum principal stress on a certain plane is zero then
   (a) $\tau_{xy} = \sqrt{\sigma_x + \sigma_y}$ (b) $\tau_{xy} = \sqrt{\sigma_x - \sigma_y}$
   (c) $\tau_{xy} = \sqrt{\sigma_x \sigma_y}$ (d) $\tau_{xy} = \sqrt{\frac{\sigma_x}{\sigma_y}}$

9. A plane rectangular element is subjected to two normal stresses 'p_1' and 'p_2' on two mutually perpendicular planes ($p_1 > p_2$) as shown in the figure.



Which one of the following statements is NOT true in this regard?
(a) The planes BC and CD are principal planes.
(b) Shear stress will act on planes inclined to planes AB and CB.
(c) There will not be any normal stress on planes having maximum shear stress.
(d) There will not be any shear stress on planes AB and BC.

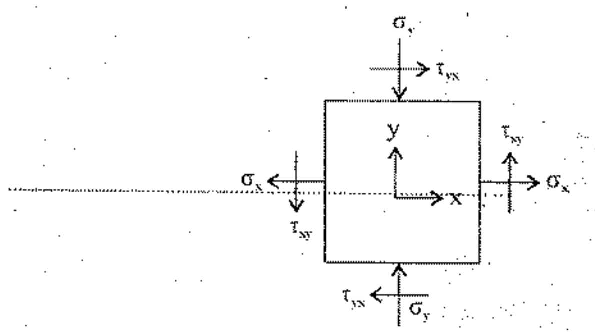
10. The rectangular block shown in the given figure is subjected to pure shear of intensity 'q'. If BE represents the principal plane and the principal stresses are σ_1, σ_2; then the values of θ, σ_1 and σ_2 will be respectively



(a) 0°, 90°; + q and − q (b) 30°, 120°; + q and − q

(c) 45°, 135°; $+\frac{q}{2}$ and $-\frac{q}{2}$ (d) 45°, 135°; + q and − q

11. The state of stresses on an element is shown in the given figure. The values of stresses are σ_x (= 32 MPa); σ_y (= − 10 MPa) and major principal stress σ_1 (= 40 MPa). The minor principal stress σ_2 is



(a) − 22 MPa
(b) − 18 MPa
(c) 22 MPa
(d) indeterminable due to insufficient data

12. A solid shaft of 100 mm diameter in a small hydraulic turbine is subjected to an axial compressive load of 100π kN and a torque of 5π kN-m. The maximum shearing stress induced in the shaft is

(a) $20\sqrt{3}$ N/mm^2 (b) $20\sqrt{8}$ N/mm^2

(c) $20\sqrt{15}$ N/mm^2 (d) $20\sqrt{17}$ N/mm^2

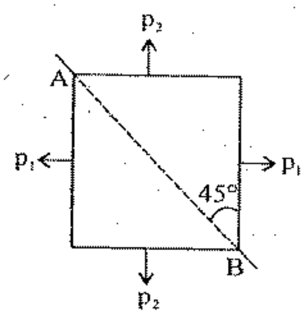
13. The principal stresses at a point in a strained material are 'p_1' and 'p_2'. The resultant stress 'p_r' on the plane carrying the maximum shear stress would be

(a) $\frac{(p_1^2+p_2^2)^{1/2}}{2}$ (b) $\left[\frac{p_1^2+p_2^2}{2}\right]^{1/2}$

(c) $\left[2\left(p_1^2+p_2^2\right)\right]^{1/2}$ (d) $2\left[p_1^2+p_2^2\right]^{1/2}$

14. If a shaft is simultaneously subjected to a torque T and a bending moment M, the ratio of maximum bending stress and maximum shearing stress is given by
(a) 2M/T (b) M/T
(c) 2T/M (d) T/M

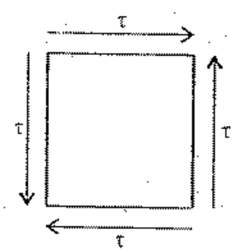
15. 'p_1' and 'p_2' are two equal tensile principal stresses. On the plane AB inclined at 45° to the plane of 'p_1' as shown in the figure below,



(a) the shear stress is maximum
(b) the normal stress is zero
(c) the shear stress is zero
(d) the normal stress is maximum

16. The radius of Mohr's circle is zero when the state of stress is such that
(a) shear stress is zero
(b) there is pure shear
(c) there is no shear stress but identical direct stresses in two mutually perpendicular directions
(d) there is no shear stress but equal direct stresses, opposite in nature, in two mutually perpendicular directions

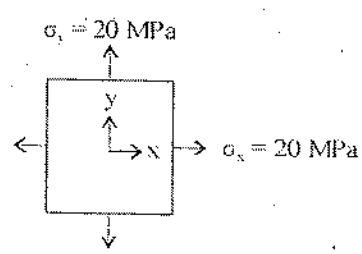
17. The given figure shows the stress condition of an element. The principal stresses are



(a) $\pm 2\tau$ (b) $\pm\frac{\tau}{2}$

(c) $\pm \tau$ (d) $\pm\frac{2\tau}{3}$

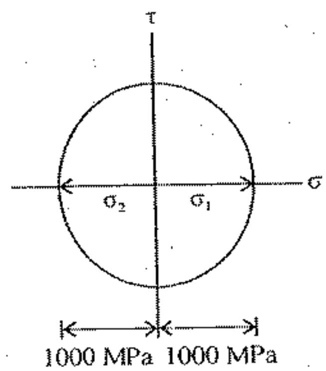
18. For the plane stress situation shown in the diagram below, what is the maximum shear stress?

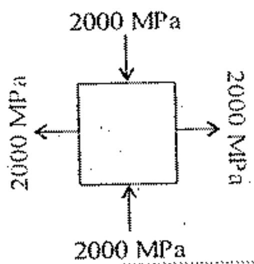


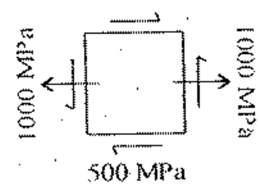
(a) Zero, when X and Y axes are rotated 45° clockwise

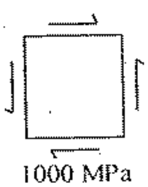
(b) Zero, at all positions of orientation of X and Y axes
(c) 20 MPa, at all positions of orientation of X and Y axes
(d) –20 MPa, when X and Y axes are rotated 45° anticlockwise

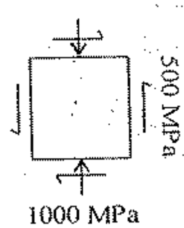
19. The Mohr's circle given above corresponds to which one of the following stress conditions



(a) 

(b) 

(c) 

(d) 

20. If the maximum principal stress for an element under bi-axial stress situation is 100 MPa (tensile) and the maximum shear stress is also 100 MPa, then what is the other principal stress?
(a) 200 MPa (tensile)
(b) 200 MPa (compressive)
(c) 100 MPa (compressive)
(d) zero

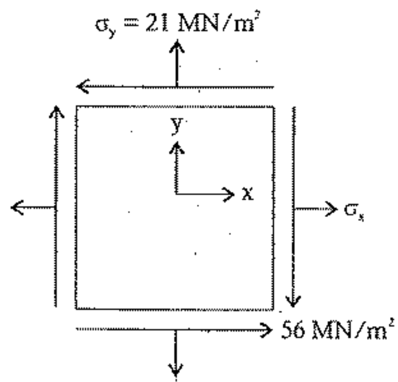
21. Mohr's stress circle helps in determining which of the following ?
1. Normal stresses on one plane.
2. Normal and tangential stresses on two planes.
3. Principal stresses in all three directions.
4. Inclination of principal planes.

Select the correct answer using the codes given below:
(a) 1 and 2 only (b) 2 and 3 only
(c) 3 and 4 only (d) 2 and 4 only

22. If a body carries two unlike principal stresses, what is the maximum shear stress?
(a) Half the difference of magnitude of the principal stresses

(b) Half the sum of the magnitude of principal stresses
(c) Difference of the magnitude of principal stresses
(d) Sum of the magnitude of principal stresses

23. Figure below shows a state of plane stress.



If the minimum principal stress is – 7 MN/m² then what is the value of σ_x?

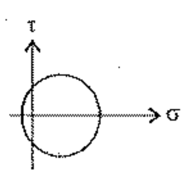
(a) 30 MN/m² (b) 68 MN/m²
(c) 98 MN/m² (d) 105 MN/m²

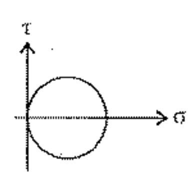
24. Consider the following statements:
If there is a state of pure shear τ at a point then
1. The Mohr's circle is tangential to the y-axis.
2. The centre of the Mohr's circle coincides with the origin.
3. Unlike principal stresses are each numerically equal to τ.
4. Principal stresses are like.

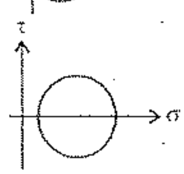
Which of these statements is/are correct?

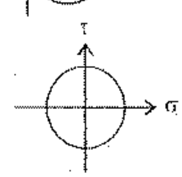
(a) 1 only (b) 1 and 2
(c) 2 and 3 (d) 3 and 4

25. A cantilever beam is loaded with a uniformly distributed load of intensity 'w' along its entire length. The span of the beam is L.
Which of the following Mohr's circle diagrams correctly represent(s) the state of stress above the neutral axis of the beam?

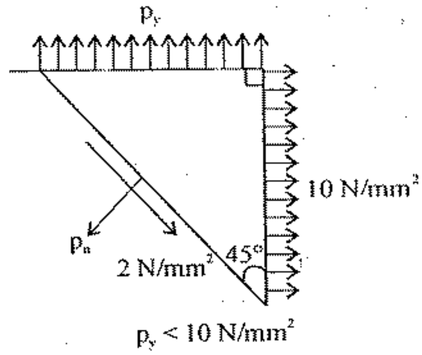
1. 

2. 

3. 

4. 

26. For the two-dimensional stresses shown in the figure below, what is the normal stress on the 45° plane?

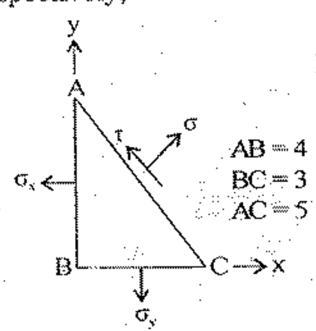


(a) 20 N/mm² (b) 12 N/mm²
(c) 4 N/mm² (d) 8 N/mm²

27. The state of two dimensional stresses acting on a concrete lamina consists of a direct tensile stress, $\sigma_x = 1.5$ N/mm², and shear stress, $\tau = 1.20$ N/mm², which cause cracking of concrete. Then the tensile strength of the concrete in N/mm² is

(a) 1.50 (b) 2.08
(c) 2.17 (d) 2.29

28. In a two dimensional stress analysis, the state of stress at a point is shown below. If $\sigma = 120$ MPa and $\tau = 70$ MPa, σ_x and σ_y, are respectively,



(a) 26.7 MPa and 172.5 MPa
(b) 54 MPa and 128 MPa
(c) 67.5 MPa and 213.3 MPa
(d) 16 MPa and 138 MPa

29. Mohr's circle for the state of stress defined by $\begin{bmatrix} 30 & 0 \\ 0 & 30 \end{bmatrix}$ MPa is a circle with

(a) center at (0, 0) and radius 30 MPa
(b) center at (0, 0) and radius 60 MPa
(c) center at (30, 0) and radius 30 MPa
(d) center at (30, 0) and zero radius

30. Consider the following statements:

1. On a principal plane, only normal stress acts.
2. On a principal plane, both normal and shear stresses act.
3. On a principal plane, only shear stress acts.
4. Isotropic state of stress is independent of frame of reference.

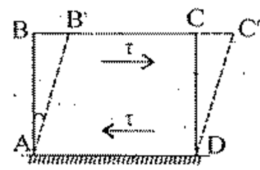
Which of the above statements is/are correct?

(a) 1 and 4 (b) 2 only
(c) 2 and 4 (d) 2 and 3

31. The side AD of the square block ABCD as shown in the given figure is fixed at the base and it is under a stage of simple shear causing shear stress τ and shear strain ϕ,

where $\phi = \dfrac{\tau}{\text{Modulus of Rigidity (G)}}$

The distorted shape is AB'C'D. The diagonal strain (linear) will be



(a) $\phi/2$ (b) $\phi/\sqrt{2}$
(c) $\sqrt{2}\,\phi$ (d) ϕ

32. A bar of uniform section is subjected to axial tensile loads such that the normal strain in the axial direction is 1.25 mm per m. If the Poisson's ratio of the material of the bar is 0.3, the volumetric strain would be
(a) 2×10^{-4} (b) 3×10^{-4}
(c) 4×10^{-4} (d) 5×10^{-4}

33. **Assertion (A)**: Normal stress of one nature (compressive or tensile) acting along one of the three orthogonal axes of a member will produce strains of the same nature in its direction and strains of opposite nature along the other two directions.
Reason (R): Sum of the strains along the three orthogonal axes equals volumetric strain.
Of these statements
(a) both A and R are true and R is the correct explanation of A
(b) both A and R are true but R is not a correct explanation of A
(c) A is true but R is false
(d) A is false but R is true

34. In the Mohr's circle for strains, radius of Mohr's circle gives the
(a) minimum value of normal strain
(b) maximum value of normal strain
(c) maximum value of shear strain
(d) half of maximum value of shear strain

35. Consider the following statements :
Mohr's strain circle can be drawn
1. for plane stress conditions
2. if strains in three directions are known
3. if strains on two mutually perpendicular planes are known
Which of these statements are correct?
(a) 1, 2 and 3 (b) 2 and 3 only
(c) 1 and 2 only (d) 1 and 3 only

$u = (-2x + 8y) \times 10^{-6}$ unit

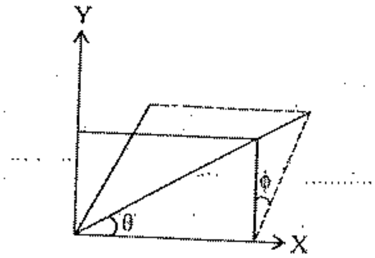
$v = (-3x + 5y) \times 10^{-6}$ unit

What is the shearing strain?

(a) 9×10^{-6} (b) 7×10^{-6}

(c) 5×10^{-6} (d) 3×10^{-6}

37. In the given figure showing the X-Y quarter plane, ϵ_x and ϵ_y are linear strains in the respective directions and ϵ_θ is the linear strain in the direction at an inclination of θ of X to Y. The shear strain ϕ is defined as shown. The critical value of θ is given by $\theta = \alpha$ where $\tan 2\alpha$ is equal to



(a) $\dfrac{\phi}{\epsilon_x - \epsilon_y}$ (b) $\dfrac{2\phi}{\epsilon_x - \epsilon_y}$

(c) $\dfrac{\phi}{\epsilon_x + \epsilon_y}$ (d) $\dfrac{2\phi}{\epsilon_x + \epsilon_y}$

38. If ϵ_x and ϵ_y are the maximum and minimum strains, respectively, in the neighbourhood of a point in a stressed material, then what is the expression for the maximum principal stress ?

(a) $E\epsilon_x$ (b) $E(\epsilon_x + \mu\epsilon_y)$

(c) $\dfrac{E(\epsilon_x + \mu\epsilon_y)}{1-\mu^2}$ (d) $\dfrac{E(\epsilon_y + \mu\epsilon_x)}{1-\mu^2}$

39. The components of strain tensor at a point in the plane strain case can be obtained by measuring longitudinal strain in following directions

(a) along any two arbitrary directions

(b) along any three arbitrary directions

(c) along two mutually orthogonal directions

(d) along any arbitrary direction

40. At a point in a structure, there are two mutually perpendicular tensile stresses of 800 kg/cm² and 400 kg/cm². If the Poisson's ratio m = 0.25, what would be the equivalent stress in simple tension according to maximum principal strain theory?

(a) 1200 kg/cm² (b) 900 kg/cm²

(c) 700 kg/cm² (d) 400 kg/cm²

41. According to maximum shear stress failure criterion, yielding in material occurs when

(a) maximum shear stress = $\frac{1}{2}$ yield stress

(b) maximum shear stress = $\sqrt{2} \times$ yield stress

(c) maximum shear stress = $\sqrt{3} \times$ yield stress

(d) maximum shear stress = $2 \times$ yield stress

42. In a two dimensional stress system, it is assumed that the principal stress σ_1 and σ_2 are such that $\sigma_1 > \sigma_2$, then according to the maximum shear stress theory, the failure occurs when (where σ_y is the yield stress, μ is the Poisson's ratio and E is the modulus of elasticity)

(a) $\frac{1}{E}(\sigma_1 - \mu\sigma_2) \geq \frac{\sigma_y}{E}$

(b) $\sigma_1^2 + \sigma_2^2 + 2\mu\sigma_1\sigma_2 \geq \sigma_y^2$

(c) $\sigma_1 - \sigma_2 \geq \sigma_y$

(d) $\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2 \geq \sigma_y$

43. According to the distortion energy theory, failure will NOT occur when (symbols have the usual meaning)

(a) $\left[\frac{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2}{2}\right]^{\frac{1}{2}} \leq \sigma_0$

(b) $\left[(\sigma_1-\sigma_2)^2+4\tau^2\right]^{\frac{1}{2}} \leq \sigma_0$

(c) $\left[(\sigma_1^2+\sigma_2^2+\sigma_3^2)-\frac{1}{m}(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1)\right]^{\frac{1}{2}} \leq \sigma_0$

(d) $\left[\frac{(\sigma_1-\sigma_2)+(\sigma_1-\sigma_2)^2+4\tau^2}{2}\right]^{\frac{1}{2}} \leq \sigma_0$

44. All the failure theories give nearly the same result

(a) when one of the principal stresses at a point is large in comparison to the other
(b) when shear stresses act
(c) when both the principal stresses are numerically equal
(d) for all situations of stress

45. A cube is subjected to equal tensile stress on all the three faces. If the yield stress of the material is 'σ_y' then based on the strain energy theory, the maximum tensile stress will be

(a) $\frac{\sigma_y}{\sqrt{3(1-2\mu)}}$ (b) $\frac{\sigma_y}{\sqrt{3(2-\mu)}}$

(c) $\frac{\sigma_y}{\sqrt{3(1-\mu)}}$ (d) $\frac{\sigma_y}{\sqrt{3(1+\mu)}}$

46. The limit of proportionality of a certain steel sample is 300 MPa in simple tension. It is subjected to principal stresses of 150 MPa (tensile), 60 MPa (tensile) and 30 MPa (tensile). According to the maximum principal stress theory, the factor of safety in this case would be

(a) 10 (b) 5
(c) 4 (d) 2

47. Match List-I (Theory of failures) with List-II (Scientists) and select the correct answer using the codes given below the lists:

List-I

A. Maximum principal stress theory

C. Maximum principal strain theory
D. Maximum distortion energy theory

List-II
1. St. Venant
2. Beltrami and Haigh
3. Tresca
4. Von-Mises
5. Rankine

Codes:

	A	B	C	D
(a)	5	3	1	4
(b)	5	1	2	4
(c)	3	5	1	2
(d)	3	1	2	5

48. In a two dimensional stress system, the two principal stresses are σ_1 of 180 N/mm^2 (tensile) and σ_2 (compressive). For the material, yield stress in simple tension and compression is 240 N/mm^2 and Poisson's ratio is 0.25. According to maximum normal strain theory, the value of σ_2 at which yielding will commence, is

(a) 240 N/mm^2 (b) 180 N/mm^2
(c) 195 N/mm^2 (d) 200 N/mm^2

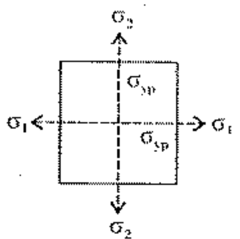
49. Consider the following statements:
Assertion (A): For a ductile material the maximum shear distortion theory is most suitable.
Reason (R): The maximum shear distortion theory of failure assumes that yielding can occur in a general three-dimensional state of stress.
Of these statements
(a) both A and R are true and R is the correct explanation of A
(b) both A and R are true but R is not a correct explanation of A
(c) A is true but R is false
(d) A is false but R is true

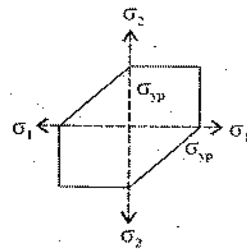
50. According to maximum shear stress criterion, at what ratio of maximum shear stress to yield stress of material does the yielding of material take place ?

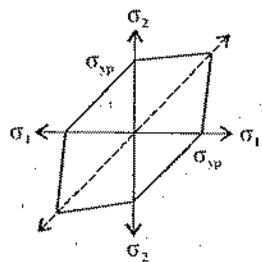
(a) 2 (b) $2/\sqrt{3}$
(c) $1/\sqrt{3}$ (d) 1/2

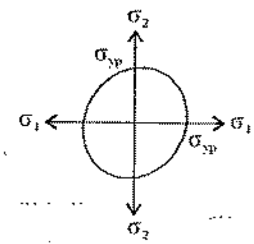
51. In a biaxial strain system ε_x and ε_y, what is the maximum engineering shearing strain?
(a) $\varepsilon_x + \varepsilon_y$ (b) $\varepsilon_x - \varepsilon_y$
(c) $\dfrac{\varepsilon_x + \varepsilon_y}{2}$ (d) $\dfrac{\varepsilon_x - \varepsilon_y}{2}$

52. Which one of the following diagrams correctly represents the Rankine or the maximum stress theory of failure?

(a) 

(b) 

(c) 

(d) 

53. If maximum principal stress σ_1 of 90 N/mm^2, σ_2 and σ_3 of values zero act on a cube of unit dimensions, then the maximum shear strain energy stored in it would be

(a) $\dfrac{337.5}{G}$ (b) $\dfrac{675}{G}$

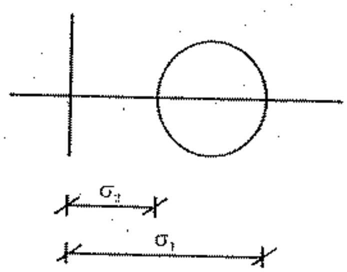
(c) $\dfrac{1350}{G}$ (d) $\dfrac{2700}{G}$

ANSWERS

1.	(b)	14.	(a)	27.	(c)	40.	(c)
2.	(b)	15.	(c)	28.	(c)	41.	(a)
3.	(d)	16.	(c)	29.	(d)	42.	(c)
4.	(a)	17.	(c)	30.	(a)	43.	(a)
5.	(c)	18.	(b)	31.	(a)	44.	(a)
6.	(d)	19.	(c)	32.	(d)	45.	(a)
7.	(b)	20.	(c)	33.	(b)	46.	(d)
8.	(c)	21.	(d)	34.	(d)	47.	(a)
9.	(c)	22.	(b)	35.	(b)	48.	(c)
10.	(d)	23.	(d)	36.	(c)	49.	(b)
11.	(b)	24.	(c)	37.	(a)	50.	(d)
12.	(d)	25.	(a)	38.	(c)	51.	(b)
13.	(b)	26.	(d)	39.	(b)	52.	(a)

SOLUTION...

1. (b)



Given statements are properties of principal stress is statement 3.

$$\tau_{max} = \frac{\sigma_1 - \sigma_2}{2}$$

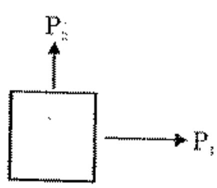
It is the difference, not the sum. Hence statement 3 is wrong.

2. (b) $$\frac{15+9}{2} \pm \sqrt{\left(\frac{15-9}{2}\right)^2 + 4^2}$$

$$\sigma_1, \sigma_2 = 12 \pm 5$$

$\sigma_1 = 17\ kN/cm^2$, $\sigma_2 = 7\ kN/cm^2$ { +ve means tenslies }

3. (d)

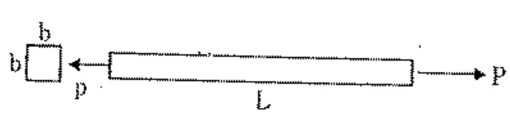


Observe that both are tensile.

$$\tau_{xy} = \left(\frac{\sigma_1 - \sigma_2}{2}\right), \ \tan\theta = \frac{\tau_{xy}}{2}$$

Max. shear stress will always occurs at 45° and 135° to principal axis.

4. (a)



$$\varepsilon_V = \varepsilon_L + 2\varepsilon_b = \frac{P}{E} + \left(-2\mu\frac{P}{E}\right)$$

$$\varepsilon_V = \frac{P}{E}(1 - 2\mu)$$

$$E = 2G(1 + \nu)$$

$$E = 3K(1 - 2\nu)$$

$$\text{Strain Energy} = \frac{1}{2} \times \sigma \times \varepsilon \times \text{volume}$$

$$= \frac{P^2}{2E} \times \text{Volume}$$

$$= \frac{1}{2} \times \text{stress} \times \text{strain}$$

5. (c)

$$\sigma = \frac{P}{A}$$

$$\text{max stress}, \ \sigma = \frac{P_{constant}}{A_{2\ min}}$$

∴ Max. normal stress occurs at the minimum c/s area statement 1 is wrong.

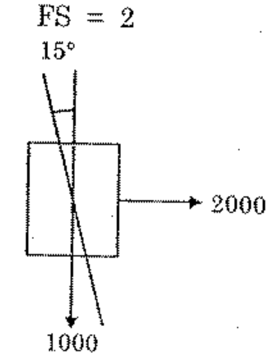
6. (d) Max. shear stress theory

absolute maximum shear stress,

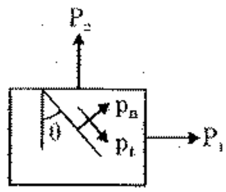
$$\tau_{max} = \frac{\sigma_1 - \sigma_2}{2} \le \frac{\left(\frac{f_y}{FS}\right)}{2}$$

$$\frac{1200 - (-300)}{2} \le \frac{\left(\frac{3000}{FS}\right)}{2}$$

∴ $$FS = 2$$



7. (b)



$$P_t = +\left(\frac{P_1 - P_2}{2}\right)\sin 2\theta$$

$$= +\left(\frac{2000 - 1000}{2}\right)\sin 2(15)$$

$$= +250\ \text{kg/cm}^2$$

[+ve/–ve is dependent on the assumed direction of P_n & P_t]

Alternate: Try it by mohr circle

8. **(c)** $\sigma_2 = 0$

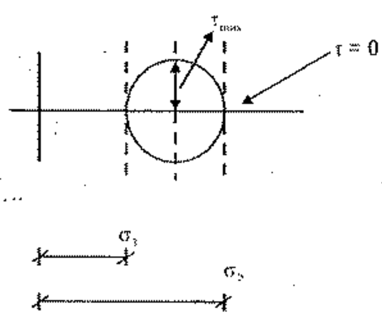
$$\sigma_2 = \left(\frac{\sigma_x + \sigma_y}{2}\right) - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = 0$$

$$\left(\frac{\sigma_x + \sigma_y}{2}\right)^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2$$

$$2\sigma_x\sigma_y = -2\sigma_x\sigma_y + 4\tau_{xy}^2$$

$$\tau_{xy} = \sqrt{\sigma_x\sigma_y}$$

9. (c)



On planes of principal stresses, shear stress = 0. But not vice versa.

For e.g. :

at τ_{max}, 'σ' exists

10. (d) $\sigma_{1,2} = 0 \pm \sqrt{0 + q^2}$

$\sigma_{1,2} = \pm q$, [occurs at $\tan 2\theta = \frac{2\tau_{xy}}{0} = \infty$]

11. (b) Can be solved from basics.

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

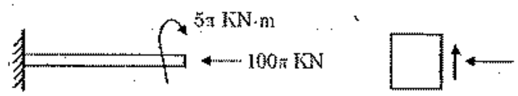
(or) alternatively, using the property

$$\sigma_1 + \sigma_2 = \sigma_x + \sigma_y$$

$$+32 - 10 = 40 + \sigma_2$$

$$\sigma_2 = -18\ \text{Mpa}$$

12. (d)



Shear stresses developed due to two loads are calculated individually.

Due to comp. load

$$\sigma_1 = \frac{100\pi \times 10^3}{\pi/4 \times 100^2} = 40$$

$$\tau_{max} = \frac{\sigma_1}{2} = 20\ \text{Mpa}$$

Due to Torsional Load

$$\tau = \frac{Tr}{J} = \frac{5\pi \times 10^6}{\frac{\pi}{16}(100)^3}$$

$$= 16 \times 5\ \text{N/mm}^2$$

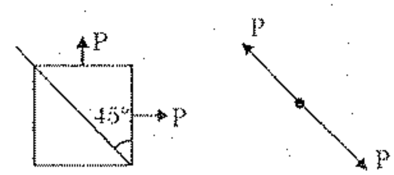
Now max. shear stress is to combination of these two.

$$\tau_{max} = \sqrt{20^2 + 80^2} = 20\sqrt{17}\ \text{N/mm}^2$$

14. (a)

$$\frac{f}{\tau} = \frac{32\frac{M}{\pi d^3}}{16\frac{T}{\pi d^3}} = \frac{2M}{T}$$

15. (c)

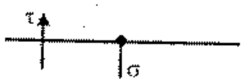


For resultant along 45° Resolving along the 45° plane,

Net = 0. ⇒ No shear stress.

Approach 2 :

$$\tau_{xy} = \frac{\sigma_1 - \sigma_2}{2}\sin 2\theta + \tau_{xy}^{\nearrow 0}\cos 2\theta = 0$$

16. (c) 

σ_1/σ_2 Mohr circle for given condition

17. (c) $\sigma_{1,2} = 0 \pm \sqrt{0+\tau^2} = \pm\tau$

18. (b)

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \frac{\sigma_x - \sigma_y}{2}$$

$$\sigma_1 = \sigma_x = 20 \text{ Mpa}$$

$$\sigma_2 = \sigma_y = 20 \text{ Mpa}$$

$$\tau_{max} = \frac{\sigma_1 - \sigma_2}{2} = 0$$

∴ Zero shear stress for all position.

19. (c) τ_{max} = radius of Mohrs circle

= 1000 Mps

20. (c)

$$\frac{\sigma_1 - \sigma_2}{2} = \tau_{xy}$$

$$+\frac{100 - \sigma_2}{2} = 100$$

$$\sigma_2 = 100$$

$$\Rightarrow \quad -\sigma_2 = 100 \text{ Mpa (compressive)}$$

23. (d)

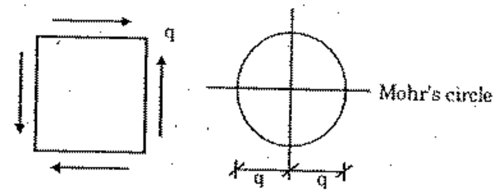
$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$-7 = \frac{\sigma_x + 21}{2} - \sqrt{\left(\frac{\sigma_x - 21}{2}\right)^2 + 56^2}$$

$$\left(\frac{\sigma_x - 21}{2}\right)^2 + 56^2 = \left(\frac{\sigma_x + 35}{2}\right)^2$$

On solving, $\sigma_x = 105$ MN/m²

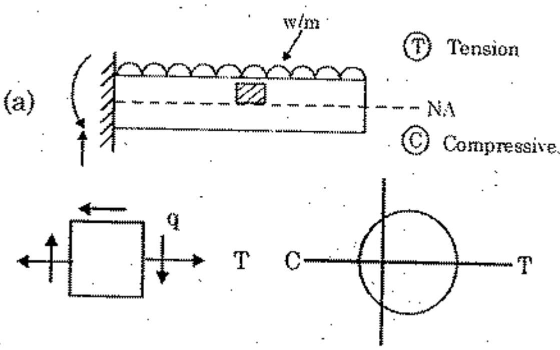
24. (c) Pure shear case



$$\sigma_{1,2} = 0 \pm \sqrt{0+q} = \pm q$$

[Unlike, equal stresses]

25. (a)

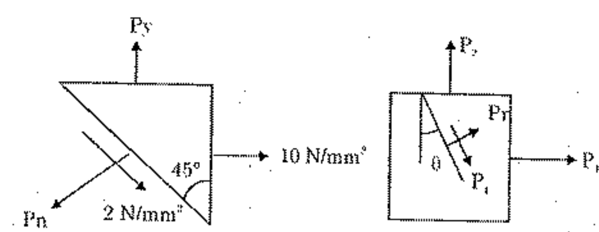


For the given loading,

Hints :

1. above NA, due to moment tension exists.
2. due to load, shear also exists.

26. (d)



$$2 = \frac{10 - P_y}{2} \times 1$$

$$P_y = 6 \text{ N/mm}^2$$

$$P_n = \frac{P_1 + P_2}{2} + \frac{P_1 - P_2}{2}\cos 2\theta$$

$$P_t = \frac{P_1 - P_2}{2}\sin 2\theta$$

$$P_n = \frac{10+6}{2} + \frac{10-6}{2}\cos 90^\circ$$

$$= 8\ \text{N/mm}^2$$

27. (c) $\sigma_{1/2} = \frac{\sigma_x + 0}{2} \pm \sqrt{\left(\frac{\sigma_x - 0}{2}\right)^2 + \tau_{xy}^2}$ –

$$\sigma_1 = \frac{1.5}{2} + \sqrt{\left(\frac{1.5}{2}\right)^2 + (1.2)^2}$$

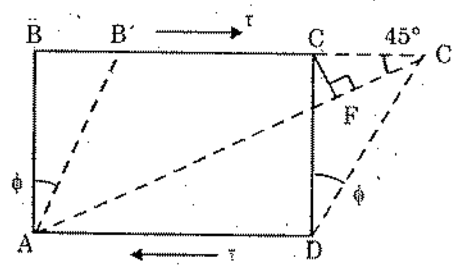
= 2.17 MPa [Tensile strength of concrete]

29. (d) $\begin{bmatrix} 30 & 0 \\ 0 & 30 \end{bmatrix} = \begin{bmatrix} \sigma_{max} & 0 \\ 0 & \sigma_{min} \end{bmatrix} \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{bmatrix}$

$\tau_{xy} = 0$

i.e., radius of Mohr's Circle = 0

31. (a)



$$\tan\phi = \frac{BB'}{AB} = \phi$$

$$\text{diagonal strain} = \frac{FC'}{AF} = \frac{CC'\cos 45^\circ}{\frac{CD}{\cos 45^\circ}}$$

$$= \frac{CC'}{CD} \times \left(\frac{1}{2}\right) = \phi \times \frac{1}{2}$$

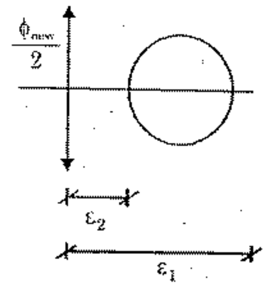
32. (d) $\varepsilon_V = \varepsilon_L + 2\varepsilon_b$

$$= (1-2\mu)\varepsilon_L$$

$$= (1 - 2 \times 0.3) \times 1.25$$

$$= 0.4 \times 1.25 = \frac{4}{10} \times \frac{5}{4} = 0.5$$

34. (d)



$$\text{Radius} = \frac{\varepsilon_1 - \varepsilon_2}{2} = \frac{\phi_{max}}{2}$$

35. (a) Strains on two mutually perpendicular planes (both axial and shear) are known then Mohr circle can be drawn.

36. (c) $\frac{\partial V}{\partial x} + \frac{\partial U}{\partial y} = \gamma_{xy}$

$(-3+8)10^{-6} = 5\times 10^{-6}$ Unit

37. (a) $\tan 2\alpha = \frac{2\frac{\phi}{2}}{\varepsilon_x - \varepsilon_y} = \frac{\phi}{\varepsilon_x - \varepsilon_y}$

40. (c) Max. principal strain theory.

$$\varepsilon_1 \le \varepsilon_y$$

$$\frac{\sigma_1 - \mu(\sigma_2 + \sigma_3)}{E} \le \frac{f_y}{E}$$

$$\frac{+800 - 0.25(400)}{E} \le \frac{\sigma_{eq}}{E}$$

$$\sigma_{eq} = 700\ \text{kg/cm}^2$$

44. (a) It will become uniaxial condition.

45. (a) Max. strain energy theory :

$$\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\mu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) \le f_y^2$$

Cube subjected to equal tensile stresses.

$$\sigma_1 = \sigma_2 = \sigma_3$$

$$3\sigma^2 - 2\mu(3\sigma^2) \le f_y^2$$

$$f_y = 3\sigma\sqrt{(1-2\mu)}$$

$$\sigma = \frac{f_y}{3\sqrt{(1-2\mu)}} \quad [\text{here } f_y = \sigma_y]$$

46. (d) Max. principal stress theory

$$\sigma_1 \le \frac{f_y}{F.O.S.}$$

$$150 \le \frac{300}{FOS}$$

$$FOS = 2$$

48. (c) Max. normal strain theory

$$\varepsilon_1 \le \varepsilon_y$$

$$\frac{\sigma_1 - \mu(\sigma_2 + \sigma_3)}{E} \le \frac{f_y}{E}$$

(a) when $\sigma_1 = 180, f_y = 240$

$$+180 - \mu\sigma_2 \le 240$$

$$-0.25 \times \sigma_2 \le 60$$

$$\sigma_2 \le 240 \text{ Mpa}$$

(b) When $\sigma_1 = 180, f_y = -240$

Using the other eqn.

$$\Rightarrow \sigma_2 - 0.25(180) \le -240$$

$$\sigma_2 \le -195 \text{ Mpa}$$

In this case, we need to check for various combination at which yielding will commence first.

∴ lowest = 195 Mpa (comp.)

50. (d) Max. shear stress theory

$$\tau_{max} = \frac{\sigma_1 - \sigma_2}{2} < \frac{f_y}{2}$$

51. (b)

$$\frac{\phi_{max}}{2} = \frac{\varepsilon_1 - \varepsilon_2}{2}$$

53. (b)

$$\sigma_1 = 90$$

$$\sigma_2 = \sigma_3 = 0$$

Max. shear strain energy

$$\frac{1}{2E}\left[(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2\right] \le \frac{f_y^2}{E}$$

$$U = \frac{\mu+1}{6E}\left[(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2\right]$$

$$E = 2G(1+\mu)$$

$$\Rightarrow U = \frac{1}{12G}\left[(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2\right]$$

$$= \frac{1}{12G}\left[90^2 + 0^2 + 90^2\right]$$

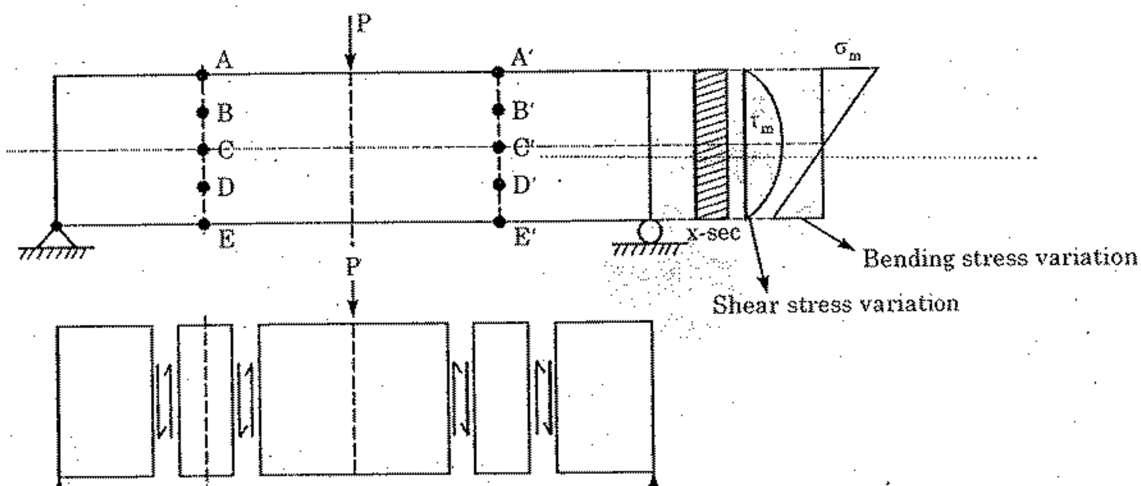
$$= \frac{8100}{12G} = \frac{675}{G}$$

5

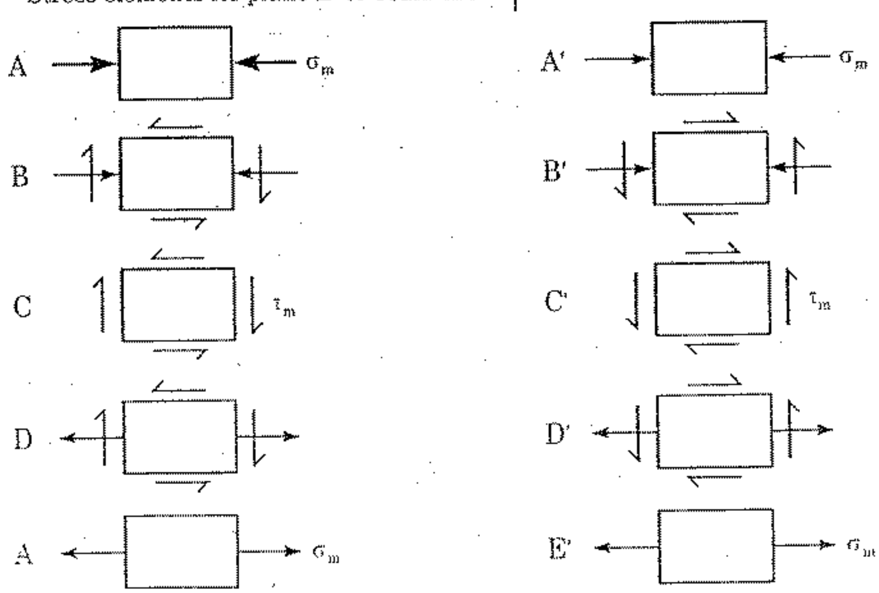
Combined Stress

ANALYSIS OF COMBINED BENDING AND TRANSVERSE SHEAR STRESS

Let a beam is loaded as shown below. In this case any point at any section is subjected to both bending and shear stresses.



Stress elements for plane ⊥r to beam are as shown below



- Corresponding to these stress elements we can find out max principal stress (σ_{max}) and max shear stress (τ_{max})

  If we have to find out max principal stress and max shear stress any where in the beam, then the points that needs to be investigated are:

  (a) For rectangular sections

  (i) Points of max bending stress and point of max shear stresses on the section of max BM. i.e. extreme points and neutral axis locations on the section of max BM.

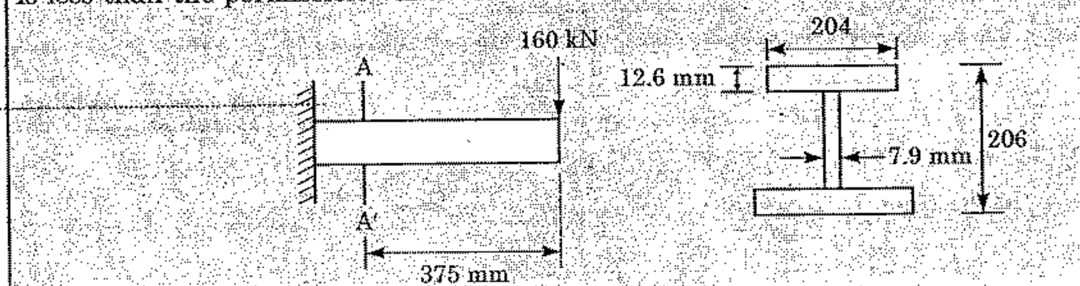
  (ii) Point of max bending stress and point of max shear stress on the section of max shear force.

  (b) For flanged sections, in addition to above points we also need to investigate the junction of web and flange.

Note: *Points discussed above will, in most of the cases, result in correct result. However for certain loadings the above conclusion may not hold true.*

Example 1

A 160 kN force is applied as shown below. Determine max normal stress at section A-A′ and check if it is less than the permissible value of 150 MPa. Take z = 512×10^{-6} m³ and I = 52.7×10^{6} mm⁴.



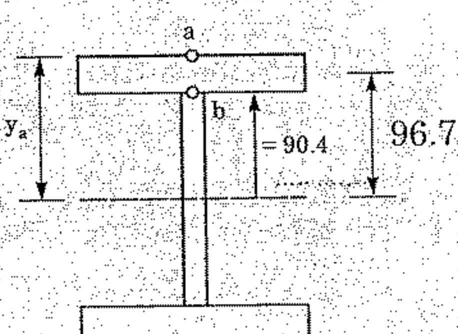
Sol: $\sigma_a = \dfrac{M_a}{z} = \dfrac{60\ \text{kNm}}{512 \times 10^{-6}\text{m}^3}$

$= 117.2$ MPa

$\sigma_b = \dfrac{M_b}{Z} = \dfrac{117.2}{103} \times 90.4$

$\sigma_b = 102.9$ MPa

$\tau_a = 0$



$\tau_b = \dfrac{VA\bar{y}}{Ib} = \dfrac{160 \times 10^3\text{N} \times (204 \times 12.6)(96.7)\ \text{mm}^3}{52.7 \times 10^6 \text{mm}^4 \times 7.9\ \text{mm}}$

$= 95.5$ N/mm² $= 95.5$ MPa

$$\sigma_{max}\big|_a = \frac{\sigma_a + 0}{2} + \sqrt{\left(\frac{\sigma_a - 0}{2}\right)^2 + 0} = 117.24 \text{ Pa}$$

$$\sigma_{max}\big|_b = \left(\frac{102.9}{2}\right) + \sqrt{\left(\frac{102.9}{2}\right)^2 + (95.5)^2}$$

$= 159.9$ MPa $> \sigma_{permissible}$

Example 2

A prismatic beam ABC is simply supported at A and B. AB = 20 m, BC = 1 m, C is free end. The entire beam is uniformly loaded with 10 kN/m. The cross section of the beam is I with the following particulars:

Flanges : width = 150 mm

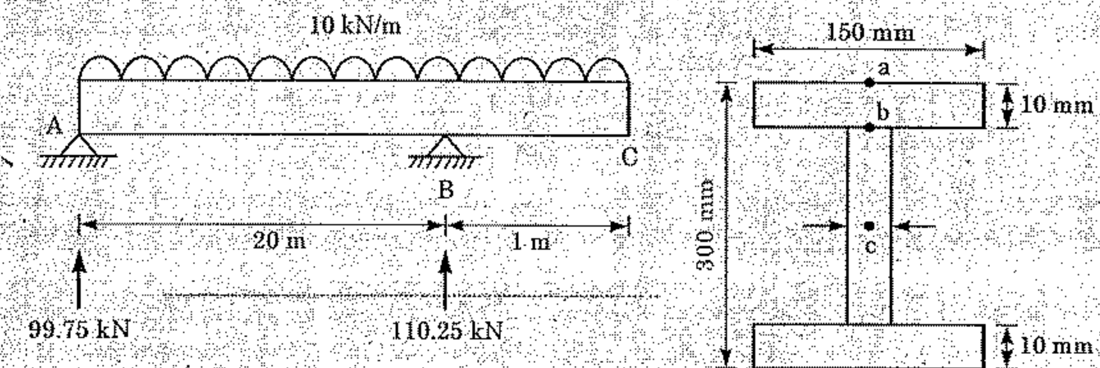
Thickness = 10 mm

Web: thickness = 10 mm

Overall depth of the beam = 300 mm.

Determine the maximum value of principal stresses occurring anywhere in the beam. Specify the location.

Sol:

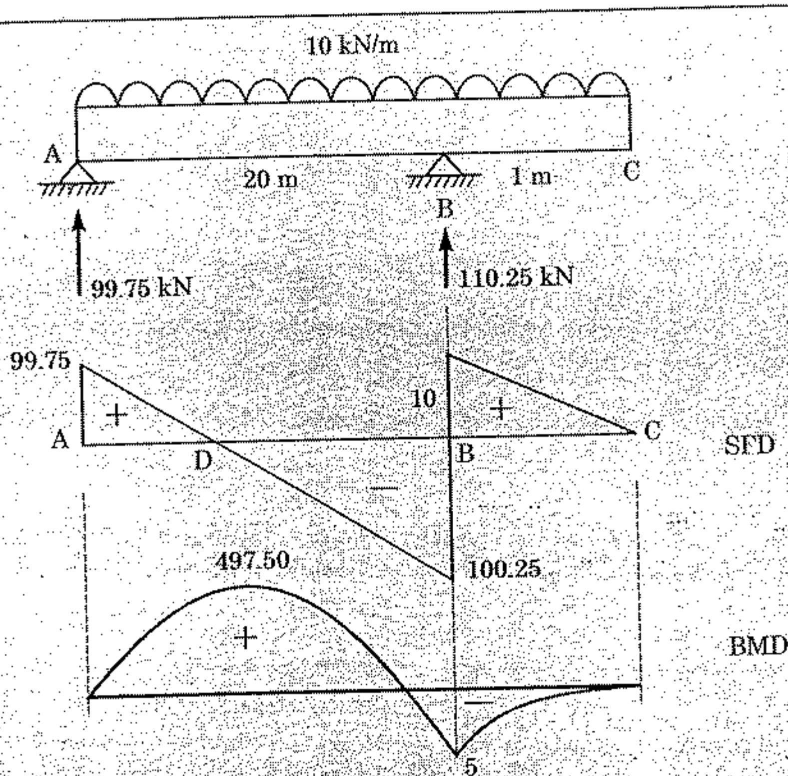


To findout max principal stress anywhere in the beam. The various points to be investigated are

(a) Section of max bending moment — Point a, Point b, Point b

(b) Section of max. shear force — Point a, Point b, Point b

However the most critical location that needs to be investigated are point 'a' and 'b' at the section of max. BM.



Let us investigate the point of max BMD (Point D)

SF = 0 at D,

$$\text{Max bending stress} = \frac{My}{I} = \text{Max principal stress}$$

$$= \frac{497.50\times10^6\times150}{\left[\frac{150\times(300)^3}{12} - \frac{140\times(280)^3}{12}\right]}$$

$$= \frac{497.5\times10^6\times150}{81.393\times10^6}$$

$= 916.844$ N/mm^2 = Max principal stress

Let us investigate the point of max SF

In this case let us investigate points c, b, a

$$\text{Shear stress at NA} = \frac{VA\bar{y}}{Ib} = \frac{100.25\times10^3\times[10\times140\times70+150\times10\times145]}{81.393\times10^6\times10}$$

$= 38.86$ N/mm^2

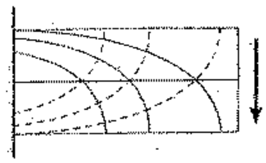
At NA there will be simple shear condition

$\Rightarrow$ Max principal stress = 38.86 N/mm^2

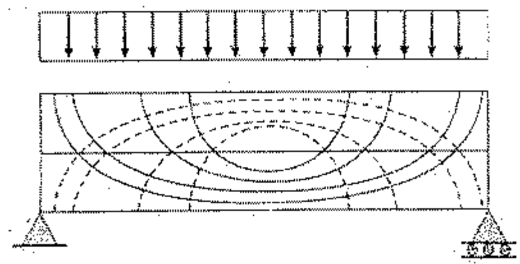
We should also investigate point 'b'. But in this case, the shear stress will be less than that at N.A. and bending stress will also be very small because BM is very small.

Hence, max principal stress any where in the beam = 916.844 N/mm^2.

STRESS TRAJECTORIES



Cantilever



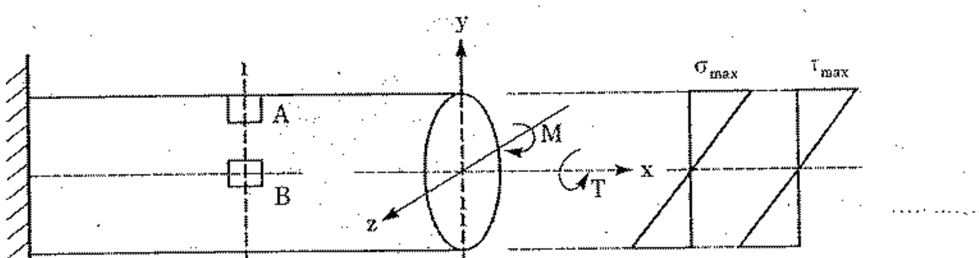
Simply supported

If every point in a beam is analysed such that we can locate the directions of max tensile stress and max compressive stress at all points, then we can draw two sets of curves on the beam faces such that tangent to 1st set of curves at any point represents the direction of max tensile stress and tangent to other sets of curves at any point represents the direction of max compressive stress. These two sets of curves combinedly are called stress trajectories.

Properties of Stress Trajectories

1. Tangent to solid line represents the direction of max tensile stress.
2. Tangent to dashed line represents the direction of max compressive stress.
3. In brittle material such as concrete, failure will occur in tension along planes perpendicular to tensile stress trajectory. Hence steel reinforcement should be placed intersecting these planes.
4. Stiffners attached to the web of plate-girder will be effective in preventing buckling only if they intersect the lines ⊥r to the compressive stress trajectories.

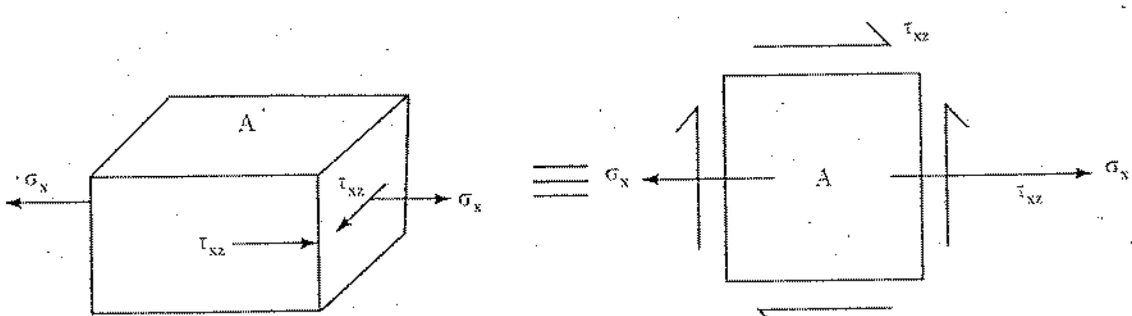
COMBINED BENDING AND TORSION

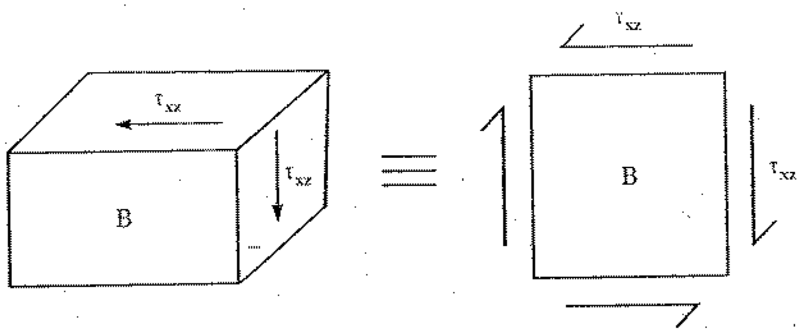


Plain stress conditions

Bending stress variation

Torsional shear stress variation





At point A

$$\tau_{xz} = \tau_{max} = \frac{Tr_{max}}{J} = \frac{T\,D/2}{\frac{\pi D^4}{32}} = \frac{16\,T}{\pi D^3}$$

$$\boxed{\tau_{max} = \frac{16\,T}{\pi D^3}}$$

$$\sigma_x = \sigma_{max} = \frac{My_{max}}{I} = \frac{M \times D/2}{\frac{\pi D^4}{64}} = \frac{32\,M}{\pi D^3}$$

$$\boxed{\sigma_{max} = \frac{32\,M}{\pi D^3}}$$

At point B

$$\tau_{xy} = -\frac{16T}{\pi D^3}$$

Principal Stresses at A

$$\sigma_{1/2} = \frac{\sigma_{max}}{2} + \sqrt{\left(\frac{\sigma_{max}}{2}\right)^2 + (\tau_{max})^2}$$

$$\sigma_{1/2} = \frac{16\,M}{\pi D^3} \pm \sqrt{\left(\frac{16M}{\pi D^3}\right)^2 + \left(\frac{16T}{\pi D^3}\right)^2}$$

$$\boxed{\sigma_{1/2} = \frac{16}{\pi D^3}\left[M \pm \sqrt{M^2 + T^2}\right]}$$

$$\boxed{\tau_{max} = \frac{16}{\pi D^3}\sqrt{M^2 + T^2}}$$

τ_{max} occur at 45° to principal stress.

At point B

We have condition of simple shear

$$\boxed{\sigma_{1/2} = \pm\frac{16T}{\pi D^3}}$$

EQUIVALENT TORQUE OR MOMENT

- **Equivalent torque** is that torque which, while acting alone, produces max shear stress equal to the max shear stress due to combined action of bending and torsion

$$\frac{16T_e}{\pi D^3} = \frac{16}{\pi D^3}\sqrt{M^2 + T^2}$$

$$\boxed{T_e = \sqrt{M^2 + T^2}}, \; T_e = \text{equivalent torque}$$

- **Equivalent moment** is that moment which while acting alone produces max normal stress equal to the max principal stress due to combined action of bending and torsion.

$$\frac{32M_e}{\pi D^3} = \frac{16}{\pi D^3}\left(M + \sqrt{M^2 + T^2}\right)$$

$$\boxed{M_e = \frac{1}{2}\left(M + \sqrt{M^2 + T^2}\right)}, \; M_e = \text{equivalent bending moment}$$

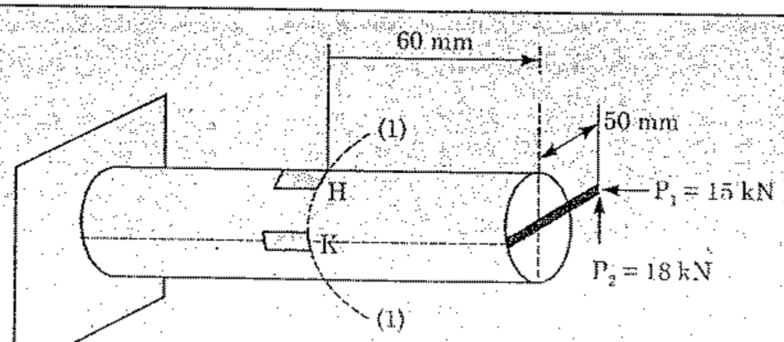
- These equivalent torque and bending moment are used for design of the section i.e. to find out diameter of the section.
- It should be clearly noted that when BM and torque are acting simultaneously, and we have to design a section, then we are not interested in the max stress at a point.
- Rather, we are interested in the max possible value of normal stress and shear stress any where in the section.
- This is the concept on which equivalent moment M_e and equivalent torque T_c has been defined.

i.e. $\boxed{\dfrac{M_e(D/2)}{\dfrac{\pi D^4}{64}} \le \sigma_{permissible}}$, and

$$\boxed{(T_e \times D/2)\Big/\frac{\pi D^4}{32} = \tau_{permissible}}$$

- Thus, while designing a section we will ensure that max normal stress due to M_e is less than the max permissible normal stress and at the same time, max shear stress corresponding to T_e must be less then max permissible shear stress.

Example: 3



Radius of cylinder = 20 mm

1. Find normal and shear stress at K.
2. Principal stress at K.
3. Max shear stress at K.

Sol:

At section (1)-(1) the various forces, moments and torques are as shown in the figure above.

At section (1)-(1)

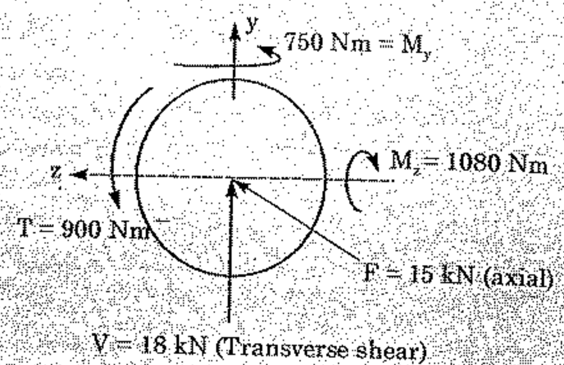
Axial force = 15 kN

Transverse shear force = 18 kN

Torque = 18 × 50 kN mm = 900 Nm

Bending moment M_y = 15 × 50 kNmm = 750 Nm

Bending moment M_z = 18 × 60 kNmm = 1080 Nm



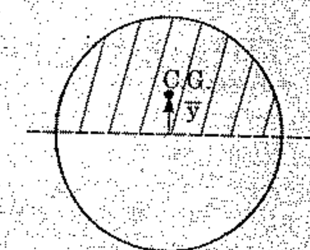
Area of the Section $= \pi(0.020)^2$

$= 1.257 \times 10^{-3}\ m^2$

$$I_y = I_z = \frac{\pi(0.020)^4}{4} = 125.7 \times 10^{-9}\ m^4$$

$$J = 2 \times I_y = 251.3 \times 10^{-9}\ m^4$$

$$\bar{y} = \frac{4r}{3\pi}$$

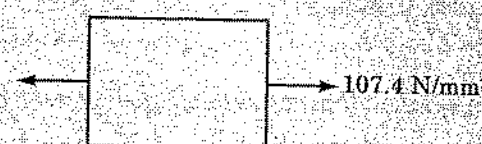


$$A\bar{y} = \left(\frac{1}{2}\pi r^2\right)\left(\frac{4r}{3\pi}\right)$$

$$= \frac{2r^3}{3} = \frac{2}{3}(0.020)^3$$

$$= 5.33 \times 10^{-6}\ m^3$$

Normal Stress at K

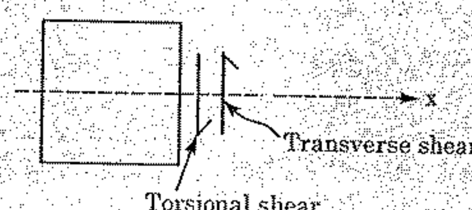


$$\sigma_x = -\frac{F}{A} + \frac{M_y r}{I_y} + 0$$

$$= -11.9\ \frac{N}{mm^2} + \frac{750\ Nm \times 0.020\ m}{125.7 \times 10^{-9} m^4}$$

$$= (-11.9 + 119.33) = 107.4\ N/mm^2$$

Shear Stress at K

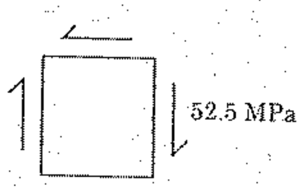


$$\text{Torsional shear} = -\frac{Tr}{J}$$

$$= -\frac{900\ Nm \times 0.020\ m}{251.3 \times 10^{-9} m^4} = -71.6\ N/mm^2$$

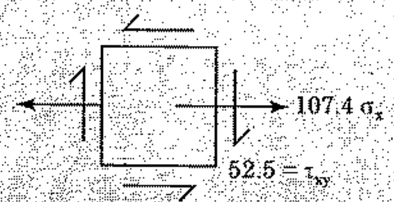
$$\text{Transverse shear stress} = \frac{VA\bar{y}}{Ib} = \frac{18 \times 10^3 N \times 5.33 \times 10^{-6} m^3}{125.7 \times 10^{-9} m^4 \times 2 \times 0.020\ m}$$

$$= 19.1\ MPa$$



Net shear stress = −71.6 + 19.1 = −52.5 MPa

Principal Stresses



$$\sigma_{1/2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \left(\tau_{xy}\right)^2}$$

$$= \frac{107.4}{2} \pm \sqrt{\left(\frac{107.4}{2}\right)^2 + (52.5)^2}$$

$$= 53.7 \pm 75.1$$

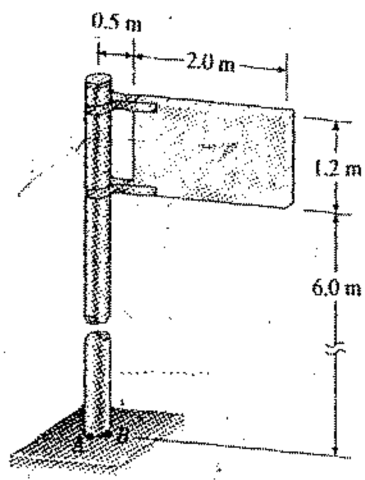
$$\sigma_{max} = 128.8 \text{ N/mm}^2$$

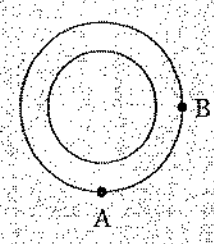
$$\sigma_{min} = -21.4 \text{ N/mm}^2$$

Max Shear Stress = 75.1 N/mm²

Example 4

A sign of dimensions 2.0m × 1.2m is supported by a hollow circular pole having outer diameter 220mm and inner diameter 180 mm. The sign is offset 0.5m from the centerline of the pole and its lower edge is 6.0m above the ground.



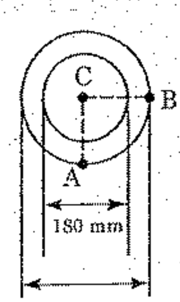


Determine the principal stresses and maximum shear stresses at points A and B at the base of the pole due to a wind pressure of 2.0 kPa against the sign.

Sol: Stress resultants: The wind pressure against the sign produces a resultant force W that acts at the midpoint of the sign and is equal to the pressure p times the area A over which it acts:

$$W = pA = (2.0 \text{ kPa})(2.0\text{m} \times 1.2\text{m}) = 4.8 \text{ kN}$$

The line of action of this force is at height h = 6.6 m above the ground and at distance b = 1.5 m from the centerline of the pole.



The wind force acting on the sign is statically equivalent to a lateral force W and a torque T acting on the pole. The torque is equal to the force W times the distance b:

T = Wb = (4.8 kN)(1.5 m) = 7.2 kN.m

The stress resultants at the base of the pole consist of a bending moment M, a torque T, and a shear force V. Their magnitudes are

M = Wh = (4.8 kN)(6.6m) = 31.68 kN.m

T = 7.2 kN.m V = W = 4.8 kN

Examination of these stress resultants shows that maximum bending stresses occur at point A and maximum shear stresses at point B.

The stress σ_A is obtained from the flexure formula:

$$\sigma_A = \frac{M(d_2/2)}{I}$$

in which d_2 is the outer diameter (220mm) and I is the moment of inertia of the cross section. The moment of inertia is

$$I = \frac{\pi}{64}\left(d_2^4 - d_1^4\right) = \frac{\pi}{64}\left[(220\text{mm})^4 - (180\text{mm})^4\right] = 63.46\times10^{4}\ \text{m}^4$$

in which d_1 is the inner diameter. Therefore, the stress σ_A is

$$\sigma_A = \frac{Md_2}{2I} = \frac{(31.68\ \text{kN.m})(220\text{mm})}{2(63.46\times10^{-6}\text{m}^4)} = 54.91\ \text{MPa}$$

The torque T produces shear stresses τ_1 at points A and B. We can calculate these stresses from the torsion formula:

$$\tau_1 = \frac{T(d_2/2)}{I_P}$$

in which I_P is the polar moment of inertia:

$$I_P = \frac{\pi}{32}\left(d_2^4 - d_1^4\right) = 2I = 126.92\times10^{-6}\ \text{m}^4$$

Thus,

$$\tau_1 = \frac{Td_2}{2I_P} = \frac{(7.2\ \text{kN.m})(220mm)}{2(126.92\times10^{-6}\text{m}^4)} = 6.24\ \text{MPa}$$

Finally, we calculate the shear stresses at points A and B due to the shear force V. The shear stress at point A is zero, and the shear stress at point B is obtained from the shear formula for a circular tube.

$$\tau_2 = \frac{4V}{3A}\left(\frac{r_2^2 + r_2 r_1 + r_1^2}{r_2^2 + r_1^2}\right)$$

in which r_2 and r_1 are the outer and inner radii, respectively, and A is the cross-sectional area:

$$r_2 = \frac{d_2}{2} = 110\text{mm} \qquad r_1 = \frac{d_1}{2} = 90\text{ mm}$$

$$A = \pi\left(r_2^2 - r_1^2\right) = 12{,}570\text{ mm}^2$$

$$\therefore\ \tau_2 = 0.76\text{ MPa}$$

Stress elements

For both elements, the y axis is parallel to the longitudinal axis of the pole and the x axis is horizontal. At point A the stresses acting on the element are

$$\sigma_x = 0 \qquad \sigma_y = \sigma_A = 54.91\text{ MPa} \qquad \tau_{xy} = \tau_1 = 6.24\text{ MPa}$$

At point B the stresses are

$$\sigma_x = \sigma_y = 0 \qquad \tau_{xy} = \tau_1 + \tau_2 = 6.24\text{ MPa} + 0.76\text{ MPa} = 7.00\text{ MPa}$$

Since there are no normal stresses acting on the element, point B is in pure shear.

Principal stress and maximum shear stresses at point A.

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

Substituting $\sigma_x = 0, \sigma_y = 54.91$ Mpa, and $\tau_{xy} = 6.24$ MPa, we get

$$\sigma_{1,2} = 27.5\text{ MPa} \pm 28.2\text{ MPa}$$

or $\sigma_1 = \mathbf{55.7\ MPa}$ $\quad \sigma_2 = \mathbf{-0.7\ MPa}$

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\tau_{max} = \mathbf{28.2\ MPa}$$

Because the principal stresses σ_1 and σ_2 have opposite signs, the maximum in-plane shear stresses are larger than the maximum out-of-plane shear stresses. Therefore, the maximum shear stress at point A is 28.2 MPa.

Principal stresses and maximum shear stresses at point B. The stresses at this point are $\sigma_x = 0, \sigma_y = 0$, and $\tau_{xy} = 7.0$ MPa. Since the element is in pure shear, the principal stresses are

$$\sigma_1 = 7.0\text{ MPa} \quad \sigma_2 = -7.0\text{ MPa}$$

and the maximum in-plane shear stress is

$$\tau_{max} = 7.0\text{ MPa}$$

Example 5

A solid steel circular shaft is required to carry a torque of 40 kN-m and a bending moment of 20 kN-m. Determine the size of the shaft by any two theories of failure.

Factor of safety = 2.0

E = 200 kN/mm²

Yield stress = 250 N/mm², μ = 0.3

Sol:

Torque = 40 kNm f_y = 250 N/mm²

BM = 20 kNm μ = 0.3

E = 2 × 10⁵ N/mm²

F.o.s. = 2

Determine size of shaft by any two method.

Let us take max principal stress theory and max shear stress theory.

(i) Max principal stress theory

$$\text{Max principal stress} \le \left(\frac{f_y}{\text{F.o.s.}}\right)$$

$$\text{Equivalent BM} = \frac{1}{2}\left(M + \sqrt{M^2 + T^2}\right)$$

$$= 32.36 \text{ kNm}$$

$$\text{Max principal stress} = \frac{32\,\text{Meq}}{\pi D^3} \le \frac{250}{2}$$

$$\frac{32 \times 32.36 \times 10^6}{\pi \times D^3} \le 125$$

$$D \ge \left(\frac{32 \times 32.36 \times 10^6}{3.14 \times 125}\right)^{1/3}$$

$$\boxed{D \ge 138.179 \text{ mm}}$$

(ii) Max shear stress theory

$$\text{Max shear stress} \le \frac{f_y}{2 \times \text{FoS}}$$

$$\Rightarrow \quad \frac{16\,T_e}{\pi D^3} \le \frac{f_y}{2 \times \text{F.o.s.}}$$

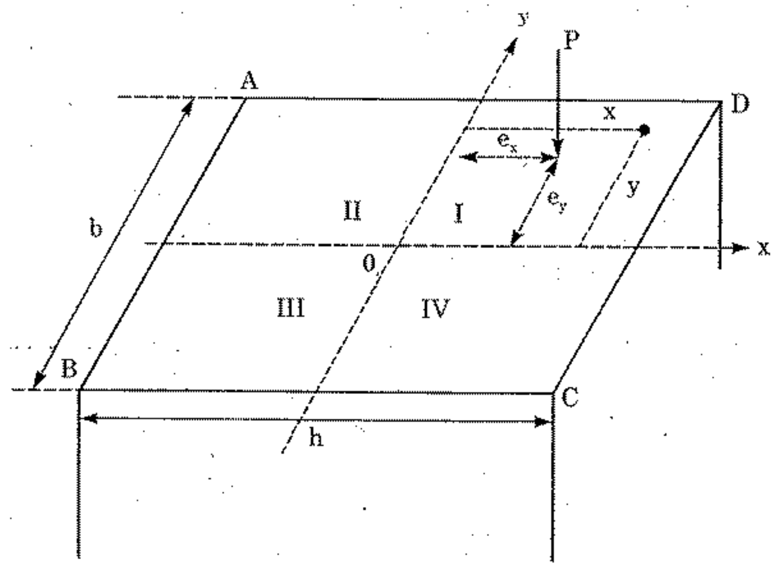
$$T_e = \sqrt{M^2 + T^2} = 44.721 \text{ kNm}$$

$$\Rightarrow \quad D^3 \ge \left(\frac{16 \times 44.721 \times 10^{-6} \times 2 \times 2}{3.14 \times 250}\right)$$

$$\Rightarrow \quad D \ge 153.9 \text{ mm}$$

COMBINED BENDING AND AXIAL FORCE

- The figure shown below shows the section of a column in which there is no chance of buckling.
- When there is no chance of buckling, normal stress at any location is given by



$$\sigma = -\frac{P}{A} - \frac{(Pe_x)\,x}{I_y} - \frac{(Pe_y)\,y}{I_x}$$

- In the above expression compression is (–) ve and tension is (+) ve
- By putting e_x, x, e_y and y with their algebraic sign, the value of σ can be calculated. If σ is (+) ve ⇒ normal stress is tensile and if σ comes out to be (–) ve, it means that σ at that point is compressive.
- Note that x, y, e_x and e_y are (+) ve in 1st quadrant e_x, x is (–) ve in 2nd quadrant, e_y, y is (+) ve in 2nd quadrant etc.
- To determine neutral axis or line of zero stress, we equate σ = 0

$$\Rightarrow \quad \frac{P}{A}\left[1 + \frac{e_x x}{r_y^2} + \frac{e_y y}{r_x^2}\right] = 0$$

$$\Rightarrow \quad \boxed{1 + \left(\frac{e_x}{r_y^2}\right)x + \left(\frac{e_y}{r_x^2}\right)y = 0}$$, equation of neutral axis

where, r_x and r_y are radius of gyrations of the section about x and y axis respectively

e_x = eccentricity along x-axis

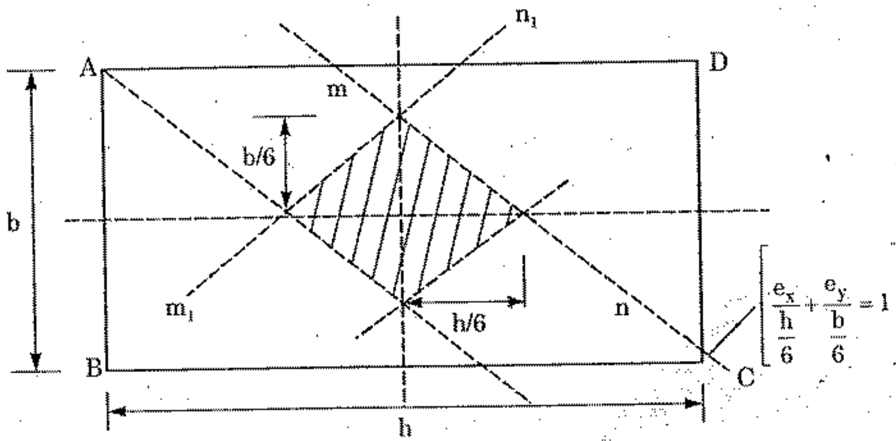
e_y = eccentricity along y-axis

- If N.A. is to pass through point B, then $\sigma_B = 0$

$$\Rightarrow \quad \text{By putting } x = -\frac{h}{2} \text{ and } y = -\frac{b}{2}$$

$$0 = -\frac{P}{bh} - \frac{Pe_x\left(-\frac{h}{2}\right)}{\frac{bh^3}{12}} - \frac{Pe_y\left(-\frac{b}{2}\right)}{\frac{hb^3}{12}}$$

$$\Rightarrow \boxed{\frac{e_x}{h/6} + \frac{e_y}{b/6} = 1}$$

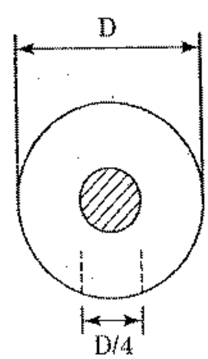


By simple analysis it can be proved that:

- if loading is to the left of line mn, stress at B will be compressive.
- if loading is to the right of mn, stress at B will be tensible.
- mn is the locus of points of application of 'P' for which corner B will have zero stress.
- On similar line, it can be shown that if loading is inside the shaded area, there will be not tension any where in the x-section.

This area is called Kern of the section.

- **Kern of Circular Section**



Note: $\frac{P}{\frac{\pi D^2}{4}} - \frac{Pe\,D/2}{\frac{\pi D^4}{64}} \geq 0$ for no tension any where in the section.

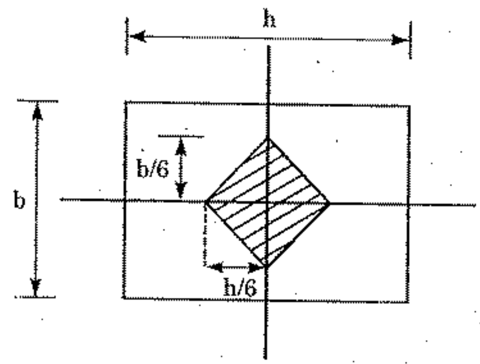
$$\Rightarrow \quad \frac{4P}{\pi D^2}\left(1-\frac{8e}{D}\right) \ge 0 \text{ for no tension}$$

$$\Rightarrow \quad \frac{8e}{D} \le 1$$

$$\Rightarrow \quad e \le D/8$$

$\Rightarrow$ Kern of a circular section will be a circle of dia $\frac{D}{4}$.

- **Kern of Rectangular Section**



Kern is of Rhompus shape

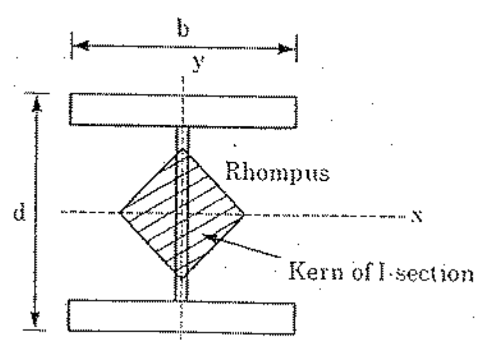
- **Kern of I-section**

$$\sigma = -\frac{P}{A} - \frac{(Pe_x)\,x}{I_y} - \frac{(Pe_y)\,y}{I_x}$$

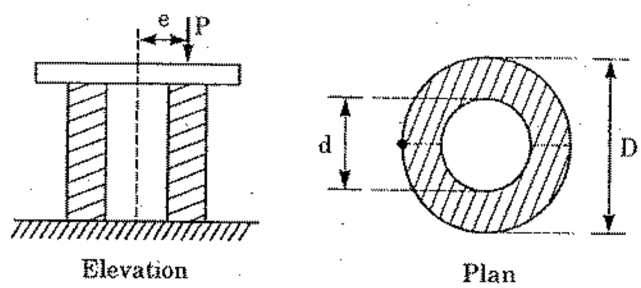
For zero stress

$$\frac{e_x \times x}{\left(\frac{I_y}{A}\right)} + \frac{e_y \times y}{\left(\frac{I_x}{A}\right)} = -1$$

By putting $x = \frac{b}{2}$ and $y = \frac{d}{2}$ we have $\frac{e_x \times b}{2k_x^2} + \frac{e_y \times d}{2k_y^2} = -1$. This equation represents the boundary line of Kern for I-section.



Kern of Hollow Circular Section



For no tension anywhere in the x-section of hollow section.

$$\sigma = \frac{-4P}{\pi(D^2 - d^2)} + \frac{Pe\,D/2}{\frac{\pi}{64}\left(D^4 - d^4\right)} \le 0$$

$$\Rightarrow \quad \frac{32\,Pe}{\pi\frac{(D^4 - d^4)}{D}} \le \frac{4P}{\pi(D^2 - d^2)}$$

$$e \le \frac{4P}{\pi(D^2 - d^2)} \times \frac{\pi(D^4 - d^4)}{D \times 32P}$$

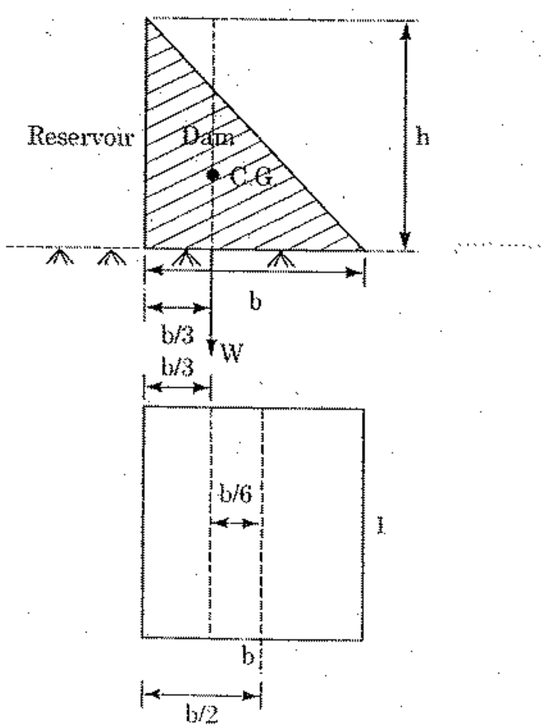
$$\Rightarrow \quad e \le \frac{D^2 + d^2}{8D}$$

$$\Rightarrow \quad \boxed{\text{dia of Kern} = \frac{D^2 + d^2}{4D}}$$

Kern is circular

DAMS AND RETAINING WALLS

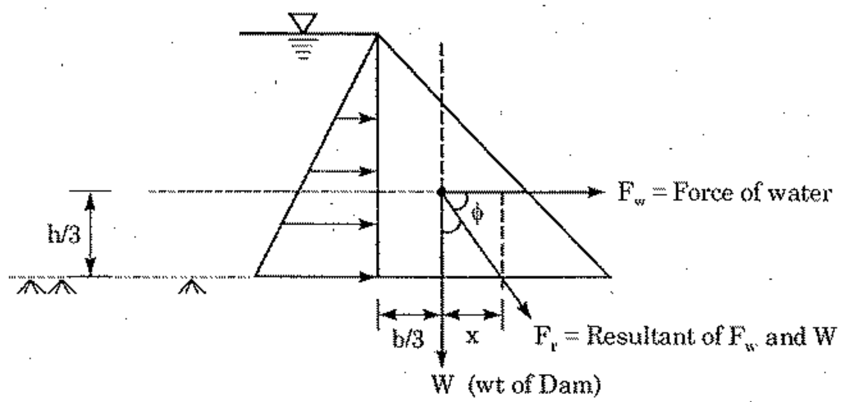
Empty Condition



- Hence under empty condition eccentricity $= \frac{b}{6}$

  $\Rightarrow$ No tension will develop at base because tension develops only when $e > \frac{b}{6}$.

Full Condition:



- Resultant of F_w and W makes an angle ϕ with F_w

  $\Rightarrow \tan\phi = \frac{W}{F_w}$

- if γ_m = unit wt of material

  γ_w = unit wt of water

  then,

$$\tan\phi = \frac{\frac{1}{2}\times b\times h\times 1\times\gamma_m}{\frac{\gamma_w h\times h\times 1}{2}} = \frac{b}{h}\cdot\frac{\gamma_m}{\gamma_w}$$

$$\tan\phi = \frac{h}{3x}$$

$$\Rightarrow \quad x = \frac{h}{3\tan\phi} = \frac{h\times h\gamma_w}{3\times b\gamma_m}$$

- Eccentricity of resultant load

$$= \left(\frac{b}{3} + x - \frac{b}{2}\right)$$

$$= \frac{b}{3} + \frac{h^2}{3b}\left(\frac{\gamma_w}{\gamma_m}\right) - \frac{b}{2} = -\frac{b}{6} + \frac{h^2\gamma_w}{3b\,\gamma_m}$$

- For no tension at base, $\left(-\frac{b}{6} + \frac{b^2}{3b\cdot G_m}\right) \le \frac{b}{6}$, $G_m = \frac{\gamma_m}{\gamma_w}$ = sp. gravity of dam material

$$\Rightarrow \quad \frac{h^2}{3b\,G_m} \le \frac{b}{3}$$

$$\boxed{G_m \ge \frac{h^2}{b^2}}$$

OBJECTIVE QUESTIONS

1. A hollow circular column of internal diameter 'd' and external diameter '1.5d' is subjected to compressive load. The maximum distance of the point of application of load from the centre for no tension is
   (a) d/8 (b) 13d/48
   (c) d/4 (d) 13d/96

2. Match List-I (Given sections) with List-II (Shape of the core) to ensure no tension condition and select the correct answer using the codes given below the lists:

List-I	List-II
A. Rectangular	1. Circle
B. I-section	2. Annulus
C. Hollow circular	3. Rhombus
D. Square	4. I-section
	5. Square
	6. Rectangular

Codes:

	A	B	C	D
(a)	4	6	1	2
(b)	3	3	1	5
(c)	6	4	2	5
(d)	3	4	5	1

3. Which one of the following rules ascertains the maximum permissible eccentricity of loads on circular column so that stresses will always be compressive?
   (a) Middle fourth rule
   (b) Middle third rule
   (c) Middle half rule
   (d) Middle two-third rule

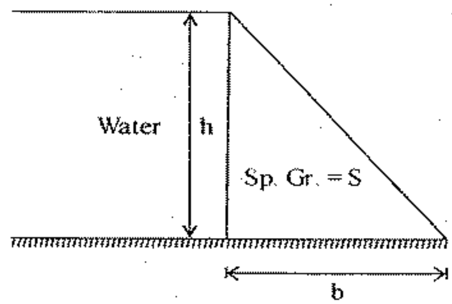
4. **Assertion (A)**: In the vertical face of triangular retaining wall of any h/b ratio, the eccentricity is b/6 for dead storage condition.
   Reason (R): For triangular retaining wall under full storage level (retained by vertical face) the eccentricity will be b/6 only if h/b ratio is $\sqrt{S}$ (where S is specific gravity of the material of the wall).
   Of these statements
   (a) both A and R are true and R is the correct explanation of A
   (b) both A and R are true but R is not a correct explanation of A
   (c) A is true but R is false
   (d) A is false but R is true

5. If ρ is the specific gravity of the material used in the design of a masonry dam of triangular section, then the ratio between the height and base width of the dam for structural safety and stability is equal to
   (a) $\sqrt{2\rho}$ (b) $\sqrt{\rho}$
   (c) $\frac{1}{\rho}$ (d) $\frac{1}{\sqrt{\rho}}$

6.

7.

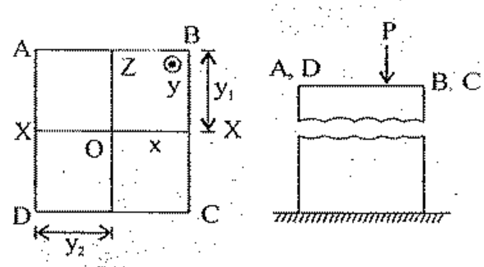
6. Assertion (A): In the retaining wall shown in the given figure when the ratio $\frac{h}{b} = \sqrt{S}$, the eccentricity is $\frac{b}{6}$ whether the storage is nil or full.



Reason (R): The resultant force will pass through the centroid of the pressure distribution diagram on the base, for the nil or full storage if the ratio $\frac{h}{b} = \sqrt{S}$.

Of these statements
(a) both A and R are true and R is the correct explanation of A
(b) both A and R are true but R is not a correct explanation of A
(c) A is true but R is false
(d) A is false but R is true

7. A column ABCD ($2y_1 \times 2y_2$) of rectangular section carries a load P at Z having the coordinates (x, y) as shown in the given figure.



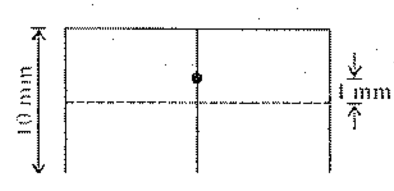
Plan Elevation

If the compressive stresses are taken as positive and area $A = 2y_1 \times 2y_2 = 4y_1 y_2$ and the moment of inertia about X and Y axis being I_{xx} and I_{yy} respectively, then the stress at the corner D is

(a) $\frac{-P}{A} + \frac{Py}{I_{xx}}y_1 + \frac{Px}{I_{yy}}y_2$ (b) $\frac{-P}{A} - \frac{Py}{I_{xx}}y_1 - \frac{Px}{I_{yy}}y_2$

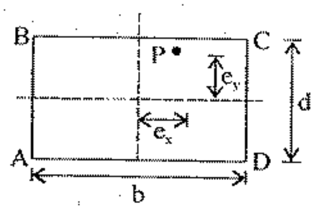
(c) $\frac{-P}{A} + \frac{Py}{I_{yy}}y_1 + \frac{Px}{I_{xx}}y_2$ (d) $\frac{-P}{A} - \frac{Py}{I_{yy}}y_1 - \frac{Px}{I_{xx}}y_2$

8. A tie bar (20 mm × 10 mm) carries a tensile load of 1 kN as shown in the figure below. Under this load, the maximum intensity of stress over the mean value will increase by



(a) 20% (b) 40%
(c) 60% (d) 80%

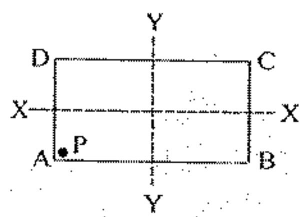
9. The rectangular column shown in the figure below carries a load P having eccentricities 'e_x' and 'e_y' along the X-axis and Y-axis respectively. The stress at any point (x, y) is given by



(a) $\frac{P}{bd}\left[1+\frac{12e_y \times y}{d^2}+\frac{12e_x \times x}{b^2}\right]$ (b) $\frac{P}{bd}\left[1+\frac{12e_y \times y}{b^2}+\frac{12e_x \times x}{d^2}\right]$

(c) $\frac{P}{bd}\left[1+\frac{6e_y \times y}{d^2}+\frac{6e_x \times x}{b^2}\right]$ (d) $\frac{P}{bd}\left[1+\frac{6e_y \times y}{b^2}+\frac{6e_x \times x}{d^2}\right]$

10. A reinforced concrete footing loaded with a concentrated load P as shown in the given figure produces maximum bending stresses of 10 kN/m² and 15 kN/m² due to eccentri cities about XX and YY axes respectively. If the direct stress due to load acting at P is 18 kN/m² (compressive), then the intensity of resultant stress at corner B will be



(a) 13 kN/m² tensile (b) 13 kN/m² compressive
(c) 31 kN/m² compressive (d) 31 kN/m² tensile

11. If the eccentricity of total self-weight W of a masonry dam at its base is equal to one-fourth of base width B, then the maximum pressure at the base is given by
(a) 2W/3B (b) 4W/3B
(c) 5W/2B (d) 8W/3B

12. A column base is subjected to moment. If the intensity of bearing pressure due to axial load is equal to stress due to the moment, then the bearing pressure between the base and the concrete is
(a) uniform compression throughout
(b) zero at one end and compression at the other end
(c) tension at one end and compression at the other end
(d) uniform tension throughout

13. A short hollow CI column section A is 150 cm² and the section modulus Z = 10 × 10⁵ mm³ carries
(i) an axial load of 250 kN, and
(ii) a load of 50 kN on a bracket, the load line being 500 mm from the axis of column
The maximum and minimum stress intensities are
(a) 50 N/mm² tensile and 10 N/mm² compressive
(b) 45 N/mm² compressive and 5 N/mm² tensile
(c) 55 N/mm² compressive and 5 N/mm² tensile

(d) 60 N/mm^2 tensile and 10 N/mm^2 compressive

14. A section of a solid circular shaft with diameter D is subjected to bending moment M and torque T. The expression for maximum principal stress at the section is

(a) $\frac{2M + T}{\pi D^3}$ (b) $\frac{16\pi}{D^3}\left(M + \sqrt{M^2 + T^2}\right)$

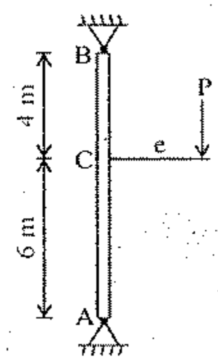
(c) $\frac{16\pi}{D^3}\left(\sqrt{M^2 + T^2}\right)$ (d) $\frac{16}{\pi D^3}\left(M + \sqrt{M^2 + T^2}\right)$

15. As per the maximum principal stress theory, when a shaft is subjected to a bending moment M and torque T. And σ is the allowable stress in axial tension, then the diameter d of the shaft is given by

(a) $d^3 = \frac{16}{\pi\sigma}\left[M + \sqrt{M^2 + T^2}\right]$ (b) $d^3 = \frac{4}{\pi\sigma}\left[M + \sqrt{M^2 + T^2}\right]$

(c) $d^3 = \frac{32}{\pi\sigma}\left[M + \sqrt{M^2 + T^2}\right]$ (d) $d^3 = \frac{8}{\pi\sigma}\left[M + \sqrt{M^2 + T^2}\right]$

16. Consider the following statements for the column with a bracket as shown in the figure given below:



1. Shear force is constant throughout
2. Maximum moment in the column is Pe.
3. The compressive axial force in the column is 0.4P.

Which of these statements is/are correct?

(a) 1, 2 and 3 (b) 1 only

(c) 1 and 3 (d) 2 only

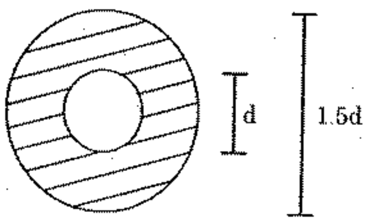
ANSWERS

1.	(b)	5.	(b)	9.	(a)	13.	(b)
2.	(b)	6.	(a)	10.	(b)	14.	(d)
3.	(a)	7.	(b)	11.	(c)	15.	(a)
4.	(b)	8.	(c)	12.	(b)	16.	(c)

SOLUTION...

1. (b) For no tension

Middle Fourth Rule for Circle

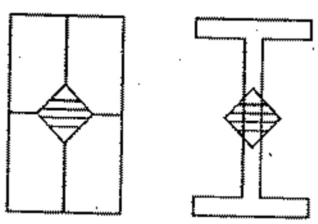


$e_{max} \le \frac{D^2 + d^2}{8D}$ [Observe that core is concentric circle]

$$\le \frac{\frac{9d^2}{4} + d^2}{8 \times \frac{3}{2}.d} \le \frac{\frac{13}{4}}{12}.d$$

$$e \le \frac{13}{48}d$$

2. (b)



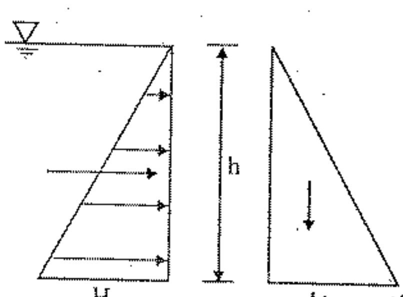
C/S - Core Shape

Rectangle } Rhombus
I section }

Circle (hollow and solid) → concentric circle

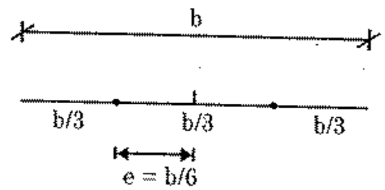
Square → Square

4. (b)



A : Middle third rule statement is given.

Dead storage condition } These are two extreme cases
Full storage condition }

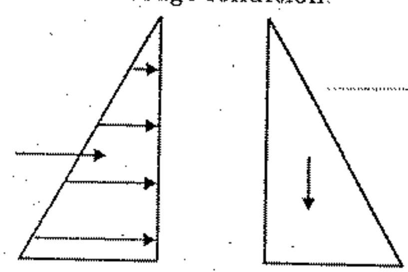


(a) For dead storage / empty case.

For no tension, resultant must pass through inner middle third pt.

$$e = \frac{b}{6} \text{ [for any } \frac{h}{b} \text{ ratio]}$$

(b) For full storage condition.



For no tension, resultant must pass through outer middle third pt.

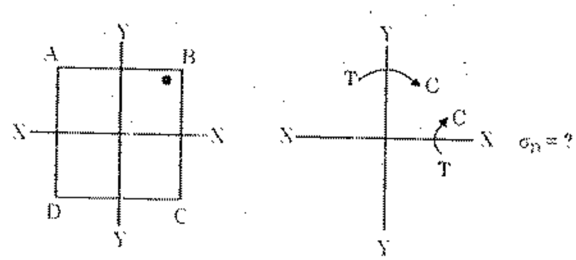
$$\bar{x} = \frac{\Sigma M}{\Sigma W}$$

$$e = \frac{b}{6} \text{ when } \frac{h}{b} = \sqrt{S}$$

[Considering no uplift]

Derived Elementary profile.

7. (b)



∴ Both the moments cause tension at 'D'

$$\sigma_D = -\frac{P}{A} - \left[\frac{MY}{I_{xx}}\right] - \left[\frac{MX}{I_{yy}}\right]$$

$$= -\frac{P}{A} - \frac{Pyy_1}{I_{xx}} - \frac{Px}{I_{yy}} \cdot y_2$$

8. (c)

$$\sigma = \frac{P}{A} + \frac{M}{Z} = \frac{P}{A} + 0.6\frac{P}{A}$$

$$\frac{M}{Z} = \frac{P \times e}{\left(\frac{bd^2}{6}\right)} = \frac{P}{bd} \times \frac{1\,mm}{\left(\frac{10}{6}\right)}$$

$$= \frac{P}{bd} \times \left(\frac{3}{5}\right)$$

$$\sigma = \left(1 + \frac{3}{5}\right)\frac{P}{A} = 1.6\ P/A$$

9. (a)

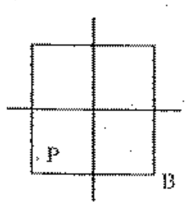
$$\sigma = \frac{P}{A} + \frac{MY}{I_X} + \frac{MX}{I_y}$$

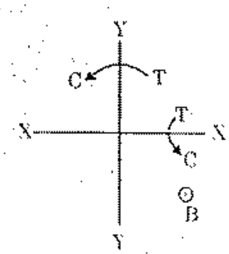
[Remember this format this eliminates option many a times.]

$$\sigma = \frac{P}{A} + \frac{(Pe_y)y}{\frac{bd^3}{12}} + \frac{(Pe_x)x}{\frac{b^3d}{12}}$$

$$= \frac{P}{A}\left[1 + \frac{12 e_y \cdot y}{d^2} + \frac{12 e_x \cdot x}{b^2}\right] \quad [A = bd]$$

10. (b)



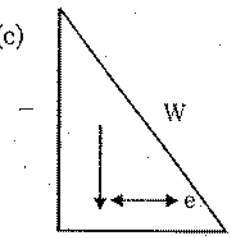


$$\sigma_B = +\left(\frac{P}{A}\right) + \left(\frac{MY}{I_x}\right) - \left(\frac{MX}{I_y}\right)$$

$$= +18 + 10 - 15$$

$$= +13 = 13 \text{ compressive}$$

11. (c)



$$\sigma_{max} = \frac{P}{A} + \frac{M}{Z} = \frac{W}{B \times 1} + \frac{W \times \left(\frac{B}{4}\right)}{1 \times \frac{B^2}{6}}$$

$$= \frac{W}{B}\left[1 + \frac{6}{4}\right] = \frac{5}{2}\frac{W}{B}$$

12. (b)

$$\frac{P}{A} = \frac{M}{Z}$$

$$\sigma = \frac{P}{A} \pm \frac{M}{Z} = 0 \text{ at one end}$$

comp at other end

13. (b)

$$A = 150\ cm^2$$

$$Z = 10^6\ mm^3$$

$$P_1 = 250\ kN$$

$$P_2 = 50\ kN,\ M = 50 \times 0.5\ kN\text{-}m$$

$$\frac{P}{A} = \frac{300 \times 10^3}{150 \times 100}\frac{N}{mm^2} = 20\ MPa$$

$$\frac{M}{Z} = \frac{50 \times 0.5 \times 10^6}{10^6} N/mm^2$$

$$= 25\ MPa$$

$$\sigma_{max} = 45\ MPa$$

$$\sigma_{min} = -5\ MPa$$

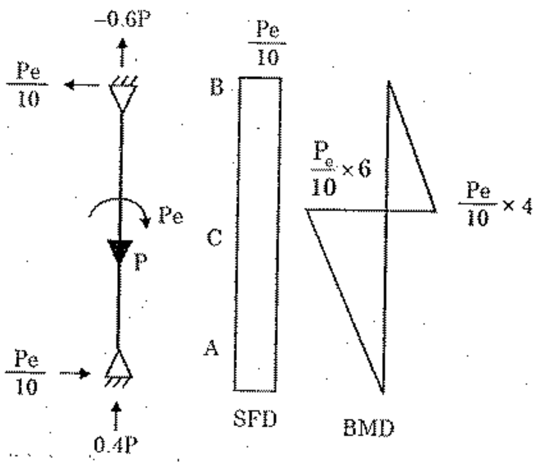
14. (d)

$$\sigma_{1,2} = \frac{f}{2} \pm \sqrt{\frac{f^2}{4} + \tau^2}$$

$$f = \frac{32M}{\pi d^3}, \quad \tau = \frac{16M}{\pi d^3}$$

$$\sigma = \frac{16}{\pi D^3}[M + \sqrt{M^2 + T^2}]$$

16. (c)



Tension in BC = 0.6 P

Comp. in AC = 0.4 P

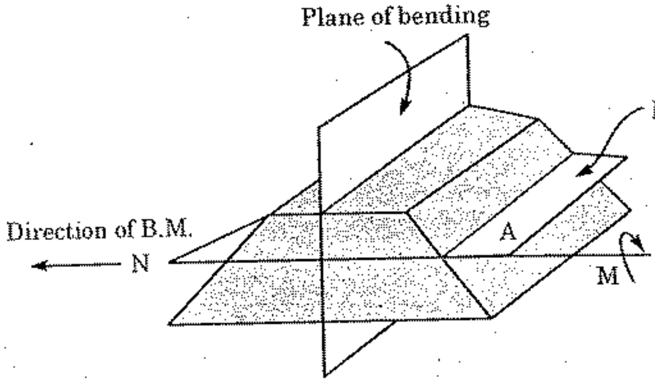
Max BM $\neq$ Pe

$\therefore$ Statement 2 is wrong

6

Bending Stress in Beam

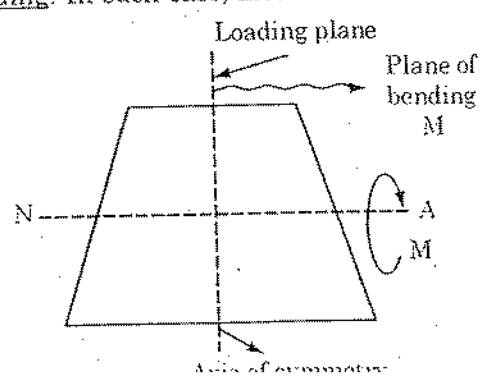
INTRODUCTION



- In the figure shown above M is the applied bending moment.
- The direction of bending moment is given by right hand thumb rule.
- When the direction of bending moment, as given by right hand thumb rule, is parallel to the N.A. (Neutral Axis), bending is said to occur in the plane of bending.
- If however, direction of bending moment and N.A. are different, bending does not take place in the plane of bending.
- Note that plane of bending is $\perp^r$ to the neutral axis (i.e. the axis of zero normal stress) and the plane in which bending moment acts is $\perp^r$ to the direction of moment.

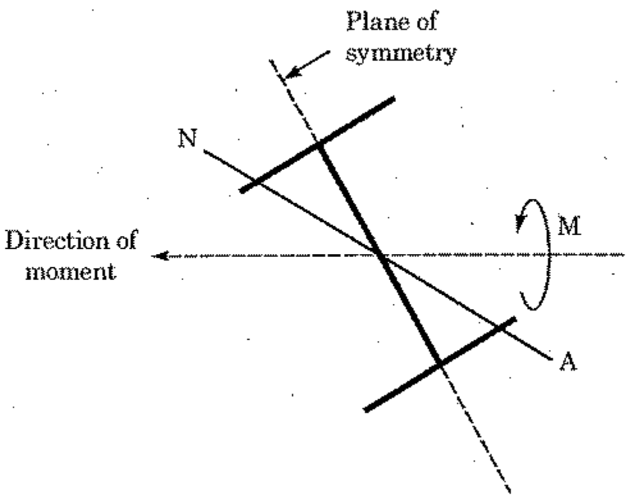
SYMMETRIC BENDING

When member possesses a plane of symmetry and loading (Bending couple) acts in the plane of symmetry, bending is called symmetric bending. In such case, member bends in the plane of couple.

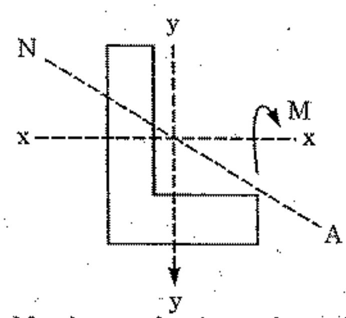


UNSYMMETRIC BENDING

When bending couple doesnot acts in the plane of symmetry of member either because they act in a different plane or because the member does not possess a plane of symmetry, the bending is called unsymmetric bending. In such case, member does not bend in the plane of couple.



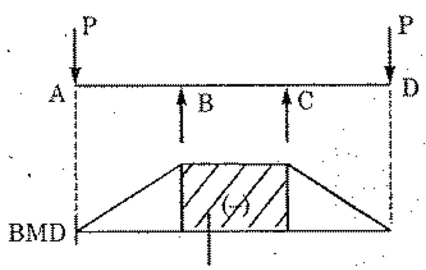
Bending not acting in the plane of symmetry



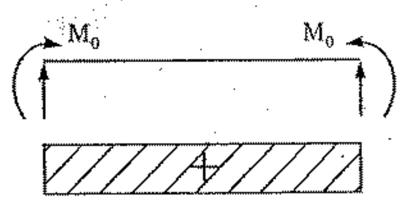
Member not having a plane of symmetry

PURE BENDING

Bending of beam under constant BM is called pure bending



Pure bending in portion BC



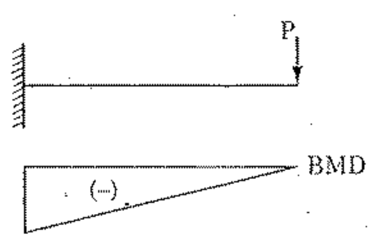
Pure bending throughout the beam

As BM is constant, SF = 0

Hence under pure bending, shear force = 0.

NON-UNIFORM BENDING

Bending of beam in the presence of shear force is called non uniform bending.

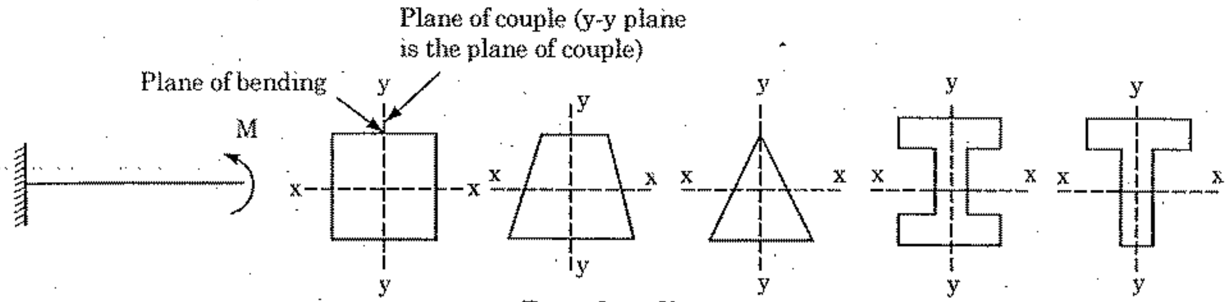


Non uniform bending case

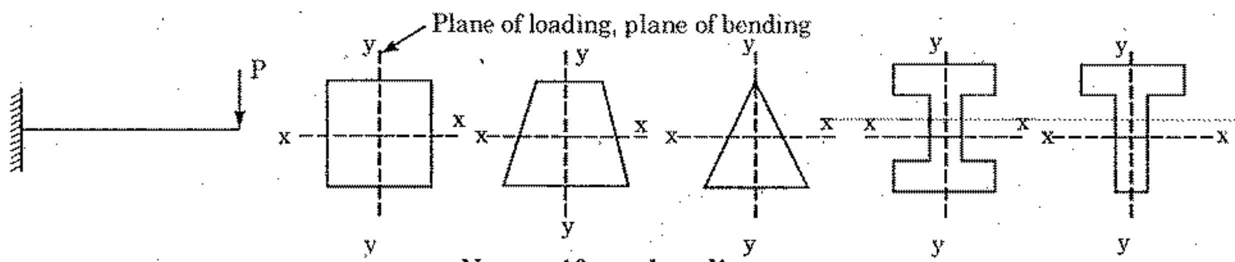
- **Analysis of Pure bending for symmetric condition** yields Flexure formula $\left(\frac{M}{I} = \frac{f}{y} = \frac{E}{R}\right)$

Thus, flexure formula will actually be applicable for pure bending only. However, it can be used for non uniform bending also (without much error).
Following examples show the conditions of pure bending, non-uniform bending and unsymmetrical bending.

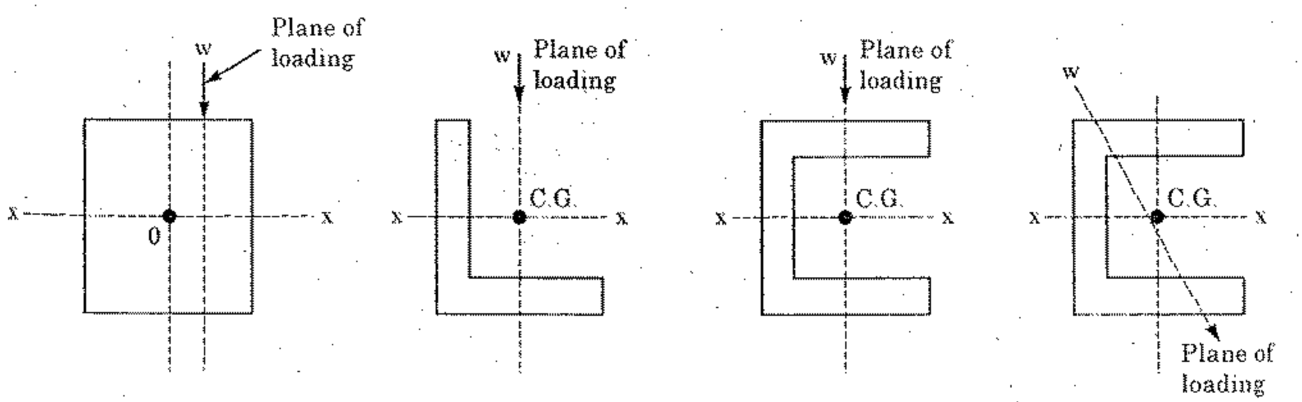
CERTAIN EXAMPLES:



Pure bending



Non-uniform bending



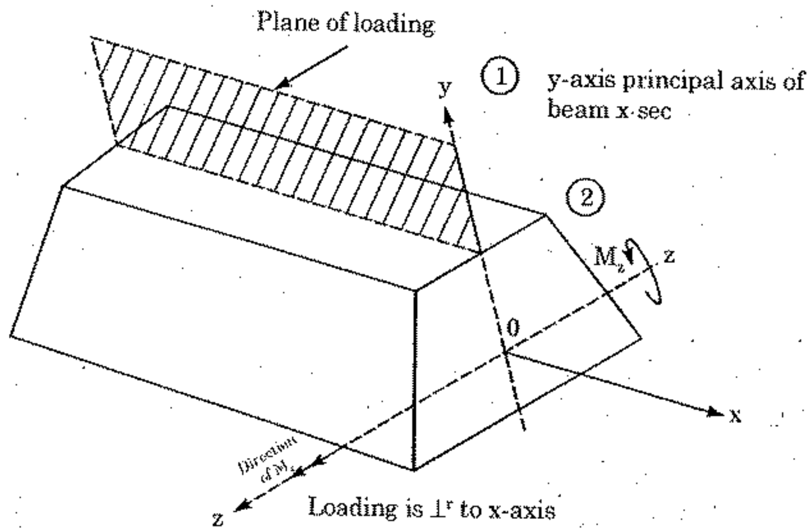
(i) Bending with twisting (ii) Bending with twisting (iii) Bending with twisting (iv) Bending with twisting

***Note:** In case of (i), (ii), (iii) and (iv) if bending is due to pure bending (i.e. bending without shear), twisting will not occur.*

ASSUMPTIONS IN THE THEORY OF PURE BENDING

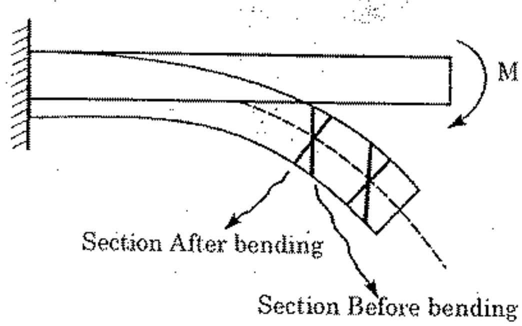
1. The plane section of the beam before bending remains plane after bending.

2. The material in the beam is homogeneous, isotropic and obeys Hooke's law.
3. Modulus of elasticity in tension and compression are equal.
4. Beam is initially straight and has constant cross-section throughout its length (i.e. beam is prismatic).
5. The plane of loading must contain a principal axis of the beam x-section and the loads must be $\perp^r$ to the longitudinal axis of beam.

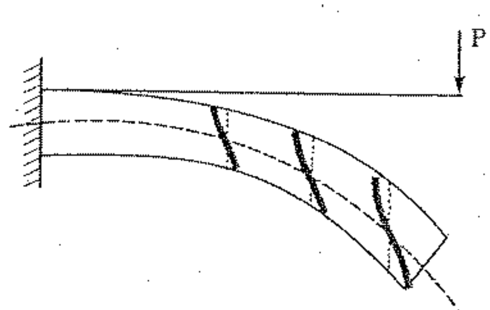


- In the above figure M_z (i.e. bending moment about z-axis) or bending moment in z-direction is $\perp^r$ to the x-axis (longitudinal axis) and the plane of loading contains a principle axis (y-axis). Thus it is case of pure bending.

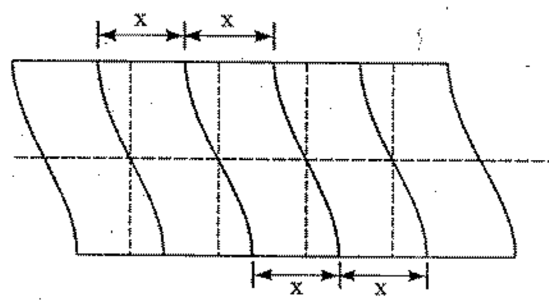
Note:



- This figure shows that Plane section before bending remains plane after bending. As shear stress is zero every where in the beam, the distortion of section will not occur.

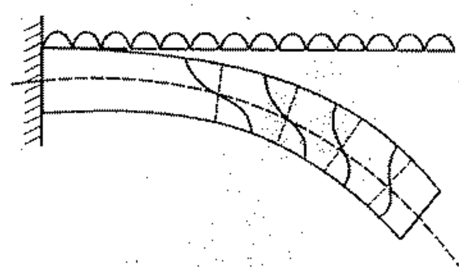


- In the presence of shear, plane section before bending does not remain plane after bending because warping of section occurs in this case. We know that shear stress leads to distortion of the section. As the shear stress variation across the section is parabolic, amount of shear stress does not remain same at a every point of a section. Hence the quantity of distortion is different at different points in a section. The max distortion being at the neutral axis location because shear stress is max at the neutral axis location. This variation of distortion along a cross-section leads to warping. The section having larger value of shear force will warp more.
- Quantity of warping at corresponding points is same at all sections in this case, because SF is constant along the length of beam. Hence normal stress/strain is not affected by shearing stresses.



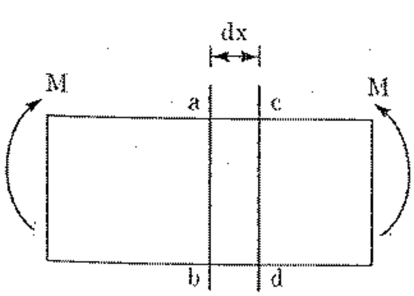
The dotted lines in the above figure shows the plane section after bending and solid lines represent section after bending and warping is a beam.

This figure, shown below, shows that quantity of warping goes on increasing towards the support as SF is increasing towards the support.

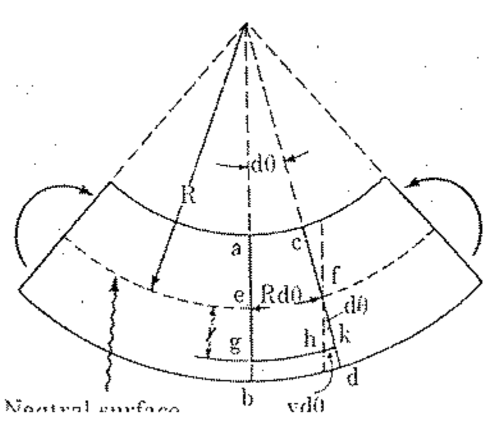


However, the effect of warping on normal stress is considered negligible for the span to depth ratio of beams normally encountered in practice. Thus even if shear stress is acting on a beam, we shall assume it to be a case of pure bending and use flexure formula for its analysis.

FLEXURE FORMULA



Unbent shape of beam



Strain (ε) in Element at a Distance 'y' from N.A.

$$\varepsilon = \frac{y\,d\theta}{ef} = \frac{y\,d\theta}{R\,d\theta}$$

$$\boxed{\varepsilon = \frac{y}{R}} \quad \text{-------------------- (i)}$$

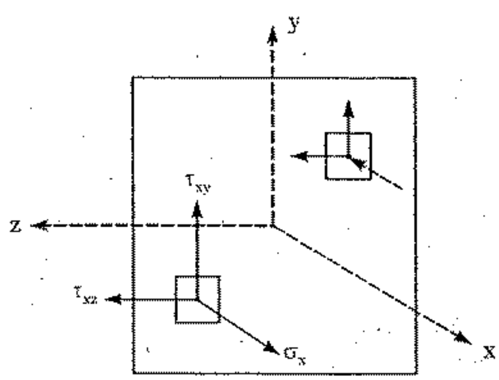
R = radius of curvature of N.A.

Thus, strain variation across the depth is linear

If Hooke's law is applicable then

Bending stress (σ) = $E\varepsilon = \frac{Ey}{R}$

$$\boxed{\sigma = \frac{Ey}{R}} \quad \text{-------------------- (ii)}$$



Area = dA

$$\int \sigma_x \cdot dA \cdot y = M_z \quad \text{-------------------- (A)}$$

$$\Rightarrow \int \frac{Ey}{R} \cdot y\,dA = M_z$$

$$\Rightarrow \frac{E}{R}\int y^2 dA = M_z$$

$$\Rightarrow \boxed{\frac{E}{R} I_z = M_z} \quad \text{-------------------- (iii)}$$

Hence from (ii) and (iii)

$$\boxed{\frac{M_z}{I_z} = \frac{\sigma}{y} = \frac{E}{R}} \quad \text{(Flexure formula) ------------ (iv)}$$

where I_z = moment of inertia about C.G. axis (about which bending occurs)

Note that

(−) Hogging moment lead to Tension at Top and Compression at bottom

(+) Sagging moment leads to Tension at bottom and compression at Top

Also from $\Sigma F_x = 0$

$$\int \sigma_x \cdot dA = 0$$

$$\Rightarrow \int_A \frac{Ey}{R} dA = 0 \text{ (if Hook's law is applicable)}$$

$$\Rightarrow \frac{E}{R}\int_A y dA = 0 \text{ (where y = distance of any point from N.A.)}$$

$$\Rightarrow \frac{E}{R}\cdot A\bar{y} = 0 \Rightarrow \bar{y} = 0$$

(where $\bar{y}$ = distance of centroidal axis from N.A.)

Thus, if Hooks law is applicable, then centroidal axis is the N.A.

$\Sigma M_x = \int(\tau_{xy}z - \tau_{xz}\cdot y)\, dA$ (Twisting moment)

- For section symmetrical about y-axis and having loading in the symmetrical plane, the element under discussion will have symmetrically placed counterpart so the above integral becomes zero.
- Thus for symmetrical section having loading in the plane of symmetry, twisting moment = 0. Otherwise, if plane of loading is not symmetrical, the beam will twist

$M_y = \int_A \sigma_x \cdot z \cdot dA = \int y\frac{E}{R}\cdot z dA = \frac{E}{R}\int yz dA$

$$M_y = \frac{EI_{yz}}{R}$$

where, I_{yz} = Product of inertia

- Product of inertia is zero only if 'y' or 'z' axis is the axis of symmetry or a principal axis.

  Thus, as in our case, plane of loading is the x-y plane which contains y-axis, (an axis of symmetry), $I_{yz} = 0$

Section Equally Strong in both Tension and Compression and of Uniform x-section

Maximum Bending Stress at a Section

$$\frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R}$$

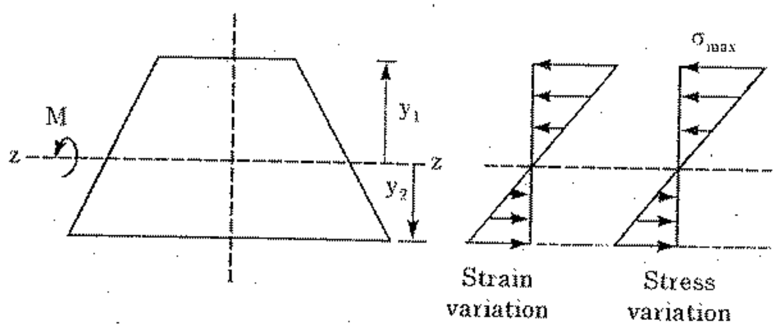
I = moment of inertia about centroidal axis about which bending occurs.

$$\Rightarrow \sigma = \frac{My}{I} = \frac{M}{(I/y)}$$

$$\sigma_{max} = \frac{M}{(I/y_{max})} = \frac{M}{Z}$$

$$\boxed{\sigma_{max} = \frac{M}{Z}}$$

where Z = section modulus about bending axis = $\frac{I}{y_{max}}$



$y_{max} = y_1$

$$\boxed{\frac{I_{zz}}{y_1} = z}$$

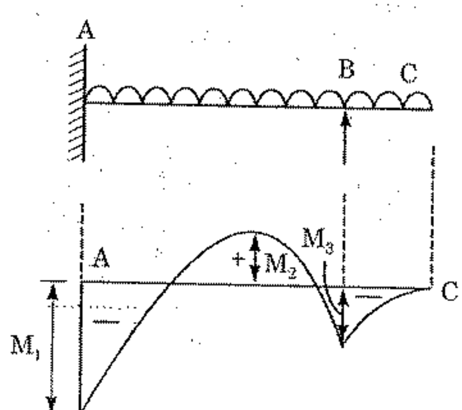
MOMENT OF RESISTANCE (M_R)

Maximum bending moment resisted by a section without undergoing failure is called moment of resistance of the section.

$$\boxed{M_R = \sigma_{permissible} \times Z}$$

where $\sigma_{permissible}$ is permissible bending stress. Its value is yields stress/factors of safety i.e. $\boxed{\frac{\sigma_y}{F.o.s.} \cdot Z}$

- Thus larger the value of section modulus, stronger is the beam.
- For designing a beam of uniform section, the section should be selected such that the moment of resistance of the section becomes equal to the applied maximum bending moment.



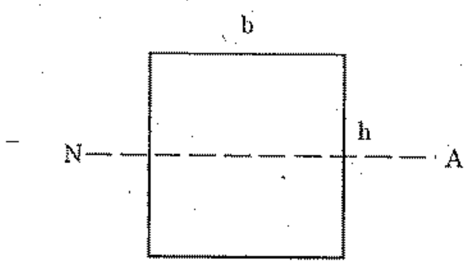
BMD

In the beam showsn above, if beam ABC is to be of uniform section then the relationship $\boxed{\text{Max.}(M_1, M_2, M_3) = \sigma_{permissible} \cdot z}$ is used to design the section.

If AB and BC can be of different section then the relationship $\boxed{\text{Max.}(M_1, M_2, M_3) = \sigma_{permissible} \cdot z}$ is used for design of section in span AB and the relationship $\boxed{M_3 = \sigma_{permissible} \times z_{BC}}$ is used for design of sections in span BC.

SECTION MODULUS (Z)

(a) Rectangular section



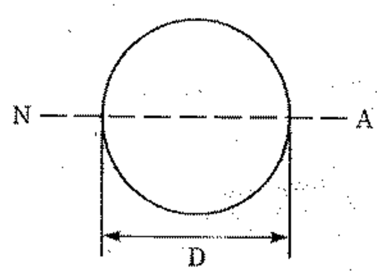
$$I_{N.A.} = \frac{bh^3}{12} = \text{moment of inertia about Neutral axis}$$

$$y_{max} = \frac{h}{2}$$

$$\boxed{Z = \frac{bh^2}{6}} = (bh)\frac{h}{6} = A \times \frac{h}{6}$$

⇒ For a given area if 'h' is more z will be more.

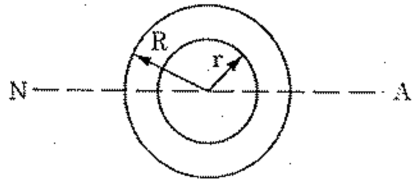
(b) Solid circular Section



$$I_{N.A.} = \frac{\pi D^4}{64}, y_{max} = \frac{D}{2}$$

$$\boxed{Z = \frac{\pi D^3}{32}}$$

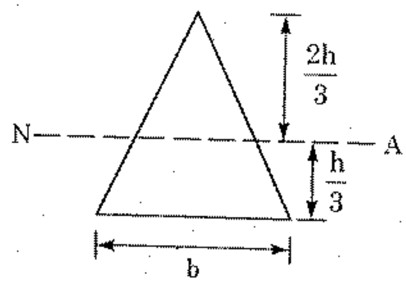
(C) Hollow Circular Section



$$I_{N.A.} = \frac{\pi (R^4 - r^4)}{4}$$

$$y_{max} = R$$

$$\Rightarrow \boxed{Z = \frac{\pi (R^4 - r^4)}{4R}}$$

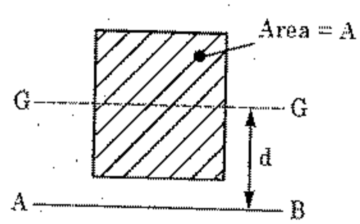


$$I_{N.A.} = \frac{bh^3}{36}, y_{max} = \frac{2h}{3}$$

$$\boxed{Z = \frac{bh^2}{24}}$$

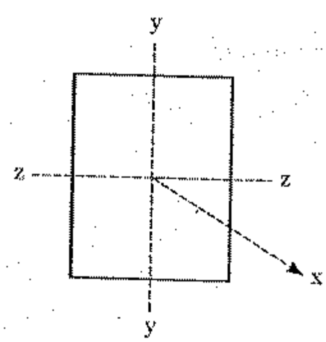
CALCULATION OF MOMENT OF INERTIA

(1)



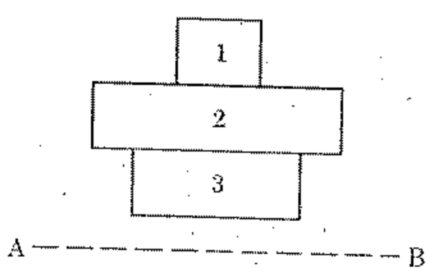
$\boxed{I_{AB} = I_{GG} + Ad^2}$ Parallel axis Theorem

(2)



$\boxed{I_{xx} = I_{yy} + I_{zz}}$ Perpendicular axis Theorem

(3)

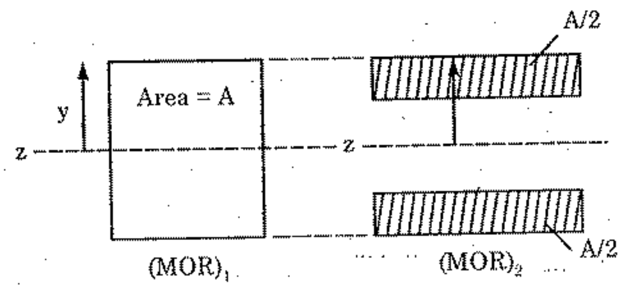


$$\boxed{I_{AB} = I_{1AB} + I_{2AB} + I_{3AB}}$$

I_{1AB} = Moment of inertia of 1st section about AB axis.

JUSTIFICATION FOR USE OF I-SECTION AS A BEAM

- Moment of resistance of a section is more when Z is more.
- But $Z = \dfrac{I}{y_{max}}$. Hence for a given y, Z is more if I is more also I is more if more area is located away from NA.

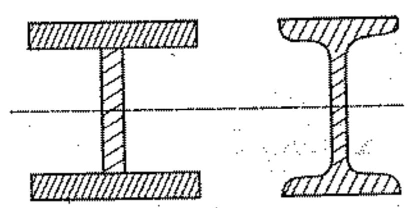


(i) (ii)

The figure shows that in Fig. (ii) more area is located away from Neutral Axis

$$\Rightarrow \boxed{MOR_2 > MOR_1}$$

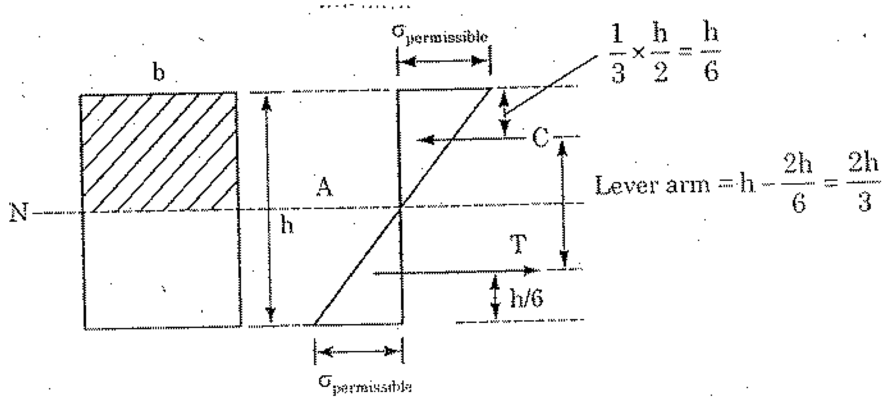
Hence instead of rectangular section, wide flanged section and I-beams are recommended. In these sections, more then 80% of bending is resisted by flange only. However, very thin web will lead to lateral instability. Hence section should be properly designed.



Wide flanged beam section I-section

From the above justification one can easily argue that out of rectangular and circular section of same x-section area rectangular section is more efficient in bending.

MOMENT OF RESISTANCE FOR A RECTANGULAR SECTION



$$T = C = \frac{1}{2}\times\sigma_{permissible}\times\frac{h}{2}\times b$$

$$\text{lever arm} = \frac{2h}{3}$$

where, T = bending tension and C = Bending compression.

$\Rightarrow$ MOR = C × lever arm = T × lever arm

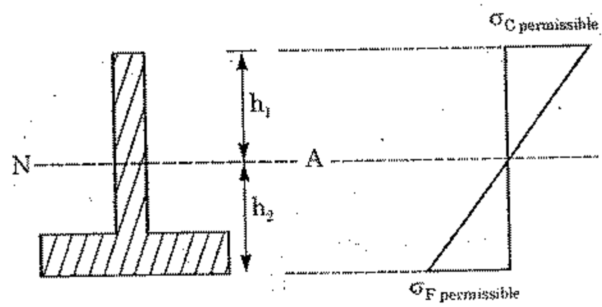
$$= \sigma_{per}\frac{bh}{4}\times\frac{2h}{3} = \sigma_{per}\frac{bh^2\times 2}{12} = \sigma_{per}\times\frac{bh^2}{6}$$

$$\boxed{MOR = \sigma_{per}\times z}$$

SECTION NOT EQUALLY STRONG IN TENSION AND COMPRESSION

Section dimension is chosen such that ratio of distances from NA to extreme fibre in tension and compression is exactly same as the ratio of allowable stresses in tension and compression. Under this provision section will be most economically utilised.

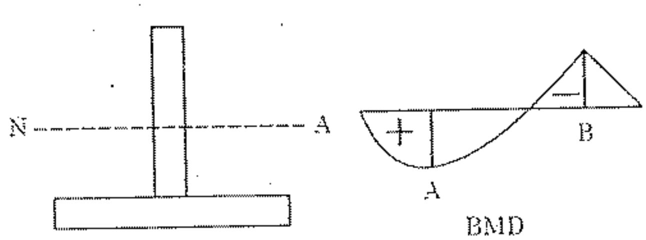
The following figure illustrates that if max permissible tensile and compressive stresses are not same then for most economical utilisation of material, the max permissible tensile and max permissible compressive stress at a section should be reached simultaneously.



Thus, $\boxed{\dfrac{\sigma_{C\,per}}{\sigma_{T\,per}} = \dfrac{h_1}{h_2}}$

Example: Cast iron < Strong in compression / Weak in tension

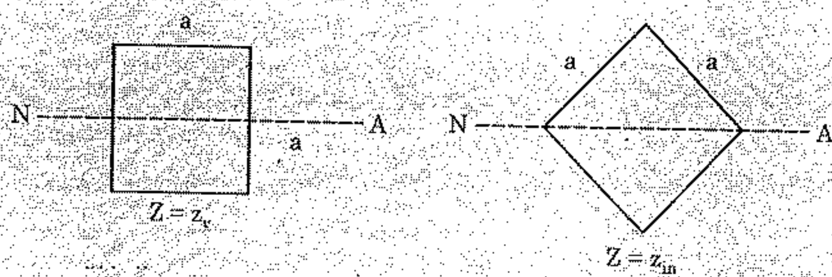
However if there is reversal of curvature, it is not necessary that max bending tensile stress and max bending compressive stress will occur at the same section. For example, if a beam is provided like the figure shown below throughout the span, then it may be possible that $\sigma_{tensile\ max}$ occurs at A and $\sigma_{compressive\ max}$ occurs at B. In such cases it will not be possible to utilise the material economically.



Example 1

Find the ratio of section modulus for a square section when it is placed such that neutral axis is parallel to one side to when neutral axis is along adiagonal.

Sol:



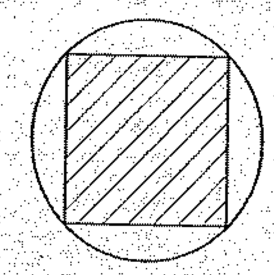
$$\frac{z_{in}}{z_v} = \frac{\frac{a^4}{12}\Big/\frac{a}{\sqrt{2}}}{\frac{a^4}{12}\Big/\frac{a}{2}} = \frac{\frac{a}{2}}{\frac{a}{\sqrt{2}}} = \frac{a}{2} \times \frac{\sqrt{2}}{a} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \frac{z_{in}}{z_v} = 0.707$$

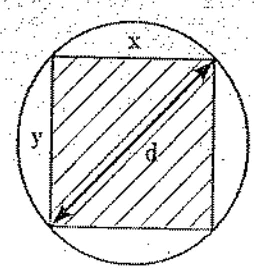
$$\Rightarrow \quad \boxed{\frac{z_v}{z_{in}} = 1.414}$$

Example 2

Find the strongest section that can be cut from a circle of dia 'd'.



Sol:



For a rectangular section cut from a circular section as shown in figure, the section modulus is

$$z = \frac{xy^2}{6}$$

$$z = \frac{x(d^2 - x^2)}{6}$$

From max Z, $\frac{dZ}{dx} = 0$

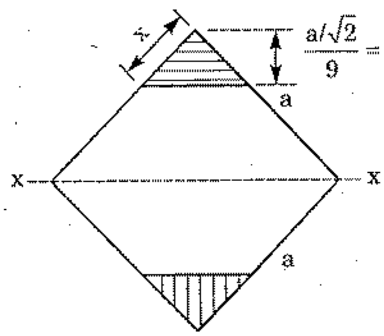
$$\Rightarrow \quad d^2 - 3x^2 = 0$$

$$\Rightarrow \quad \boxed{x = \frac{d}{\sqrt{3}}}$$

$$\boxed{y = \sqrt{d^2 - \frac{d^2}{3}} = \frac{\sqrt{2}\,d}{\sqrt{3}}}$$

$$\boxed{z_{max} = \frac{\frac{d}{\sqrt{3}} \times \frac{2d^2}{3}}{6} = \frac{d^3}{9\sqrt{3}}}$$

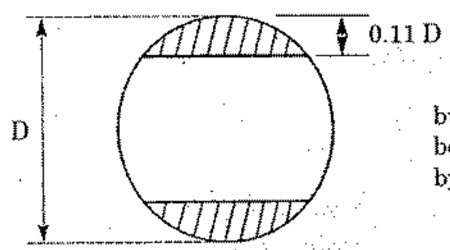
SOME INTERESTING RESULTS



$\frac{a/\sqrt{2}}{9} = 0.0785\,a$

by cutting $x = \frac{a}{9}$

i.e. depth of 0.0785a, beam can be made stronger by 5.35% for bending about x-x-axis



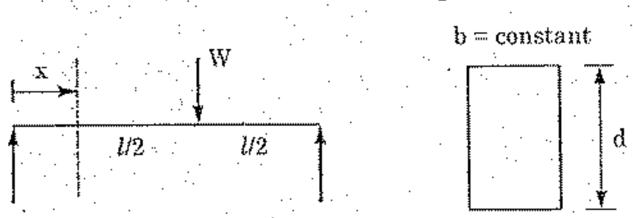
but cutting 0.11 D from top and bottom strength can be increased by 0.7% in bending

BEAM OF CONSTANT STRENGTH OR FULLY STRESSED BEAM

To minimise the quantity of material and thereby to have the lightest possible beam, we can vary the dimensions of cross-section such that max stress at every x-section of the beam is equal to max allowable bending stress in beam. The beam so obtained is called fully stressed beam or a beam of constant strength.

Example 3

For a simply supported rectangular beam loaded with a point load W at mid span, if width is constant. Find the variation of depth so that a beam of constant strength is obtained.



Sol: Design stress = f (say)

if max stress is to be constant throughout the beam, then

$$\Rightarrow \frac{My_{max}}{I} = f = \text{constant}$$

$$\Rightarrow \frac{\frac{W}{2}x \times \left(\frac{d_x}{2}\right)}{\frac{bd_x^3}{12}} = f = \text{constant},$$

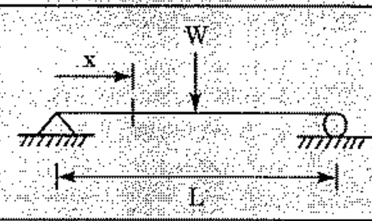
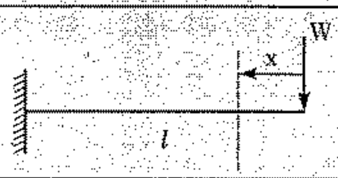
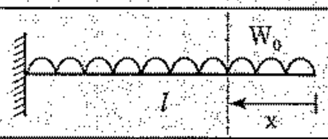
where d_x = depth of the beam at a distance x from support. [The limit of x is $0 < x < l/2$]

$$\Rightarrow \frac{3Wx}{bd_x^2} = f$$

$$\boxed{d_x = \sqrt{\frac{3W}{bf}x}} \Rightarrow d \propto \sqrt{x}$$

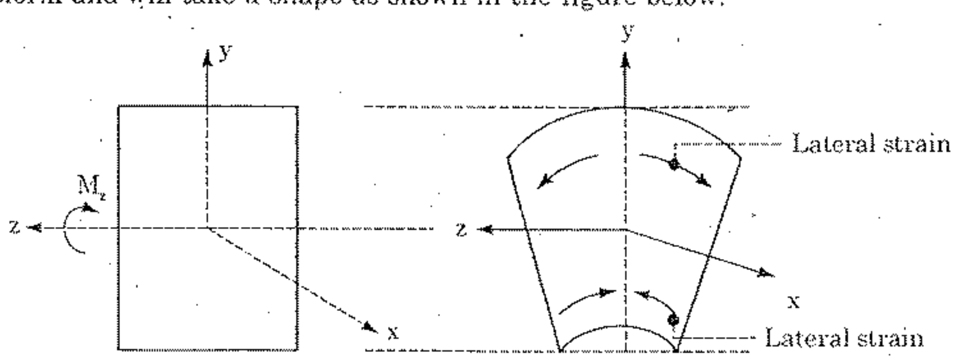
at $x = \frac{l}{2}$ i.e. at mid span, $d_{l/2} = \sqrt{\frac{3Wl}{2bf}}$

Note: *Following results must be remembered.*

Sr. No.	Rectangular beam loading	Max bending stress	b = constant	d = constant
1.		$\frac{3Wx}{b_x d_x^2} = f$	$d_x \propto \sqrt{x}$	$b_x \propto x$
2.		$\frac{6Wx}{b_x d_x^2} = f$	$d_x \propto \sqrt{x}$	$b_x \propto x$
3.		$\frac{3W_0x^2}{b_x d_x^2} = f$	$d_x \propto x$	$b_x \propto x^2$

ANTICLASTIC CURVATURE

We know that due to Poisson's effect, longitudinal strain leads to lateral strain. Hence a rectangular x-section will deform and will take a shape as shown in the figure below.



When the beam bends about z axis the radius of curvature is given by 'R'

where, $\varepsilon_x = \frac{y}{R}$, $\varepsilon_z = -\mu\varepsilon_x$

$$\Rightarrow \quad |\varepsilon_z| = |\mu\varepsilon_x|$$

If the radius of curvature of transverse bending due to Poisson's effect is R' then

$$\varepsilon_z = \frac{y}{R'} = \frac{\mu y}{R}$$

$$\Rightarrow \quad \boxed{R' = \frac{R}{\mu}}$$

This radius of curvature R' is called anticlastic radius of curvature.

BEAM OF COMPOSITE SECTION

A beam section composed of two different materials is called beam of composite section

The two different materials, can be

(a) Rigidly connected

(b) Simply placed one over the other.

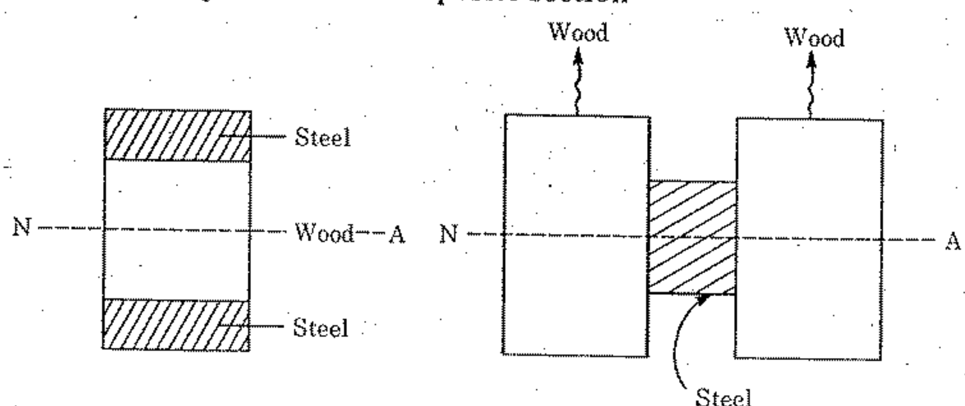
If M is the total BM resisted by the section then $\boxed{M = M_1 + M_2}$

where M_1 = BM resisted by 1st material

M_2 = BM resisted by 2nd material

(a) Rigidly Connected Composite Section

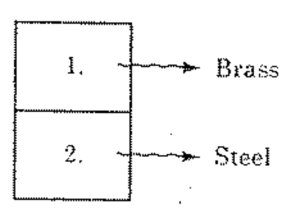
(i) Symmetrically connected composite section



The above beams are also called fliched beams.

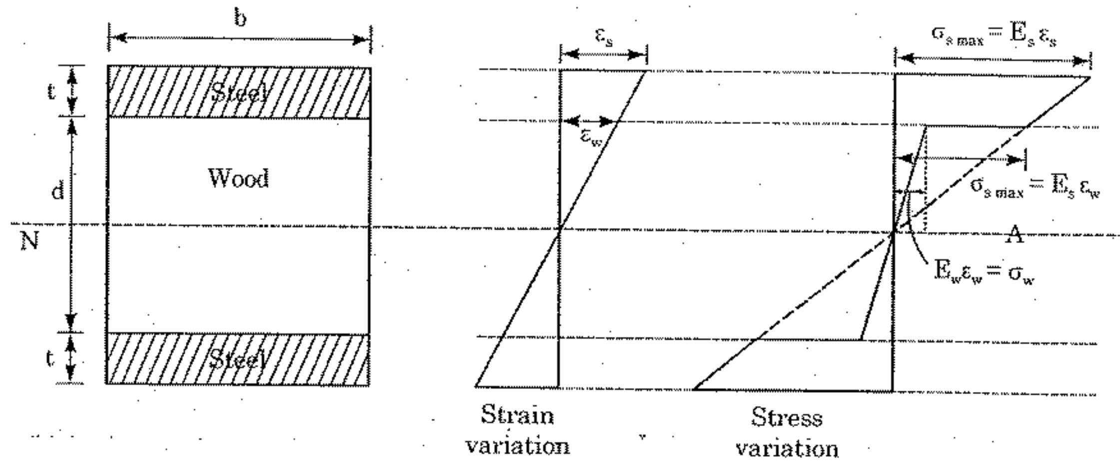
For symmetrical section, location of N.A. to easy.

(ii) Unsymmetric Connections



In this case location of N.A. becomes difficult.

(a) (i) Symmetrically connected composite section (Top and Bottom Symmetry)



The meaning of connected composite section is that the component materials will have the same radius of bending i.e. they will bend together.

Moment of resistance of composite section = $M_{R\ wood} + M_{R\ steel}$ .... (i)

$$M_{R\,wood} = \sigma_w \cdot Z_w = \sigma_w \times \frac{bd^2}{6} \quad \text{............ (ii)}$$

$$M_{R\,steel} = \sigma_{s\,max} \frac{b(d+2t)^2}{6} - \sigma_s \frac{bd^2}{6} \quad \text{............ (iii)}$$

where $M_{R\ wood}$ = moment resisted by wood; $M_{R\ steel}$ = moment resisted by steel

Any problem framed on the above type of composite section will be solved based on the strain and stress variations shown above.

For example, it we have to find out the max BM that the above section can resist, then we find out the MOR of the section under two conditions:

(i) When wood reaches its max permissible stress.

(ii) When steel reaches its max permissible stress.

The minimum of the two moment of resistances is the actual MOR of the section.

Thus, **when wood reaches its maximum permissible stress**

$$\sigma_w = \sigma_{wood\ per} \quad \text{(i.e. max permissible stress in wood)}$$

$$\sigma_s = \frac{\sigma_{wood\ per}}{E_w} \times E_s = m\,\sigma_{wood\ per}$$

$$m = \text{modular ratio} = \frac{E_s}{E_w}$$

$$\sigma_{s\,max} = \frac{\sigma_s \times \left(\frac{d}{2} + t\right)}{\frac{d}{2}} = \frac{m\,\sigma_{wood\ per}\,(d+2t)}{d}$$

$$\text{MOR} = \sigma_{wood\ per}\frac{bd^2}{6} + \left(m\sigma_{wood\ per}\frac{(d+2t)}{d}\right) \times \frac{b(d+2t)^2}{6} - \left((m\sigma_{wood\ per})\right) \times \frac{bd^2}{6}$$

where steel reaches its maximum permissible stress

$$\sigma_{smax} = \sigma_{steel\ per}$$

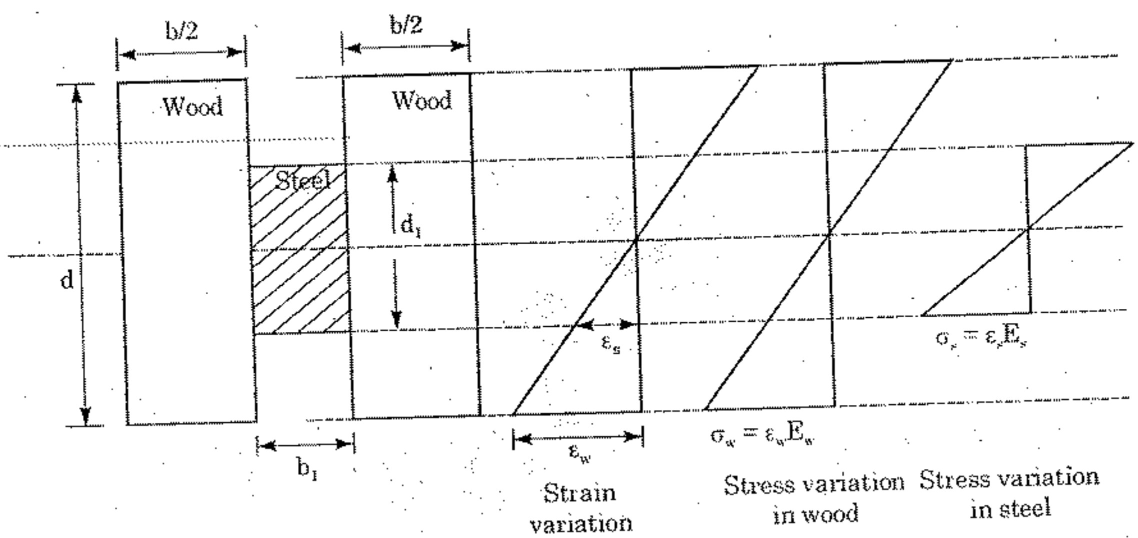
$$\Rightarrow \quad \sigma_s = \frac{\sigma_{steel\ per}\left(\frac{d}{2}\right)}{\left(\frac{d}{2}+t\right)}$$

$$\Rightarrow \quad \sigma_w = \frac{\sigma_{steel\ per}\left(\frac{d}{2}\right)}{\left(\frac{d}{2}+t\right)E_s} \times E_w = \sigma_{steel\ per}\left(\frac{d}{d+2t}\right)\times\frac{1}{m}$$

$$\Rightarrow \quad \boxed{MOR = \sigma_{steel\ per}\frac{(d)}{(d+2t)}\times\frac{1}{m}\times\frac{bd^2}{6} + \sigma_{steel\ per}\times\frac{b(d+2t)^2}{6} - \frac{\sigma_{steel\ per}(d)}{(d+2t)}\times\frac{bd^2}{6}}$$

Note that one should not try to remember these formula understanding of concept only is required.

Vertical Symmetry



Moment of resistance of complete section = MOR of steel (i.e. M_{Rs}) + MOR of wood (i.e. M_{Rw})

$$M_R = M_{Rw} + M_{Rs}$$

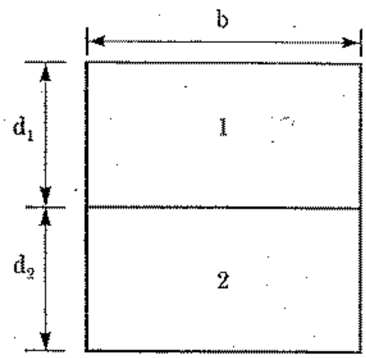
$$M_R = \sigma_w \frac{bd^2}{6} + \sigma_s \times \frac{b_1 d_1^2}{6}$$

$$\left.\begin{aligned} \sigma_w &= \frac{E_w\left(\frac{d}{2}\right)}{R} \\ \sigma_s &= \frac{E_s \times \left(\frac{d_1}{2}\right)}{R} \end{aligned}\right\} \text{Since N.A. is same for both the material}$$

R = Radius of bending of N.A.

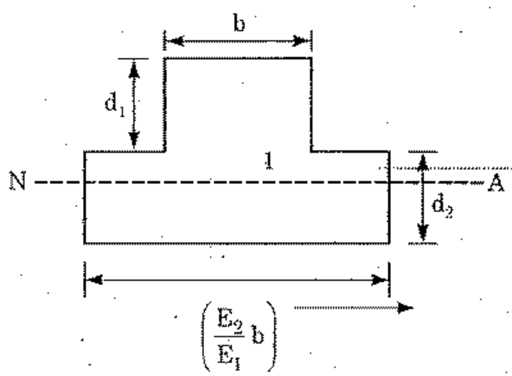
Unsymmetric Connection

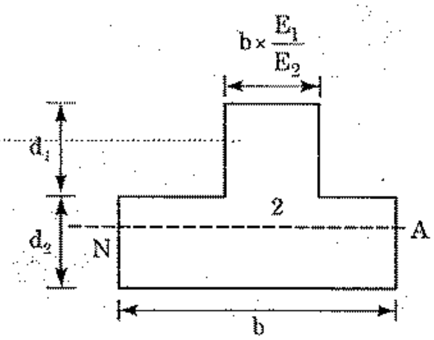
When two materials are placed over on another and rigidity connected. Analysis of this case is done as given in the steps below.



Step 1: Neutral axis is found out by drawing transformed section.

Transformed section is drawn by converting material 2 into equivalent material 1 or by converting material 1 into equivalent material 2.





2 material converted to 1

Width multiplication factor = $\frac{E_2}{E_1}$ for material 2

Width multiplication factor = 1 for material 1

1 material converted to 2

Width multiplication factor = $\frac{E_1}{E_2}$ for material 1

Width multiplication factor = 1 for material 1.

Note that for making transformed section, only width modification is done. The depth of the section is not changed.

For transformed section N.A. is found out.

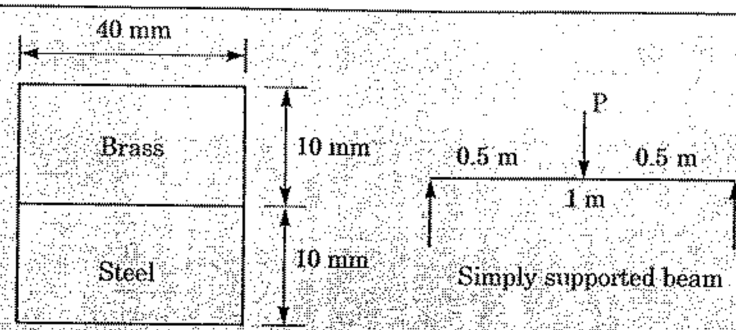
Step 2: From N.A., stress distribution is calculated for transformed section.

Step 3: From stress distribution in transformed section, stress in both the material is calculated by multiplying the stress of transformed section with width multiplication factor.

Note: *The above method can also be used for flitched beam by converting steel into an equivalent wood section. This is done by increasing the width of steel portion by a factor of* $m = \frac{E_s}{E_w}$.

Example 4

For a rigidly connected unsymmetric section find max load 'P' that can be applied.



x-section

If $E_{steel} = E_s = 2.05 \times 10^5$ N/mm²

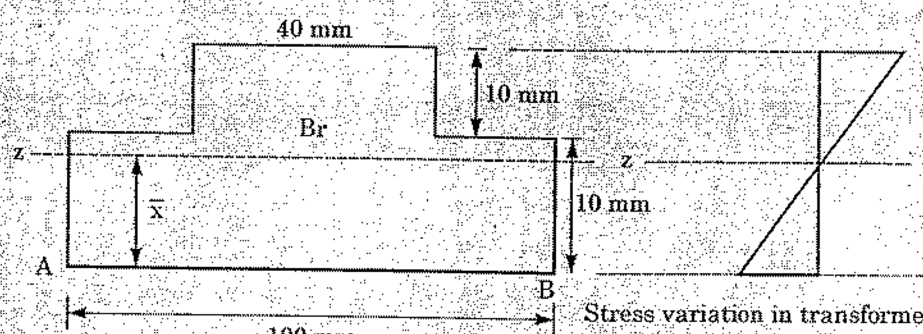
$E_{Brass} = E_B = 0.82 \times 10^5$ N/mm²

$(\sigma_{br})_{max\ permissible} = 75$ N/mm²

$(\sigma_{st})_{max\ permissible} = 112.5$ N/mm²

Sol: For transforming steel into Brass width of steel needs to be multiplied by width modification factor of steel i.e. $\frac{E_s}{E_B}$. Now N.A. of transformed section is found out by taking moment of all area about AB

$\left(\frac{E_s}{E_B} = \frac{2.05}{0.82} = 2.5\right)$



Transformed section (steel has been transformed into brass)

Stress variation in transformed section

$$(40\times10+100\times10)\,\bar{x} = (40\times10)(10+5)+100\times10\times5$$

$$\Rightarrow \quad \bar{x} = \frac{6000+5000}{1400} = \frac{11000}{1400} = \frac{110}{14} = 7.875 \text{ mm}$$

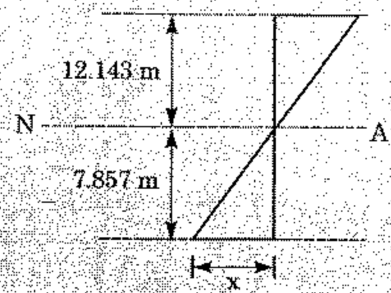
$$\bar{x} = 7.857 \text{ mm}$$

Moment of inertia of transformed section about N.A. is

$$I = \frac{1}{3}\left[100\times(7.857)^3 + 40(10+10-7.857)^3 + (100-40)(10-7.857)^3\right]$$

$$= 4.0238 \times 10^4 \text{ mm}^4$$

Maximum bending moment $= \frac{Pl}{4} = \frac{P\times1}{4} = \frac{P}{4}$ Nm



Max bending stress in brass = (Stress at extreme fibre of brass side in transformed section) × (Width modification factor for brass)

$$\Rightarrow \quad 75 = \left(\frac{E_B}{E_B}\right) \times \left(\frac{P \times 1000}{4} \times \frac{12.143}{4.0238 \times 10^4}\right)$$

$\Rightarrow \; P = 994.1 \text{ N}$

Max bending stress in steel = (Stress at extreme fibre of steel side in transformed section) × (Width modification factor of steel)

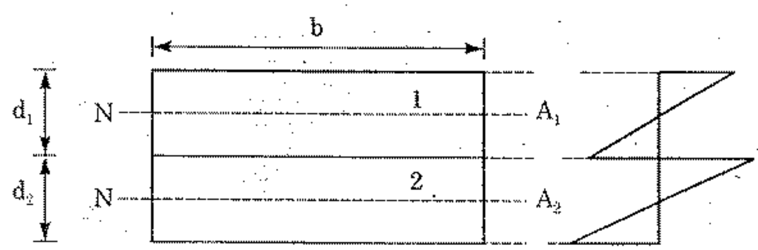
$$\Rightarrow \quad 112.5 = \left(\frac{E_{st}}{E_B}\right)\left[\frac{P \times 1000}{4} \times \frac{7.857}{4.0238 \times 10^4}\right]$$

$$= \quad 2.5 \times \left[\frac{P \times 1000}{4} \times \frac{7.857}{4.0238 \times 10^4}\right]$$

$\Rightarrow \; P = 921.83 \text{ N}$

$\Rightarrow$ Max load 'P' that can be applied is min of the above two values of P i.e. $P_{max} = 921.83$ N.

When two sections are simply placed order each other (Not connected rigidly)



$$M = M_1 + M_2 \quad \text{.............} \quad \text{(i)}$$

In this case the two components will bend independently.

However, as an approximation, we assume radius of bending (R) to be same for both the sections.

Thus, from flexure formula $\left(\frac{M}{I} = \frac{f}{y} = \frac{E}{R}\right)$

$$\frac{E_1 I_1}{M_1} = \frac{E_2 I_2}{M_2} \quad \text{.............} \quad \text{(ii)}$$

From (i) and (ii) find M_1 and M_2 in term of M. Finally, from M_1 and M_2, stresses can be calculated and safety can be checked like

$$\frac{M_1\left(\frac{d_1}{2}\right)}{\frac{bd_1^3}{12}} \le \sigma_{max\,1} \text{ and } \frac{M_2\left(\frac{d_2}{2}\right)}{\left(\frac{bd_2^3}{12}\right)} \le \sigma_{max\,2}$$

Moment of resistance in this case can be calculated as

$$MOR = (MOR)_1 + (MOR)_2$$

$$MOR = \sigma_{1\,max} \cdot z_1 + \sigma_{2\,max} \cdot z_2$$

$$= \sigma_{1max}\frac{bd_1^2}{6} + \sigma_{2max}\frac{bd_2^2}{6}$$

Modulus of Rupture (f)

Modulus of rupture is a fictitious stress obtained from

$$f = \frac{M_U y_{max}}{I}$$

where M_U = ultimate bending moment which causes failure in the beam.

- It is used to compare ultimate strengths of beam of various sizes and materials.
- Modulus of repture is always greater than actual ultimate strength σ_u.

UNSYMMETRICAL BENDING

When bending couple doesnot act in the plane of symmetry of member either because there is no plane of symmetry or because the couple acts in different plane, the bending is called unsymmetric bending. Under unsymmetric bending, we can classify the problems as:

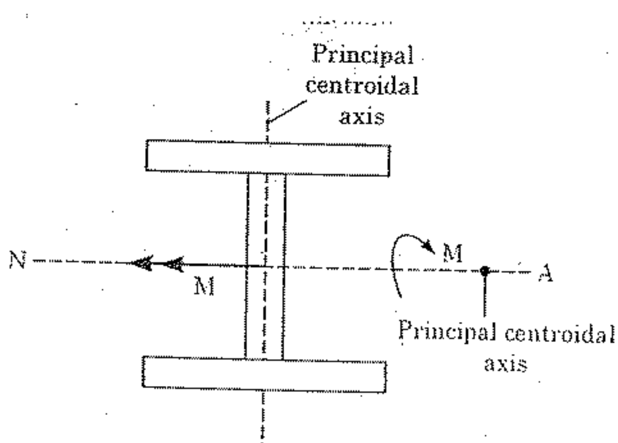
(i) Doubly symmetric section with skew loading.

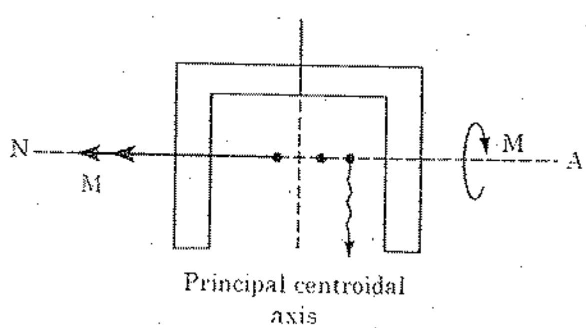
(ii) Pure bending of unsymmetric beams.

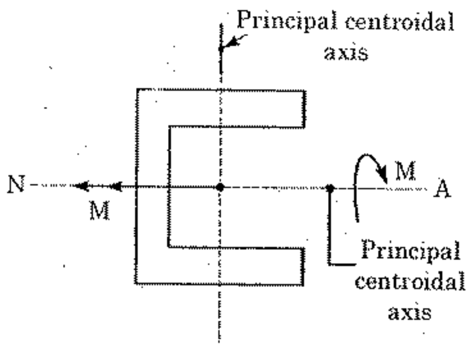
(iii) Bending of unsymmetric beams due to load causing moment and shear.

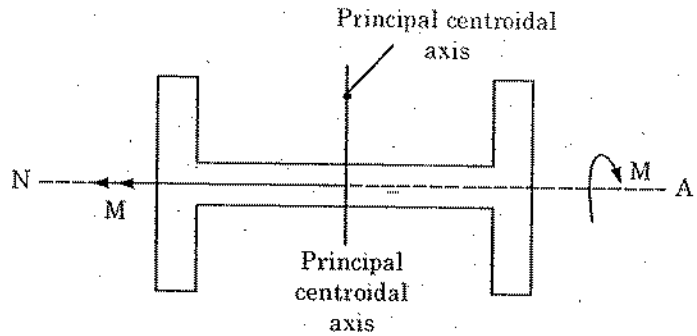
Before discussing the various cases (i), (ii) and (iii) as above, let us discuss an important observation about bending.

The important observation about bending is that, if the direction of bending couple (as obtained from right hand Thumb rule) is in the direction of a principle centroidal axis, the neutral axis will be in the direction of couple. Thus, the following figures depict this concept.





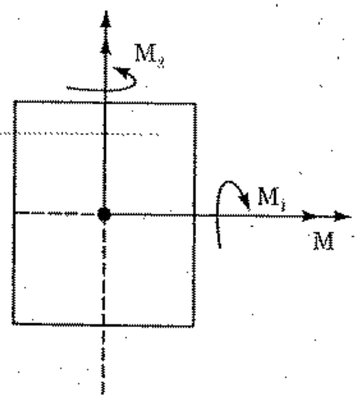




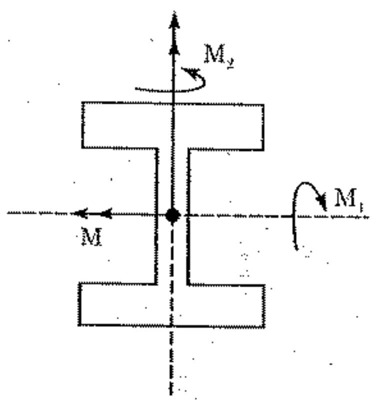
Note: *If the section is having at least one axis of symmetry, that axis and an axis ⊥ to it are the principle axes.*

Case (i): Doubly symmetric beam with Skew Loading

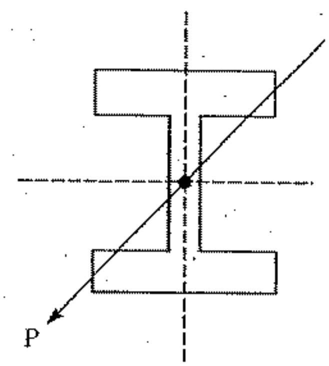
In case of doubly symmetric section, when there is either <u>pure bending</u> or when the loading passes through the C.G. of the x-section, there will be no twisting of the section.



Pure bending



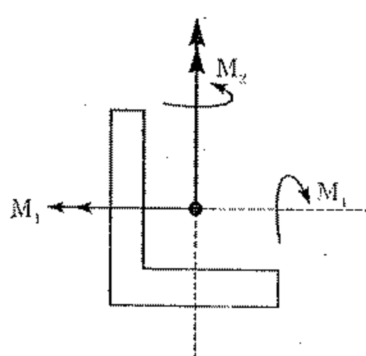
Pure bending



Load passing through the C.G. of section

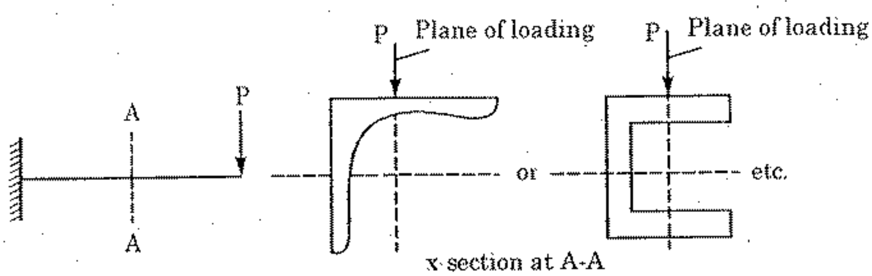
Case (ii): Unsymmetric beams with pure bending

In case of unsymmetric beams, if only pure bending occurs, there will be no twisting of the sections.



Case (iii): Bending of unsymmetric beams due to load causing moment and shear

In case of unsymmetric beams, if bending is associated with shear, then twisting of section may occur.

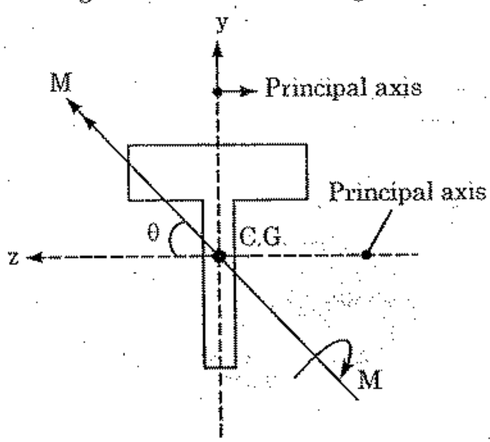


With the given plane of loading, as shown in figure, bending is associated with twisting. However twisting will not occur only if the loading passes through shear centre.

Analysis of Unsymmetric bending (for no twisting cases): To find out bending stress at any point in the x-section under unsymmetric bending, the concept discussed as below will be used.

If the direction of bending couple is in the direction of principal centroidal axis, the neutral axis will be in the direction of couple.

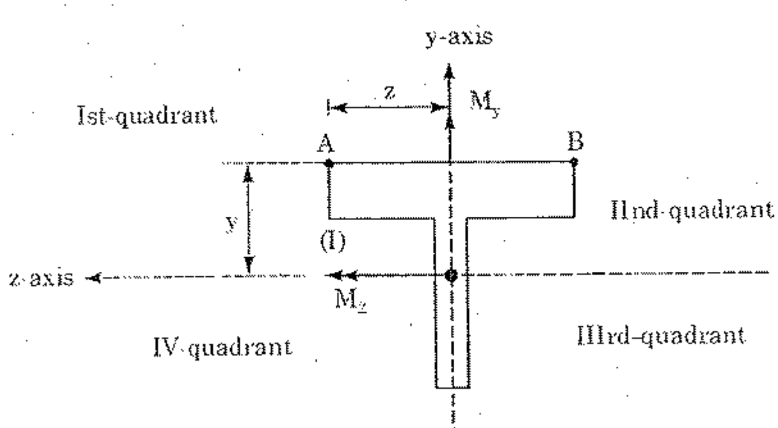
The following example illustrates the steps in the calculation of bending stress under unsymmetric bending: Let there be a section in which bending moment 'M' is acting in the direction shown.



Step 1: Locate principal axis of the cross-section.

Step 2: Find the component of moment vector about the principal axes

$$\begin{pmatrix} M_y = M\sin\theta \\ M_z = M\cos\theta \end{pmatrix}$$



Step 3:

$$\boxed{\sigma_x = -\frac{M_z y}{I_z} + \frac{M_y z}{I_y}}$$

If σ_x is (+) ve it is tensile and it is (–) ve, it is compressive. The value of M_z, y, M_y, z should be put in the above expression with proper sign. Thus for point B, y is (+) ve z is (–) ve.

Note that you don't have to remember this formula. it can be written simphy by writting the stress at any point in 1st quadrant. If we write bending stress at A in the above figure.

Bending stress at A due to M_z will be compressive

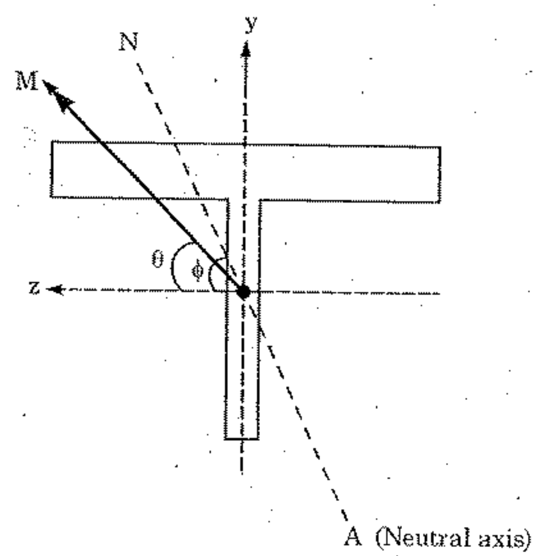
$$\Rightarrow \quad \sigma_{x_1} = -\frac{M_z y}{I_z}$$

Bending stress at A due to M_y will be tensile

$$\Rightarrow \quad \sigma_{x_2} = +\frac{M_y z}{I_y}$$

$$\Rightarrow \quad \sigma_x = \sigma_{x_1} + \sigma_{x_2} = \frac{-M_z y}{I_z} + \frac{M_y z}{I_y}$$

Step 4: Neutral axis can be located by equating $\sigma_x = 0$



$$\Rightarrow \quad \frac{-M_z y}{I_z} + \frac{M_y z}{I_y} = 0$$

$$\Rightarrow \quad y = \left(\frac{I_z}{I_y}\frac{M_y}{M_z}\right)z$$

$$= \frac{I_z}{I_y}\frac{M\sin\theta}{M\cos\theta}.z = \left(\frac{I_z}{I_y}\tan\theta\right)z$$

$$\Rightarrow \quad \boxed{y = \left(\frac{I_z}{I_y}\tan\theta\right)z} \quad \underline{\text{eq. of Neutral axis}}$$

Slope of N.A. is $\tan\phi = \frac{I_z}{I_y}\tan\theta$

Angles are measured from (+) z-direction to (+) y-direction.

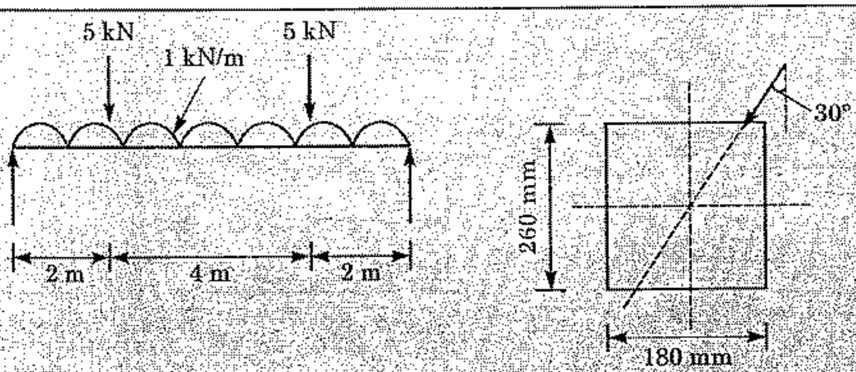
Note that $\phi > \theta$, if $I_z > I_y$

$\phi < \theta$, if $I_z < I_y$

Thus Neutral axis is always located between the couple vector and the minor principal axis.

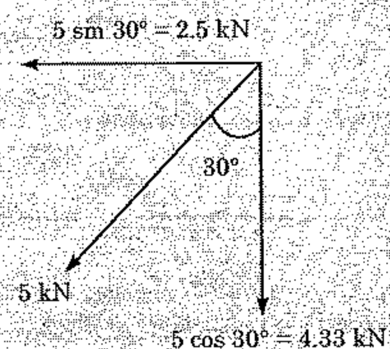
Example 5

A rectangular steel beam 180 × 260 mm in x-sec is used as simply supported beam over a span of 8m and is subjected to a udl of 1 kN/m and also concentrated loads of 5 kN at 2m from each support inclined at 30° to the vertical axis. Determine the bending stress at the four corners of the beam and location of N.A. of x-section

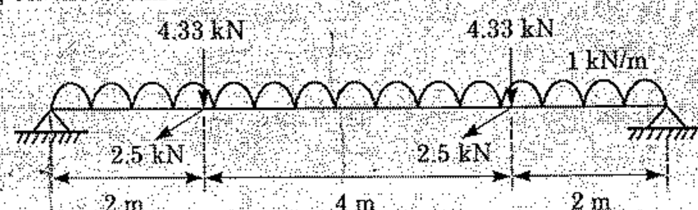


x-section of beam

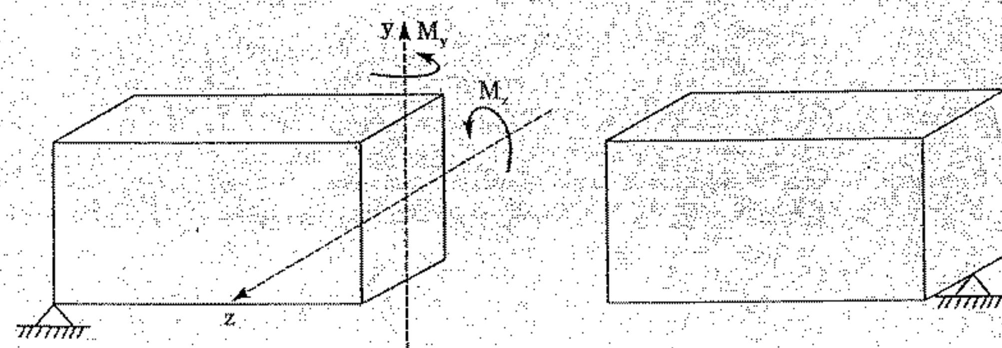
Sol: The loading can be shown as



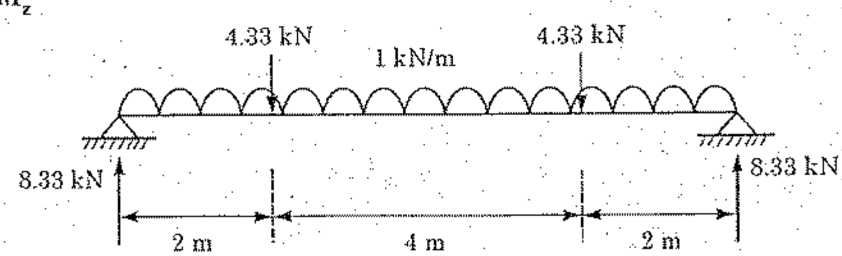
The equivalent loading on the beam is as shown below.



Max BM will be at the mid span of the beam.



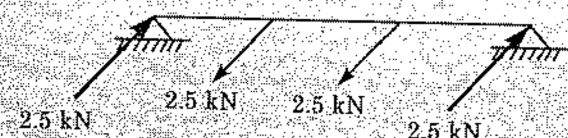
Calculation of M_z



$$M_z = 8.33\times 4 - \frac{1\times(4)^2}{2} - 4.33\times 2$$

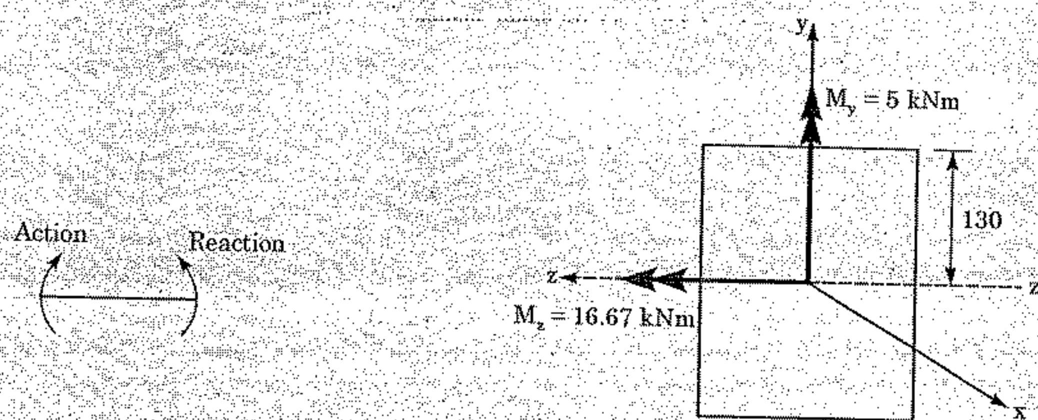
$$= 16.67 \text{ kNm}$$

Calculation of M_y



$$M_y = 2.5 \times 2 = 5 \text{ kNm}$$

Note that M_y and M_z will be (+) ve because we have shown M_y and M_z are the reaction moments at the section.



$$\sigma_x = -\frac{M_z y}{I_z} + \frac{M_y z}{I_y}$$

For neutral axis $\sigma_x = 0$

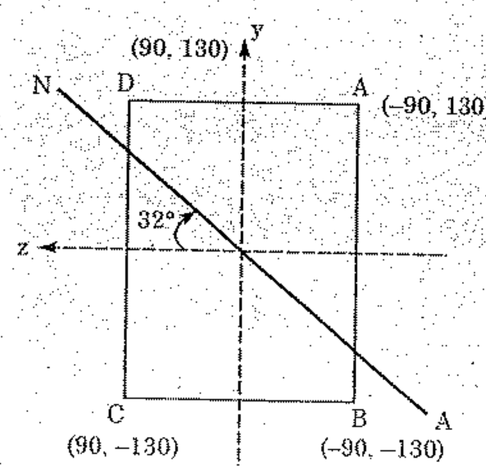
$$\Rightarrow -\frac{M_z y}{I_z} + \frac{M_y z}{I_y} = 0$$

$$\Rightarrow -\frac{16.67\, y}{(180)\times\frac{(260)^3}{12}} \times \frac{(5)\, z}{\frac{(260)(180)^3}{12}} = 0$$

$\Rightarrow$ $y = 0.626\, z$

$\Rightarrow$ $\tan\phi = 0.626$

$\phi = 32°$



Bending stress are calculated as given below

$$I_z = \frac{180\,(260)^3}{12} = 2.6364\times 10^8 \text{mm}^4$$

$$I_y = \frac{260\,(180)^3}{12} = 1.2636\times 10^8 \text{mm}^4$$

$$\sigma_x = -\frac{M_z y}{I_z} + \frac{M_y z}{I_y}$$

$$\sigma_{xA} = -\frac{16.67\times10^6\times130}{2.6365\times10^8}+\frac{(5\times10^6)\times(-90)}{1.2636\times10^8}$$

$$= -8.214 - 3.561 = -11.775 \text{ N/mm}^2 \quad \text{(compressive)}$$

$$\sigma_{xB} = -\frac{16.66\times10^6\times(-130)}{2.6368\times10^8}+\frac{(5\times10^6)(-90)}{1.2636\times10^8}$$

$$\sigma_{xB} = 8.214 - 3.561 = 4.653 \text{ N/mm}^2 \quad \text{(Tension)}$$

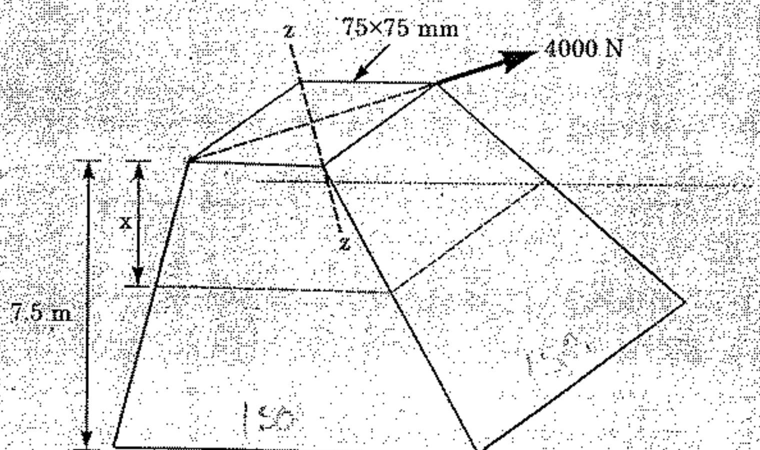
$$\sigma_{xC} = -\frac{16.66\times10^6(-130)}{2.6368\times10^8}+\frac{(5\times10^6)(90)}{1.2636\times10^8}$$

$$= 8.214 + 3.561 = 11.775 \text{ N/mm}^2 \quad \text{(Tension)}$$

$$\sigma_{xD} = -8.214 + 3.561 = -4.653 \text{ N/mm}^2 \quad \text{(Compression)}$$

Example 6

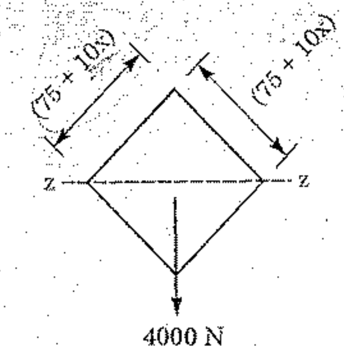
Calculate maximum bending stress for the condition shown below.



BM at a distance x from the free end = M = 4000 x

$$I_{zz} = \frac{(75)^4}{12} \text{ mm}^4 \text{ at top}$$

at any distance x meter from top



By putting x in meter, we get dimension in mm

$$I_{zz} = \frac{(75+10\,x)^4}{12}$$

Bending stress at the extreme fibre of section located at a distance x from top

$$= \frac{(4000\,x)\times\frac{(75+10\,x)}{\sqrt{2}}}{\frac{(75+10\,x)^4}{12}}$$

$$\Rightarrow \quad \sigma = \frac{4000\times 12}{\sqrt{2}}\frac{x}{(75+10x)^3}$$

$$= 33941.12\,\frac{x}{(75+10x)^3}$$

For stress to be max $\frac{d\sigma}{dx} = 0$

$$\Rightarrow \quad \frac{(75\times 10x)^3\times 1 - x\times 3(75+10x)^2\times 10}{(75+10x)^6} = 0$$

$$\Rightarrow \quad (75 + 10x) - 30x = 0$$

$$75 - 20x = 0$$

$$x = \frac{75}{20} = 3.75 \text{ m}$$

$\Rightarrow$ **Bending stress is max at x = 3.75 m**

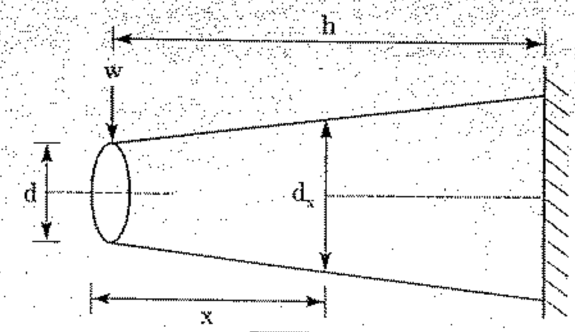
$$\text{Max bending stress} = \frac{4000\times 12}{\sqrt{2}}\times\frac{x\times 1000\ \text{Nmm}}{(75+10x)^3\,\text{mm}^3}$$

By putting x = 3.75 m

Max bending stress = **89.39 N/mm²**

Example 7

A cantilever beam has a circular x-section with the variation of diameter as shown below. Calculate the location at which max bending stress will act.



$d_x = d + tx$

Sol:

$$d_x = d + tx$$

$$I = \frac{\pi d_x^4}{}$$

$$\text{Bending stress} = \frac{wx\left(\frac{d_x}{2}\right)}{I_x}$$

$$\sigma_x = \frac{wx(d+tx)/2}{\frac{\pi(d+tx)^4}{64}}$$

$$\Rightarrow \sigma_x = \frac{32wx}{\pi(d+tx)^3}$$

For max bending stress $\frac{d\sigma_x}{dx} = 0$

$\Rightarrow \pi(d + tx)^3 \times 32w - (32wx) \times \pi \times 3(d + tx)^2 \times t = 0$

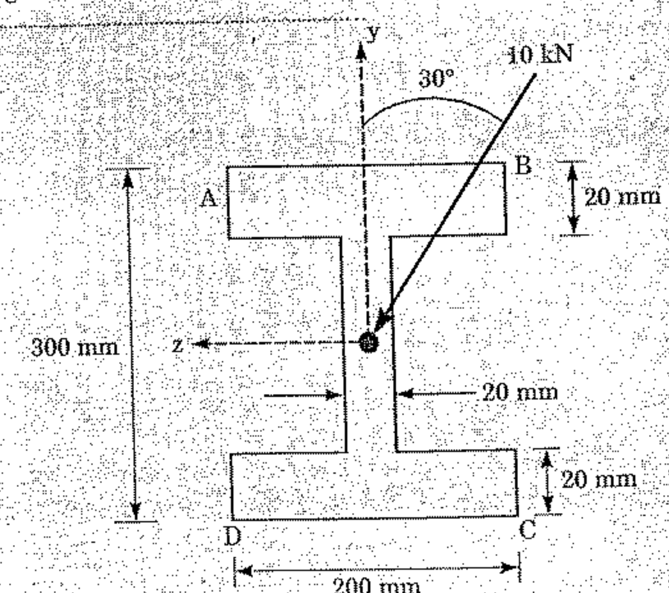
$\Rightarrow 32\pi w\,(d + tx)^2\,[d + tx - 3tx] = 0$

$\Rightarrow d - 2t\,x = 0$

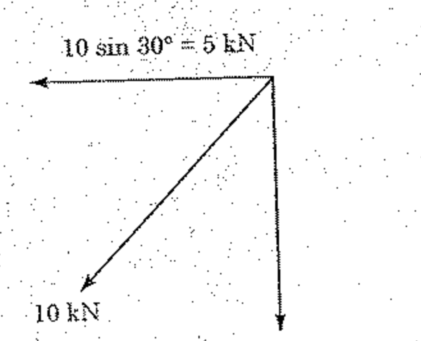
$$\Rightarrow \boxed{x = \frac{d}{2t}}$$

Example 8

A cantilever beam of span 2 m has inclined loading at the free end. The x-section of the beam is shown below. Calculate bending stress at the four corners A, B, C and D of the beam x-section at fixed end.



Sol: At Fixed end



$$M_z = -8.66 \times 2 = -17.32 \text{ kNm}$$

$$M_y = -5 \times 2 = -10 \text{ kNm}$$

$$I_z = \frac{200\,(300)^3}{12} - \frac{180(260)^3}{12}$$

$$= 186.36 \times 10^6 \text{ mm}^4$$

$$I_y = 2 \times \frac{20\,(200)^3}{12} + \frac{260\,(20)^3}{12}$$

$$= 26.84 \times 10^6 \text{ mm}^4$$

$$\sigma_x = -\frac{M_z y}{I_z} + \frac{M_y z}{I_y}$$

$$\sigma_{xA} = \frac{-(-17.32\times10^6)\times150}{186.36\times10^6} + \frac{(-10\times10^6)\times100}{26.84\times10^6}$$

$$= 13.94 - 37.25 = -23.32 \text{ N/mm}^2$$

$$\sigma_A = -23.32 \text{ N/mm}^2 \quad \text{(compressive)}$$

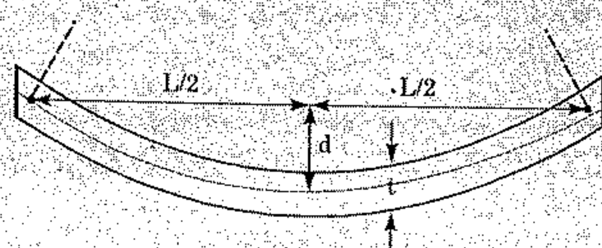
$$\sigma_B = 13.94 + 37.25 = 51.19 \text{ N/mm}^2 \quad \text{(tensile)}$$

$$\sigma_C = -13.94 + 37.25 = 23.32 \text{ N/mm}^2 \quad \text{(tensile)}$$

$$\sigma_D = -13.94 - 37.25 = -51.19 \text{ N/mm}^2 \quad \text{(compressive)}$$

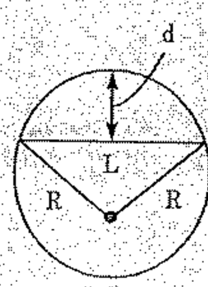
Example 9

A long rod of uniform rectangular section and thickness 't' originally straight is bent into the form of a circular arc and the displacement 'd' of the mid point of length 'L' is measured by means of a dial guage. The displacement 'd' may be regarded as small compared to length 'L'. Show that the longitudinal surface strain ε in the rod is given by $\varepsilon = \frac{4td}{L^2}$



Sol: Longitudinal strain $= \varepsilon = \frac{y}{R} = \frac{t}{2R}$

R = radius of curvature



$$R - d = \sqrt{R^2 - \frac{L^2}{4}}$$

$$(R-d)^2 = R^2 - \frac{L^2}{4}$$

$$R^2 - (R-d)^2 = \frac{L^2}{4}$$

$$(R - R + d)(R + R - d) = \frac{L^2}{4}$$

$$d(2R - d) = \frac{L^2}{4}$$

$$2dR - d^2 = \frac{L^2}{4}$$

$$\frac{2dR}{L^2} - \frac{d^2}{L^2} = \frac{1}{4}$$

$$\frac{d}{L} \text{ is very small} \Rightarrow \frac{d^2}{L^2} \text{ is negligible}$$

$$\Rightarrow \quad \frac{2dR}{L^2} = \frac{1}{4}$$

$$\frac{L^2}{8d} = R$$

$$\Rightarrow \quad \varepsilon = \frac{t}{2 \times \frac{L^2}{8d}} = \frac{4td}{L^2}$$

$$\Rightarrow \quad \boxed{\varepsilon = \frac{4td}{L^2}}$$

Example 10

The composite beam shown below is formed of a wood beam (100 mm × 150 mm actual dimensions) and a steel reinforcing plate (100 mm wide and 12 mm thick). The beam is subjected to a positive bending moment M = 6 kNm.

Using the transformed section method, calculate the largest tensile and compressive stresses in the wood (material 1) and the maximum and minimum tensile stresses in the steel (material 2) if E_1 = 10.5 GPa and E_2 = 210 GPa.

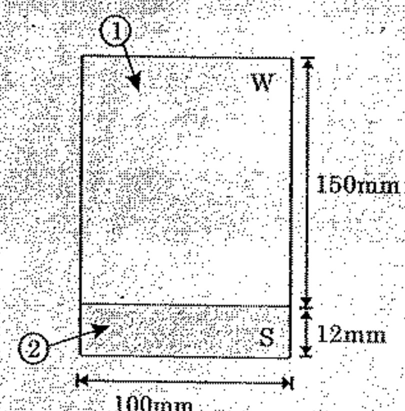
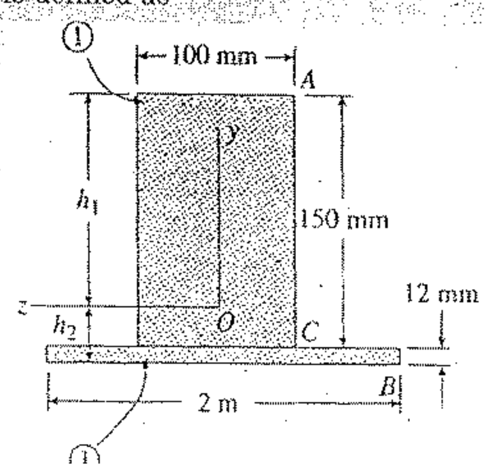


Fig. (a)

Sol. Transformed Section: We will transform the original beam into a beam of material 1, which means that the modular ratio is defined as



$$n = \frac{E_2}{E_1} = \frac{210\text{ GPa}}{10.5\text{ GPa}} = 20$$

The part of the beam made of wood (material 1) is not altered but the part made of steel (material 2) has its width multiplied by the modular ratio. Thus, the width of this part of the beam becomes n(100 mm) = 20(100 mm) = 2m in the transformed section (Fig. b)

Neutral Axis: Because the transformed beam consists of only one material, the neutral axis passes through the centroid of the cross-sectional area. Therefore, with the top edge of the cross section serving as a reference line, and with the distance y_i measured positive downward, we can calculate the distance h_1 to the centroid as follows:

$$h_1 = \frac{\sum y_i A_i}{\sum A_i} = \frac{(75\text{ mm})(100\text{ mm})(150\text{ mm}) + (156\text{ mm})(2000\text{ mm})(12\text{ mm})}{(100\text{ mm})(150\text{ mm}) + (2000\text{ mm})(12\text{ mm})}$$

$$= \frac{4869\times10^3\text{ mm}^3}{39\times10^3\text{ mm}^2} = 124.8\text{ mm}$$

Also, the distance h_2 from the lower edge of the section to the centroid is $h_2 = 162\text{ mm} - h_1 = 37.2$ mm. Thus, the location of the neutral axis is determined.

Moment of inertia of the transformed section. Using the parallel-axis theorem, we can calculate the moment of inertia I_T of the entire cross-sectional area with respect to the neutral axis as follows:

$$I_T = \frac{1}{12}(100\text{ mm})(150\text{ mm})^3 + (100\text{ mm})(150\text{ mm})(h_1 - 75\text{ mm})^2$$

$$+\frac{1}{12}(2000\text{ mm})(12\text{ mm})^3 + (2000\text{ mm})(12\text{ mm})(h_2 - 6\text{ mm})^2$$

$$= 65.3\times10^6\text{ mm}^4 + 23.7\times10^6\text{ mm}^4 = 89.0\times10^6\text{ mm}^4$$

Normal stresses in the wood (material 1): The stresses in the transformed beam (Fig. b) at the top of the cross section (A) and at the contact plane between the two parts (C) are the same as in the original beam (Fig. a).

$$\sigma_{1A} = -\frac{My}{I_T} = -\frac{(6\times10^6\text{ N. mm})(124.8\text{ mm})}{89\times10^6\text{ mm}^4} = -8.42\text{ MPa}$$

$$\sigma_{1C} = -\frac{My}{I_T} = -\frac{(6\times10^6\text{ N.mm})(-25.2\text{ mm})}{89\times10^6\text{ mm}^4} = 1.696\text{ MPa}$$

These are the largest tensile and compressive stresses in the wood (material1) in the original beam. The stress σ_{1A} is compressive and the stress σ_{1C} is tensile.

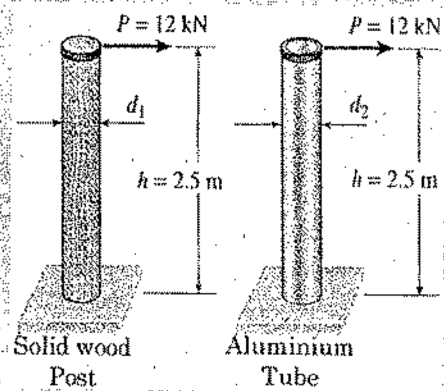
Normal stresses in the steel (material 2): The maximum and minimum stresses in the steel plate are found by multiplying the corresponding stresses in the transformed beam by the modular ratio n. The maximum stress occurs at the lower edge of the cross section (B) and the minimum stress occurs at the contact plane(C):

$$\sigma_{2B} = -\frac{My}{I_T}n = -\frac{(6\times10^6\text{N.mm})(-37.2\text{mm})}{89.0\times10^6\text{mm}^4}(20) = 50.2\text{ MPa}$$

$$\sigma_{2C} = -\frac{My}{I_T}n = -\frac{(6\times10^6\text{N.mm})(-25.2\text{ mm})}{89.0\times10^6 mm^4}(20) = 34\text{ MPa}$$

Example 11

What is the minimum required outer diameter d_2 of the aluminum tube if its wall thickness is to be one-eighth of the outer diameter and the allowable bending stress in the aluminum is 50 MPa?



Sol:

(ii) Aluminum tube: To determine the section modulus S_2 for the tube, we first must find the moment of inertia I_2 of the cross section. The wall thickness of the tube is $d_2/8$, and therefore the inner diameter is $d_2 - d_2/4$, or $0.75d_2$. Thus, the moment of inertia is

$$I_2 = \frac{\pi}{64}\left[d_2^4 - (0.75d_2)^4\right] = 0.03356d_2^4$$

The section modulus of the tube is now obtained as follows:

$$S_2 = \frac{I_2}{c} = \frac{0.03356d_2^4}{d_2/2} = 0.06712d_2^3$$

The required section modulus is

$$S_2 = \frac{M_{max}}{\sigma_{allow}} = \frac{30\text{ kN.m}}{50\text{ MPa}} = 0.0006\text{ m}^3 = 600 \times 10^3\text{ mm}^2$$

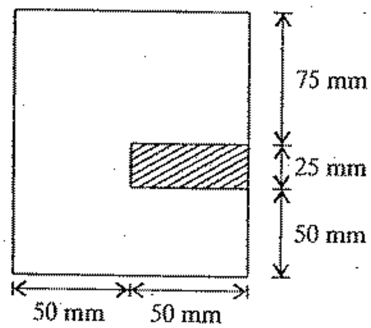
By equating the two preceding expressions for the section modulus, we can solve for the required outer diameter:

$$d_2 = \left(\frac{600\times10^3\text{ mm}^3}{0.06712}\right)^{1/3} = 208mm$$

The corresponding inner diameter is 0.75(208 mm), or 156 mm.

OBJECTIVE QUESTIONS

1. A beam with the cross-section given below is subjected to a positive bending moment (causing compression at the top) of 16 kN-m acting around the horizontal axis. The tensile force acting on the hatched area of the cross-section is



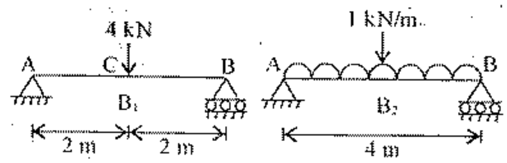
(a) zero (b) 5.9 kN
(c) 8.9 kN (d) 17.8 kN

2. A homogeneous, simply supported prismatic beam of width B, depth D and span L is subjected to a concentrated load of magnitude P. The load can be placed anywhere along the span of the beam. The maximum flexural stress developed in beam is

(a) $\frac{2}{3}\frac{PL}{BD^2}$ (b) $\frac{3}{4}\frac{PL}{BD^2}$
(c) $\frac{4}{3}\frac{PL}{BD^2}$ (d) $\frac{3}{2}\frac{PL}{BD^2}$

3. For a material which is very strong in compression and very weak in tension, the ideal shape of the cross-section to resist bending moment will be
(a) T-section (b) circular
(c) I-section (d) rectangular

4. The diagrams below show the details of two simply supported beams B_1 and B_2. EI is constant throughout the length and same for both the beams. The beams have the same area of cross-section and the same depth. What is the ratio of maximum bending stress in B_2 to that in B_1?



(a) 4 (b) 1
(c) 2 (d) 1/2

5. Which of the following are implied in the assumption of plane sections remaining plane (in simple bending)?
1. Stress is proportional to the distance from neutral axis.
2. Displacement is proportional to the distance from neutral axis.
3. Strain is zero across the cross-section.
4. Strain is directly proportional to the distance from neutral axis.

Select the correct answer using the codes given below :

(c) 3 and 4 (d) 1, 2 and 4

6. A steel beam is replaced by a corresponding aluminium beam of same cross sectional shape and dimensions, and is subjected to same loading. The maximum bending stress will
   (a) be unaltered
   (b) increase
   (c) decrease
   (d) vary in proportion to their modulus of elasticity

7. A structural beam subjected to sagging bending has a cross-section which is an unsymmetrical I section. The overall depth of the beam is 300 mm. The flange stresses in the beam are :
   $\sigma_{top} = 200\ N/mm^2$
   $\sigma_{bottom} = 50\ N/mm^2$
   What is the height in mm of the neutral axis above the bottom flange ?
   (a) 240 mm (b) 60 mm
   (c) 180 mm (d) 120 mm

8. A square beam laid flat is rotated in such a way that one of its diagonal becomes horizontal. How is its moment capacity affected ?
   (a) Increases by 41.4%
   (b) Increases by 29.27%
   (c) Decreases by 29.27%
   (d) Decreases by 41.4%

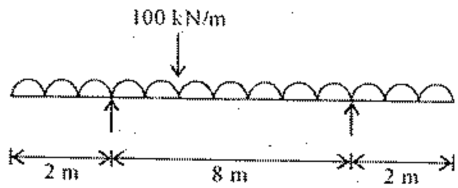
9. A cantilever beam of T cross-section carries uniformly distributed load. Where does the maximum magnitude of the bending stress occur ?
   (a) At the top of cross section
   (b) At the junction of flange and web
   (c) At the mid-depth point
   (d) At the bottom of the section

10. A 20 cm long rod of uniform rectangular section, 8 mm wide × 1.2 mm thick is bent into the form of a circular arc resulting in a central displacement of 0.8 cm. Neglecting second-order quantities in computations, what is the longitudinal surface strain (approximate) in the rod ?
   (a) 7.2×10^{-4} (b) 8.4×10^{-4}
   (c) 9.6×10^{-4} (d) 10.8×10^{-4}

11. Consider the following statements for a beam based on theory of bending:
   1. Strain developed in any fibre is directly proportional to the distance of fibre from neutral surface.
   2. For flexural loading and linearly elastic action the neutral axis passes through the centroid of cross-section.
   3. The assumption of the plane cross-section remaining plane will not hold good during inelastic action.
   4. Instances in which the neutral axis does not pass through the centroid of a cross-section include a homogenous symmetrical beam (with respect to neutral axis) and subjected to inelastic action.

   Which of these statements are correct?
   (a) 1, 2, 3 and 4 (b) 1, 2 and 4
   (c) 3 and 4 (d) 1 and 2

12. The bending moment for which the beam shown below is to be designed is



(a) 200 kN-m (b) 800 kN-m
(c) 600 kN-m (d) 640 kN-m

13. Neglecting self weight, which of the following beams will have points of contraflexure?
(a) A simply supported beam with uniformly distributed load over part of the structure
(b) An overhanging beam with loading only over supported span and not on overhangs
(c) Fixed beam subjected to concentrated load
(d) Cantilever beam subjected to uniformly varying load with zero load at free end

14. Match **List-I** with **List-II** and select the correct answer using the codes given below the lists:

List-I

A. Assumption in the theory of simple bending
B. The point at which the bending stress is maximum for any cross-section
C. The point at which the bending stress is zero for any cross section
D. The point in the cross section through which the neutral axis passes

List-II

1. Neutral axis
2. Centroid
3. The plane sections remain plane
4. Extreme fibre
5. The cross-section is circular

Codes:

	A	B	C	D
(a)	5	4	1	2
(b)	3	1	2	4
(c)	5	1	2	4
(d)	3	4	1	2

15. **Assertion (A)**: A beam of fixed length and for given weight of material, a rectangular cross section provides the greatest possible moment of resistance.
Reason (R): In a beam of I cross-section, more material is positioned near the outer fibres representing regions of greatest stress and hence is stronger than beam of rectangular cross-section.
Of these statements
(a) both A and R are true and R is the correct explanation of A
(b) both A and R are true but R is not a correct explanation of A
(c) A is true but R is false
(d) A is false but R is true

16. Which of the following points are considered while deriving the formula $\frac{M}{I} = \frac{f}{y} = \frac{E}{R}$

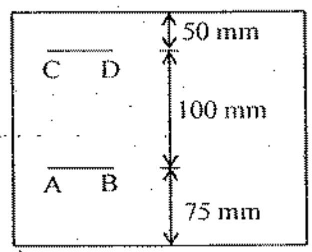
1. Type of material.
2. Transverse shear force

3. The stresses in the remaining principal direction.
4. $\sigma_y = \sigma_z = \tau_{xz} = \tau_{zx} = 0$
5. Linear variation of strain.

Select the correct answer using the codes given below:

(a) 1, 2 and 4 (b) 2, 3 and 5
(c) 4 and 5 (d) 1 and 3

17. A test is conducted on a beam loaded by end couples. The fibres at layer CD are found to lengthen by 0.03 mm and fibres at layer AB shorten by 0.09 mm in 20 mm gauge length as shown in the given figure. Taking $E = 2 \times 10^5$ N/mm^2, the flexural stress at top fibres would be



(a) 900 N/mm^2 tensile (b) 1000 N/mm^2 tensile
(c) 1200 N/mm^2 tensile (d) 1200 N/mm^2 compressive

18. The cross-section of a beam is shown in figure-I. Its I_{xx} is equal to 3×10^6 mm^4. It is subjected to a load as shown in figure-II. The maximum tensile stress in the beam would be:

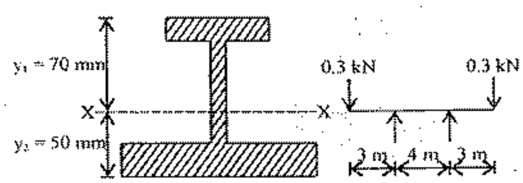


Figure-I Figure-II

(a) indeterminable as data is insufficient
(b) 21 MN/m^2
(c) 21 kN/m^2
(d) 21 N/mm^2

19. A high strength steel band-saw of 90 mm width and 0.5 mm thickness runs over a pulley of 500 mm diameter. Assuming E = 200 GPa, the maximum flexural stress developed would be

(a) 100 MPa (b) 200 MPa
(c) 400 MPa (d) 500 MPa

20. A rectangular timber beam is cut out of a cylindrical log of diameter D. The depth of the strongest timber beam will be

(a) $\sqrt{\frac{1}{2}}D$ (b) $\sqrt{\frac{2}{3}}D$

(c) $\sqrt{\frac{5}{8}}D$ (d) $\sqrt{\frac{3}{4}}D$

21. The ratio of the flexural strength of two beams of square cross-section, the first beam being placed with its top and bottom sides horizontally and the second beam being placed with one diagonal horizontally is

(a) $\sqrt{3}$ (b) $1/\sqrt{3}$
(c) $1/\sqrt{2}$ (d) $\sqrt{2}$

22. In a simply supported wooden beam under uniformly distributed load, a hole has to be made in the direction of width at midspan to provide a pipeline. From structural strength point of view, it would be advisable to have the hole made at
(a) the bottom
(b) the top
(c) mid-depth
(d) 1/4 depth either from the top or the bottom

23. The ratio of moment carrying capacity of a circular beam of diameter D and square beam of size D is
(a) $\pi/4$ (b) $3\pi/8$
(c) $\pi/3$ (d) $3\pi/16$

24. **Assertion (A):** I-Section is preferred to rectangular section for resisting bending moment.
Reason (R): In I-Section more than 80% of bending moment is resisted by flanges only.
Of these statements
(a) both A and R are true and R is the correct explanation of A
(b) both A and R are true but R is not a correct explanation of A
(c) A is true but R is false
(d) A is false but R is true

25. A cantilever of constant depth carries a uniformly distributed load on the whole span. To make the maximum stress at all sections the same, the breadth of the section at a distance x from the free end should be proportional to
(a) x (b) $\sqrt{x}$
(c) x^2 (d) x^3

26. A simply supported beam of constant width and varying depth and uniform strength is subjected to a central concentrated load. The depth of the beam 'd_x' at a distance 'x' from one of the supports is proportional to
(a) $x^{1/2}$ (b) $x^{1/3}$
(c) x (d) x^2

27. In order to construct a simply supported beam of constant strength throughout its length in flexure, the section of the beam must be
(a) prismatic
(b) non-prismatic
(c) symmetrical
(d) unsymmetrical

28. A beam of uniform strength refers to which one of the following?
(a) A beam in which extreme fibre stresses are same at all cross-sections along the length of the beam.
(b) A beam in which the moment of inertia about the axis of bending is constant at all cross-section of the beam.
(c) A beam in which the distribution of bending stress across the depth of cross-section is uniform at the all cross-sections of the beam.

29. Consider the following statements:

For each component in a flitched beam under the action of a transverse load,

1. the radius of curvature will be different
2. the radius of curvature will be the same
3. the maximum bending stress will be the same
4. the maximum bending stress will be dependent upon the modulus of elasticity of the material of the component

Which of these statements are correct?

(a) 1 and 3 (b) 1 and 4
(c) 2 and 3 (d) 2 and 4

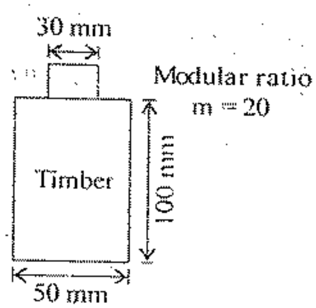
30. Consider the following statements about flitched beams:

1. A flitched beam has a composite section made of two or more materials joined together in such a manner that they behave as a unit piece and each material bends to the same radius of curvature.
2. The total moment of resistance of a flitched beam is equal to the sum of the moments of resistance of individual sections.
3. Flitched beams are used when a beam of one material, if used alone, would require quite a large cross-sectional area.

Which of these statements are correct?

(a) 1, 2 and 3 (b) 1 and 2
(c) 1 and 3 (d) 2 and 3

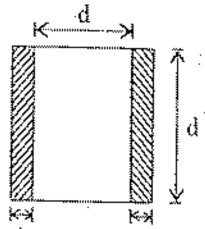
31. A timber beam of rectangular section 100 mm × 50 mm is simply supported at the ends, has a 30 mm × 10 mm steel strip securely fixed to the top surface as shown in the given figure.

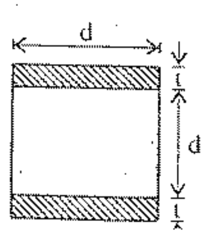


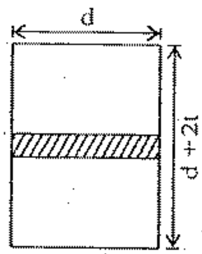
The centroid of the "equivalent timber beam" in this case from the top surface

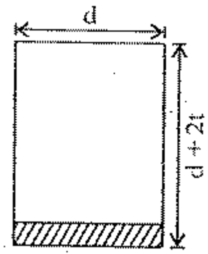
(a) is 5 mm
(b) is 30 mm
(c) is 15 mm
(d) cannot be predicted

32. The modular ratio of the materials used in the flitched beam is 10 and the ratio of the allowable stresses is also 10. Four different sections of the beam are shown in the given figures. The material shown hatched has larger modulus of elasticity and allowable stress than the rest.

1. 

2. 

3. 

4. 

Which one of the following statements is true for the beam under consideration?

(a) All the given sections would support the same magnitude of load.

(b) Sections 2, 3 and 4 would support equal loads which is more than what section 1 would support.

(c) Sections 1 and 2 would support equal loads which is more than what section 3 and 4 would support.

(d) Section 2 would support greatest load.

33. A simply supported beam is made of two wooden planks of same width resting one upon the other without friction and without connection. The upper plank is of half the thickness as compared to lower plank. The assembly is loaded by a uniformly distributed load on the entire span. The ratio of the maximum stresses developed between top and bottom planks will be

(a) 1 : 16 (b) 1 : 8

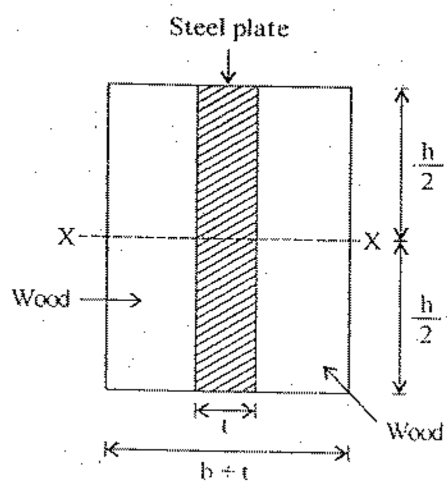
(c) 1 : 4 (d) 1 : 2

34. A beam is made of two identical metal flats soldered together. What is the ratio of stiffness of this beam to the stiffness of a beam in which the two flats are not soldered and which acts independently?

(a) 2 (b) 4

(c) 6 (d) 8

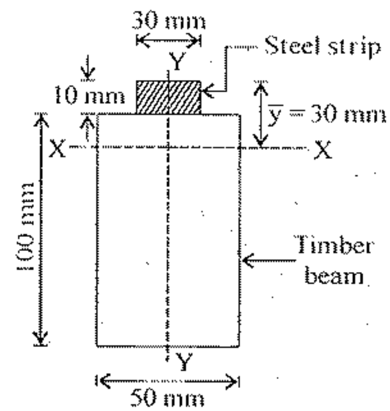
35. The figure below shows the cross-section of a flitched beam consisting of a steel plate sandwiched between two wooden blocks. The second moment of area of the composite beam about the neutral axis XX is



(a) $\frac{bh^3}{12}+\frac{mth^3}{12}$ (b) $\frac{bh^3}{12}+\frac{t(mh)^3}{12}$

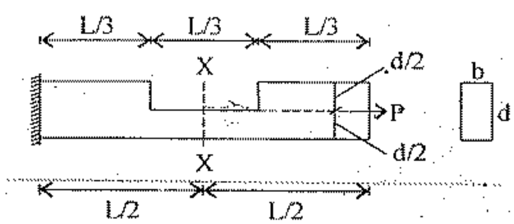
(c) $(b+t)\frac{h^3}{12}$ (d) $\frac{bh^3}{12}$

36. The cross-section of a timber beam with a steel strip is shown in the given figure. It is subjected to a sagging bending moment of 1000 Nm. I = 1250 × 10^4 mm^4 and modular ratio m = 20. The stresses



(a) + 6.4 MN/m² and – 6.4 MN/m²
(b) – 6.4 MN/m² and – 32 MN/m²
(c) + 6.4 MN/m² and – 32 MN/m²
(d) + 32 MN/m² and – 6.4 MN/m²

37. The maximum tensile stress at the section X-X shown in the figure below is



(a) $\frac{8P}{bd}$ (b) $\frac{6P}{bd}$

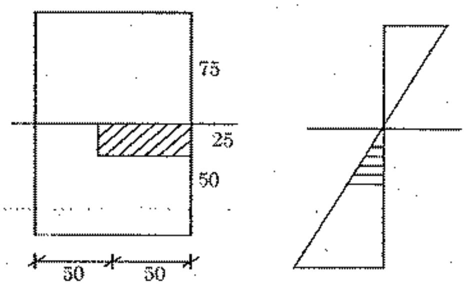
(c) $\frac{4P}{bd}$ (d) $\frac{2P}{bd}$

ANSWERS

1. (c)	10. (c)	19. (b)	28. (a)
2. (d)	11. (d)	20. (b)	29. (d)
3. (a)	12. (c)	21. (d)	30. (c)
4. (d)	13. (c)	22. (c)	31. (b)
5. (d)	14. (d)	23. (d)	32. (d)
6. (a)	15. (d)	24. (a)	33. (d)
7. (b)	16. (c)	25. (c)	34. (b)
8. (c)	17. (a)	26. (a)	35. (a)
9. (d)	18. (b)	27. (b)	36. (c)
			37. (a)

SOLUTION...

1. (c)



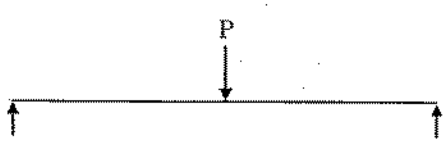
$$\sigma \text{ @ 25 mm from NA} = \frac{(16\times10^6)\times25}{\left(\frac{100\times150^3}{12}\right)}$$

$$= 14.22 \text{ N/mm}^2$$

$$\text{Tensile force (T)} = \left(\frac{1}{2}\times14.22\right)\times(25\times50)$$

$$= 8.9 \text{ kN}$$

2. (d) For max BM, 'P' must be placed at centre.

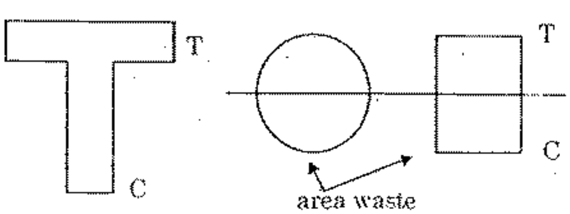


$$M_{max} = \frac{PL}{4}$$

$$\sigma = \frac{My}{I} = \frac{\frac{PL}{4}\times\frac{D}{2}}{\frac{BD^3}{12}} = \frac{3PL}{2BD^2}$$

3. (a) Very strong in compression → requires lesser C/S area in that stress portion.

Very weak in tension → Larger area @ one side



4. (d)

$$\sigma = \frac{My}{I}$$

$$\frac{\sigma_1}{\sigma_2} = \frac{M_1}{M_2}$$

$$= \frac{WL/4}{wL^2/8}$$

$$= \frac{4\times4}{4}\times\frac{8}{1\times4^2}$$

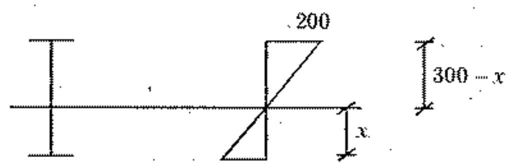
$$\sigma_1/\sigma_2 = 2$$

$$\therefore \quad \sigma_2/\sigma_1 = \frac{1}{2}$$

6. (a) σ_{max} will be same for given beam and loading in both Al and Steel.

However, note that the stress levels that the materials can take /bear before/ till failure is dependent (on material – AL & St).

7. (b)



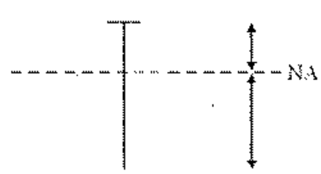
$$\sigma = \frac{My}{I}$$

$$\frac{50}{x} = \frac{200}{300-x}$$

$$300 = 5x$$

$$x = 60 \text{ mm}$$

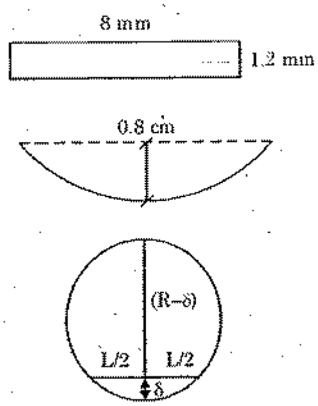
9. (d)



Bending stress → linear variation max stress at the extreme fibre.

Therefore at the bottom

10. (c)



$$\frac{f}{y} = \frac{M}{I} = \frac{E}{R} \quad \ldots (i)$$

Properties of Circle

$$\frac{L^2}{4} = (2R-\delta)\delta$$

$$= 2R\delta - \delta^2$$

$$R = \frac{L^2}{8\delta} \quad [\delta^2 \to 0] \quad \ldots (ii)$$

$$R = \frac{200^2}{8\times 8} \text{ mm}$$

$$\frac{f}{\left(\frac{1.2}{2}\right)\text{mm}} = \frac{2\times 10^5 \text{ N/mm}^2}{\left(\frac{200^2}{8\times 8}\right)}$$

$$f = 192 \text{ N/mm}^2$$

$$E = \frac{f}{\varepsilon}$$

$$\varepsilon = \frac{192}{2\times 10^5}$$

$$\varepsilon = 9.6\times 10^{-4}$$

12. (c) Beams are always designed for max. BM that may occur anywhere in the beam.

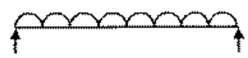
$$\text{BM @ centre} = \frac{Wl^2}{8} - \frac{Wa^2}{2}$$

$$= W\times\left(\frac{8^2}{8} - \frac{2^2}{2}\right)$$

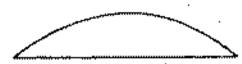
$= 100 \times 6 = 600$ kN·m → max (design B.M)

$$\text{B.M @ supp.} = \frac{Wa^2}{2} = 100\times\frac{2^2}{2} = 200 \text{ kN·m}$$

13. (c)

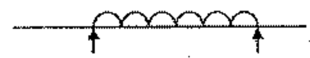


(a)



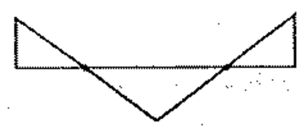
BMD

(b)



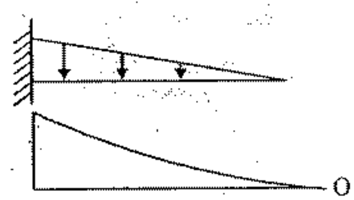
same as SSB

(c)



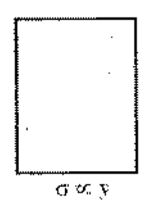
2 Nos of contraflexure

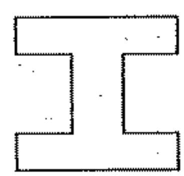
(d)



BMD

15. (d)





Stresses are more at extreme fibres

Keeping more area to resist the loads at extreme ends is more economical and stronger.

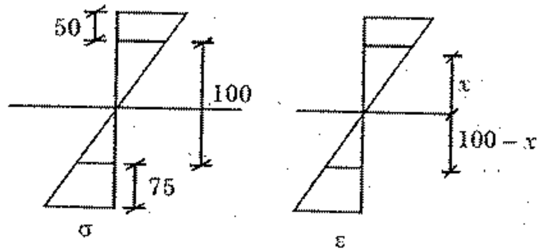
Area at NA is unused/less stressed (Under used)

16. (c) 1. Basic assumption of solid mechanics is linear variation of strain.

Hookes law is the basic law, valid based on this assumption.

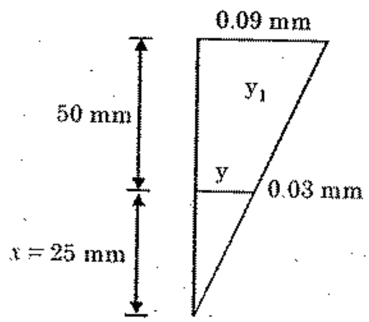
2. Bending theory is valid only for pure bending stresses.

17. (a) Pure bending test :



$$\frac{x}{0.03} = \frac{100-x}{0.09}$$
$$4x = 100$$
$$x = 25 \text{ mm}$$

Variation is linear

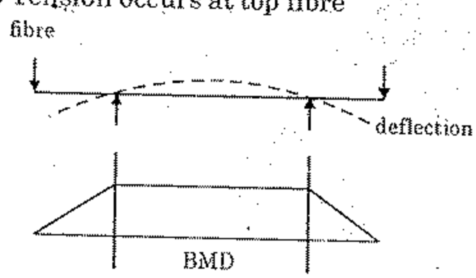


$$\frac{25}{0.03} = \frac{75}{(y)}$$
$$(y) = 0.09 \text{ mm}$$

$$\sigma = E \times \varepsilon = 2 \times 10^5 \times \frac{0.09}{20}$$

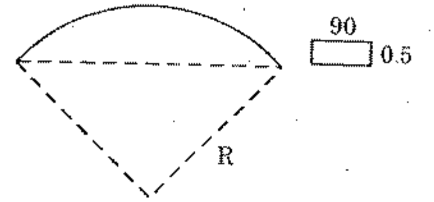
top → tensile $= 900 \text{ N/mm}^2$

18. (b) Tension occurs at top fibre



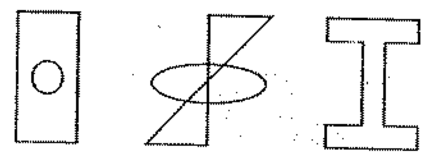
$$M_{max} = 0.3 \times 3 = 0.9 \text{ kN-m}$$
$$\frac{\sigma}{y} = \frac{M}{I}$$
$$\sigma = \frac{0.9 \times 10^6 \times 70}{3 \times 10^6} = 21 \text{ N/mm}^2$$
$$= \frac{21 \times N}{} = 21\frac{MN}{}$$

19. (b)



$$\frac{f}{y} = \frac{E}{R}$$
$$f = \frac{2 \times 10^5 \text{ N/mm}^2 \times \frac{0.5}{2}}{250 \text{ mm}}$$
$$f = 200 \text{ MPa}$$

22. (c)

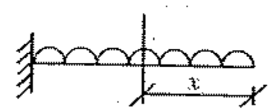


As far as flexure is concerned alone, area nearer to NA is under utilized.

∴ Hole can be suggested @ NA

i.e, even if some area is lost, flexural stresses will still be within permissible stresses.

25. (c)



$$M_{max} = \frac{Wx^2}{2}$$

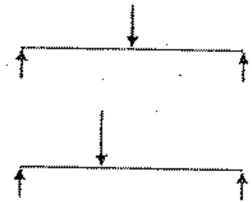
$$\sigma = \frac{MY}{I} = \frac{M}{\left(\frac{bd^2}{6}\right)}$$

$$\sigma = \frac{Wx^2/2}{bd^2/6} \text{ [For uniform strength]}$$

breadth is variable here $bd^2 \propto x^2$

$$b \propto x^2$$

26. (a)



$b = \text{const.}$

$M = \frac{Wl}{4}$ [for uniform strength everywhere]

$\sigma = \frac{6M}{bd^2}$ [$M \propto x$]

$x \propto d^2$

$d \propto \sqrt{x}$

28 (a) Constant strength $\Rightarrow$ Extreme stresses

σ = constant every where along the length

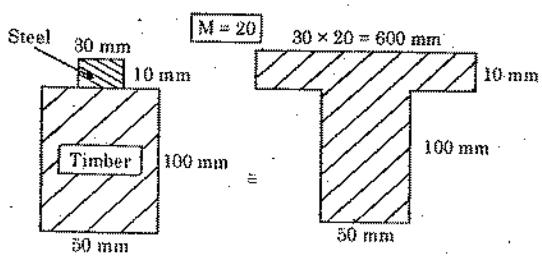
$\Rightarrow$ C/s of the member changes from point to point.

$\Rightarrow$ Non-Prismatic

29. (d) Concept of Flitched beam.

30. (a) Concept of Flitched beam.

31. (b)

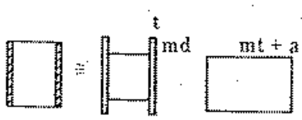


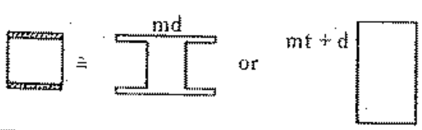
equivalent timber beams

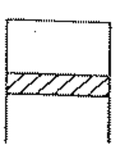
$$\bar{y}_{(\text{from top})} = \frac{600\times10\times5+5000\times(60)}{11000}$$

$$= \frac{330}{11} = 30 \text{ mm}$$

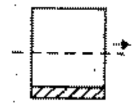
32. (d)

1\.


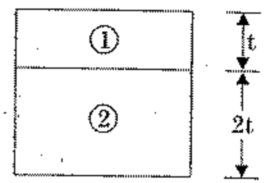
2\.


3\.


as material at NA is under utilized

4\.


33. (d) Radius of curvature will be same for both the case.



$M_1 + M_2$ = total BM at - a section

$$\frac{M_1}{E_1I_1} = \frac{M_2}{E_2I_2}$$

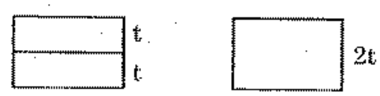
$$M_2/M_1 = I_2/I_1 = \frac{\frac{bd_2^3}{12}}{bd_1^3/12} = \frac{d_2^3}{d_1^3} = 8$$

$$M_2 = 8M_1$$

$$\therefore \quad \frac{\sigma_1}{\sigma_2} = \frac{\frac{M_1}{I_1}y_1}{\frac{M_2}{I_2}y_2} = \left(\frac{M_1}{M_2}\right)\left(\frac{I_2}{I_1}\right)\times\frac{y_1}{y_2}$$

$$\frac{\sigma_1}{\sigma_2} = \frac{y_1}{y_2} = \frac{t/2}{t} = \frac{1}{2}$$

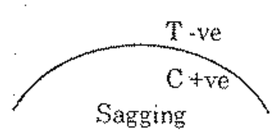
34. (b)



Stiffness = I

$$\frac{K_{\text{together}}}{K} = \frac{(2t)^3}{2\times t^3} = 4$$

36. **(c) For steel @ lower surface**



$$\sigma = \frac{MY}{I} = \frac{10^3\times(20\times10^{-3})}{1250\times10^4\times10^{-12}}$$

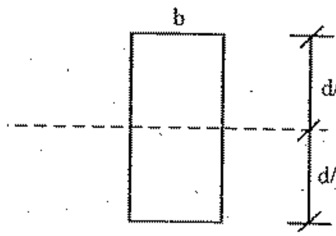
$= 32 \text{ MN/m}^2$ (T)

For timber @ lower surface

$$\sigma = \frac{10^3\,\text{N}-\text{m}\times\left(80\times10^{-3}\,\text{m}\right)}{1250\times10^4\times10^{-12}}$$

$= 6.4\ \text{MN/m}^2$ (c)

37. (a)



$$\sigma_{max/min} = \frac{P}{A}\pm\frac{M}{Z}$$

$$Z = \frac{bd^2}{6}$$

$$\sigma_{max\ tensile\ stress} = \frac{P}{\frac{bd}{2}}+\frac{P\times\frac{d}{4}}{\frac{b}{6}\times\left(\frac{d}{2}\right)^2}$$

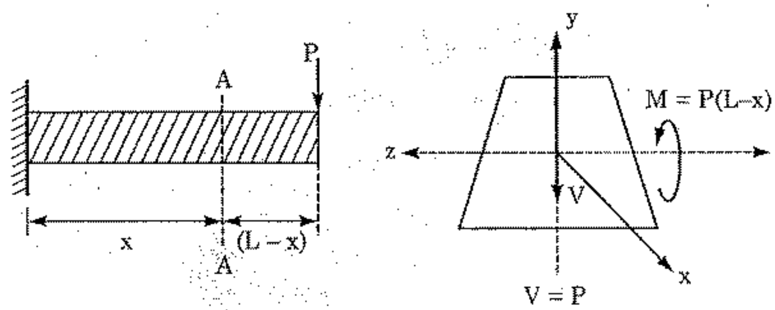
$$= \frac{P}{bd}(2+6) = \frac{8P}{bd}$$

7

Shear Stress in Beams

INTRODUCTION

- Transverse loading applied to a beam (as shown below) results in normal and shearing stresses in the beam.
- Normal stress is produced by bending and shear stress is produced by shear force V = P.
- The vertical shearing stress is accompanied by a horizontal shearing stress of equal magnitude known as complementary shear stress.
- The dominant criterion in the design of beam for strength is the max value of normal stress. However, shearing stress is dominant in case of short and deeper beams.



Section A-A

- The following figure shows the vertical and complementary shear stresses. Figure B shows that deformation produced by the shear stress. γ as shown in the figure is the shear strain.

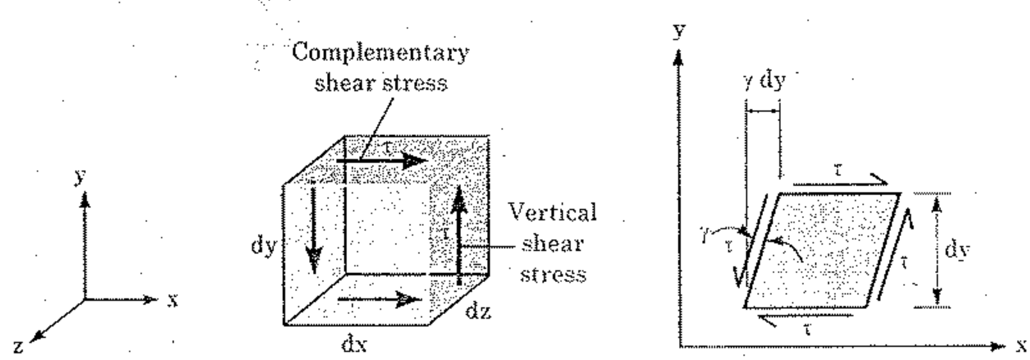
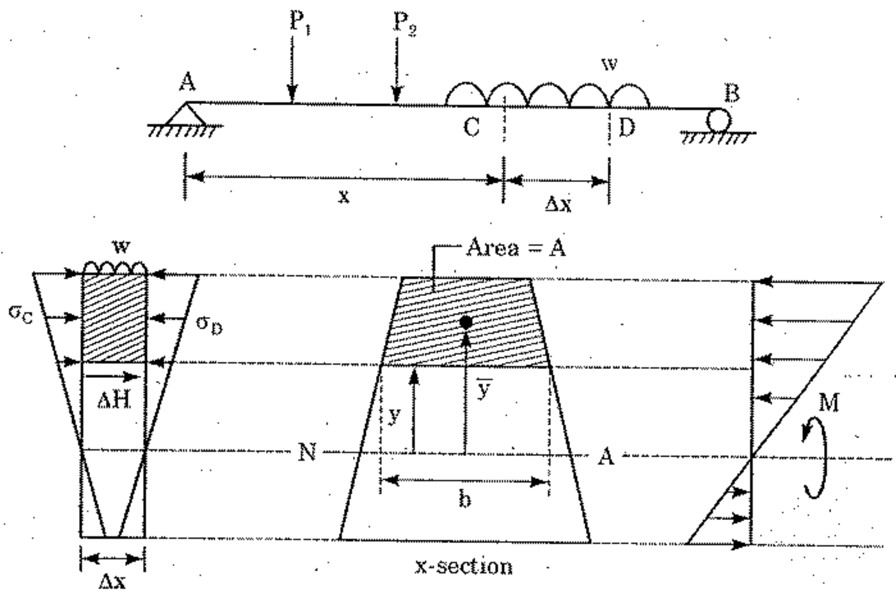


Fig. A   Fig. B

SHEAR STRESS IN PRISMATIC BEAM HAVING LOADING IN THE VERTICAL PLANE OF SYMMTRY



$$\Sigma F_{Horizontal} = 0 \Rightarrow \Delta H + \int_A (\sigma_C - \sigma_D)\, dA = 0$$

$$\Rightarrow \Delta H = \int_A (\sigma_D - \sigma_C)\, dA = \int_A \frac{(M_D - M_C)\, y}{I}\, dA$$

$$= \frac{M_D - M_C}{I} \int_A y\, dA = \frac{\Delta M}{I} \cdot A\bar{y}$$

$$\Rightarrow \Delta H = \frac{\left(\frac{dM}{dx} \times \Delta x\right) A\bar{y}}{I} = \frac{VA\bar{y}}{I} \Delta x$$

$$\Rightarrow \boxed{\Delta H = \frac{VA\bar{y}}{I} \Delta x}$$ = Shear force in length Δx of beam

- Shear force per unit length of beam $= \frac{\Delta H}{\Delta x} = \frac{VA\bar{Y}}{I}$

  Shear force per unit length of beam is called shear flow (q)

  $$\Rightarrow \boxed{q = \frac{VA\bar{y}}{I}}$$

- Shear stress at the level y from N.A. = Complementary shear stress at the leve y from N.A. $= \frac{\Delta H}{b\Delta x} = \frac{VA\bar{y}}{Ib}$

  where V = SF at the section where shear stress is to be found.

$$\boxed{\text{Shear stress at the level y from N.A.} = \frac{VA\bar{y}}{Ib}}$$

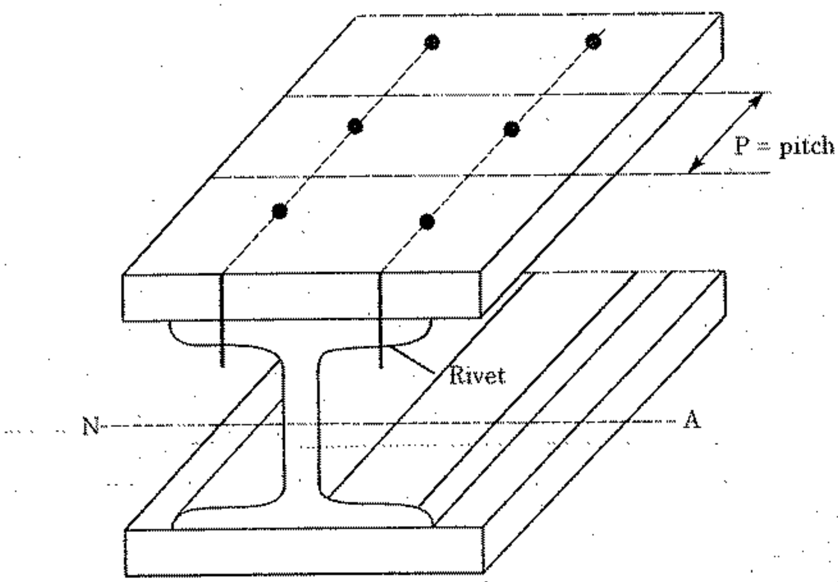
where

$A\bar{y}$ = Moment of area of section above the level at which shear stress is to be found out

I = Moment of inertia of complete section about NA

b = Width of section at the level where shear stress is to be found

An Important Application



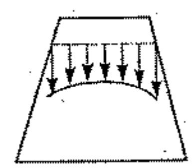
$$\boxed{\text{Shear force in one rivet} = \frac{\left(\frac{VA\bar{y}}{I}\right)\times p}{2}}$$

$$\text{Shear force in one rivet} = \frac{\text{Shear force per pitch length}}{2}$$

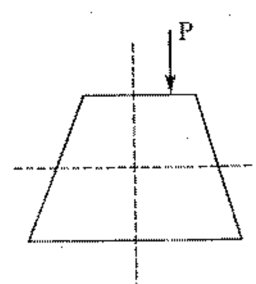
$$= \frac{\text{Shear force per unit length} \times \text{Pitch}}{2}$$

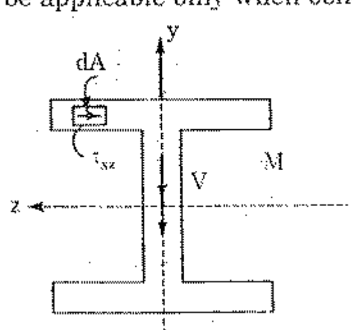
Note:

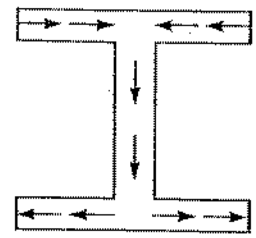
1. Shear stress $= \frac{VA\bar{y}}{Ib}$ has been derived assuming shear stress to be constant across the width. However, shear stress varies across the width as shown below. Hence the formula $\frac{VA\bar{y}}{Ib}$ shear stress, should actually be valid only for narrow sections.



2. If plane of loading does not pass through the symmetrical plane as show below, twisting of the section will occur. Shear stress $= \frac{VA\bar{y}}{Ib}$ or shear flow $= \frac{VA\bar{y}}{I}$ will be applicable only when bending occurs without twisting.



3. 



- For the loading shown in the figure, $\Sigma F_z = 0$

  $\Rightarrow \int \tau_{xz} \times dA = 0$

- This does not mean that $t_{xz} = 0$. It simply means that $\int \tau_{xz} \times dA = \bar{\tau}_{xz}.A = 0$

  $\Rightarrow \bar{\tau}_{xz} = 0$ (i.e. av. shear stress on x-face in z-direction is zero).
- Hence corresponding to loading shown in the above figure, shear stress as shown in the following figure will exist.
- Note that although $\bar{\tau}_{xz}$ in the above figure is zero, but τ_{xz} is not zero.

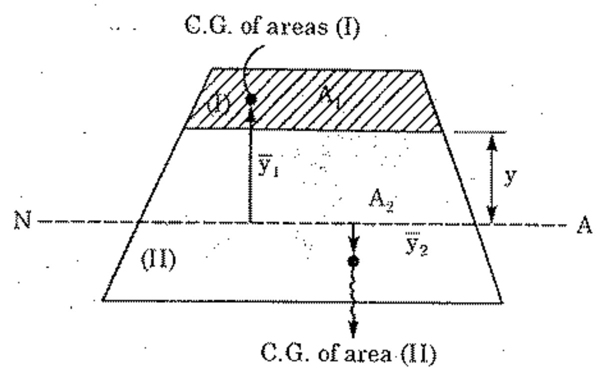
4. We know that moment of area of complete section about N.A. is equal to zero. [This is the definition of N.A. for stresses with in proportional limit].

   Hence in the following figure

   |Moment of area of Area (I) about N.A.| = |Moment of area of area (II) about N.A.|

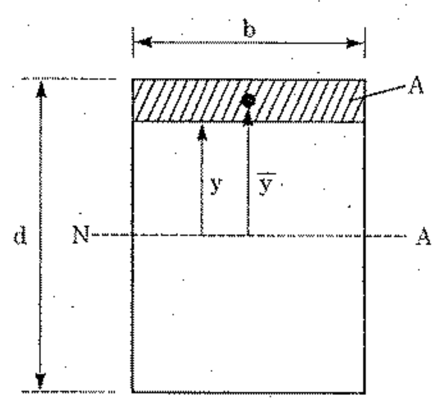
   i.e. $A_1\bar{y}_1 = A_2\bar{y}_2$

   where A_1 = shaded area, A_2 = Unshaded area



Thus, in the expression shear stress = $\frac{VA_1\bar{y}_1}{Ib}$, $A_1\bar{y}_1$ can be replaced by $A_2\bar{y}_2$, where $A_2\bar{y}_2$ is the moment of area located below the level at which we want to find out shear stress about Neutral Axis.

SHEAR STRESS IN RECTANGULAR SECTION



$$I = \frac{bd^3}{12}$$

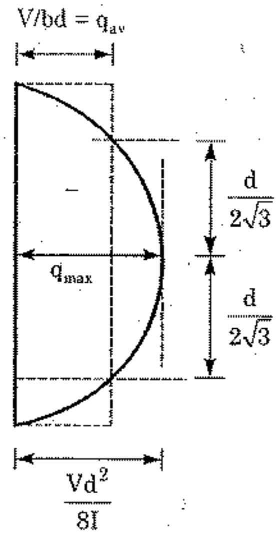
$$A\bar{y} = b\left(\frac{d}{2} - y\right)\left[y + \frac{\frac{d}{2} - y}{2}\right] = b\left(\frac{d}{2} - y\right)\left(\frac{y + \frac{d}{2}}{2}\right) = \frac{b\left(\frac{d^2}{4} - y^2\right)}{2}$$

$$\Rightarrow \text{Shear stress} = \frac{VA\bar{y}}{Ib} = \frac{Vb\left(\frac{d^2}{4} - y^2\right)}{2 \times \frac{bd^3}{12} \times b}$$

$$\Rightarrow \boxed{\text{Shear stress} = \frac{6V}{bd^3}\left(\frac{d^2}{4} - y^2\right) = \frac{V}{2I}\left(\frac{d^2}{4} - y^2\right)} \quad \text{......... (A)}$$

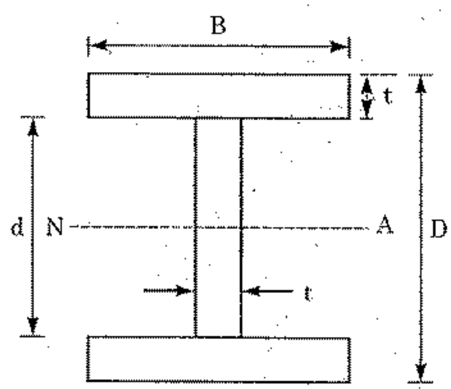
- Max shear stress occurs at y = 0 (i.e. at neutral axis)
- $(\text{Shear stress})_{max} = \frac{3}{2}\left(\frac{V}{bd}\right) = \frac{3}{2}(\text{shear stress})_{av}$
- Variation is parabolic which is as show in the figure below.

Shear Stress Distribution in Rectangular Section

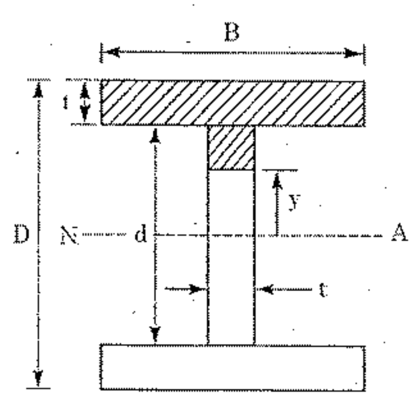


- $\boxed{q_{max} = \frac{3}{2} q_{av}}$
- Note that shear stress will be equal to average shear stress at $\frac{d}{2\sqrt{3}}$ distance from neutral axis.

SHEAR STRESS IN I-SECTION



Shear Stress in Web



- We know that

$$A\bar{y} = A_1\bar{y}_1 + A_2\bar{y}_2 + A_3\bar{y}_3 + ...$$

Hence .

$$A\bar{y} = B\left(\frac{D-d}{2}\right)\left(\frac{d}{2}+\frac{D-d}{4}\right) - t\left(\frac{d}{2}-y\right)\left(y+\frac{\frac{d}{2}-y}{2}\right)$$

$$\Rightarrow \quad A\bar{y} = \frac{B(D^2-d^2)}{8} + \frac{t\left(\frac{d^2}{4}-y^2\right)}{2}$$

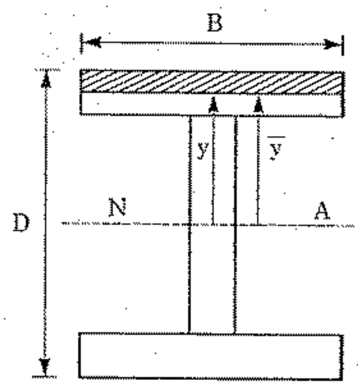
- $I = \left[\frac{BD^3}{12} - \frac{(B-t)(D-2t)^3}{12}\right]$

  $= \left[\frac{BD^3}{12} - \frac{(B-t)d^3}{12}\right]$

- $\tau = \frac{V}{I\times t}\left[\frac{B(D^2-d^2)}{8} + \frac{t\left(\frac{d^2}{4}-y^2\right)}{2}\right]$ = Shear stress in web

$$\boxed{\tau = \frac{V(D^2-d^2)}{8I}\left(\frac{B}{t}\right) + \frac{V}{2I}\left(\frac{d^2}{4}-y^2\right)}$$ = Shear stress in web

Shear Stress in Flange



$$A\bar{y} = B\left(\frac{D}{2}-y\right)\left(y+\frac{\frac{D}{2}-y}{2}\right) = \frac{B\left(\frac{D^2}{4}-y^2\right)}{2}$$

$$\tau = \text{Shear stress} = \frac{VB\left(\frac{D^2}{4}-y^2\right)}{2\times I\times B}$$

$$\boxed{\tau = \frac{V}{2I}\left(\frac{D^2}{4}-y^2\right)}$$

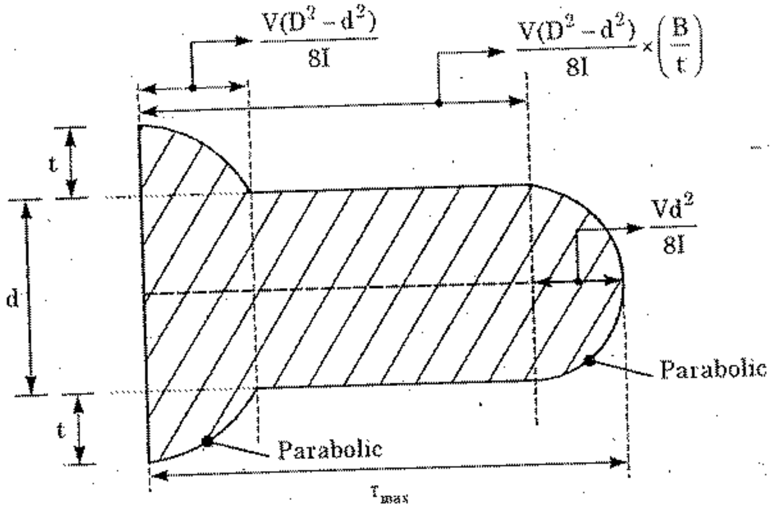
- At $y = \frac{d}{2}$

  Shear stress in flange $= \frac{V(D^2-d^2)}{8I}$

  Shear stress in web $= \frac{V(D^2-d^2)}{8I}\times\frac{B}{t}$

$\Rightarrow$ Due to Sudden change in width, transverse shear stress changes in the ratio of $\frac{B}{t}$

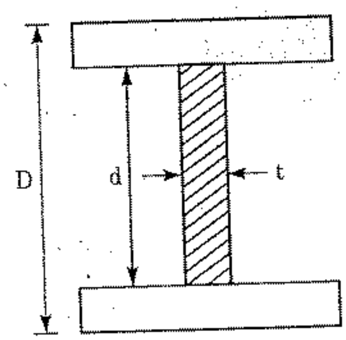
Shear Stress Distribution in I-Section



Note:

1. Normally for I-sec, approx. (80–85%) of shear is resisted by web.
2. For approximate calculation of max shear stress in I-sec, following approach is used.

$$\boxed{\tau_{max} = \frac{V}{d\times t} = \frac{V}{A_{web}}}$$

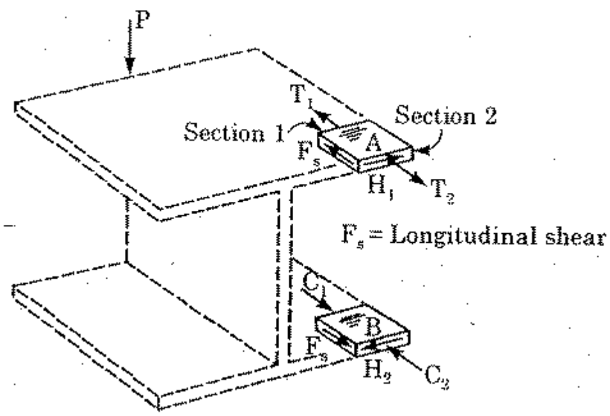


- In design of steel structure max permissible value of τ_{max} is 0.45 f_y, where f_y is equal to yield stress.

  Thus, we can use $\boxed{\frac{V}{A_{web}} \le 0.45\, f_y}$.
- However in actual practice in design, we generally calculate average shear stress
- Average shear stress is given by $\tau_{av.} = \frac{V}{Dt}$ and the acceptable limit of $\tau_{av.}$ in the design of steel structure is 0.4 f_y. Actually, the validity of $\tau_{av.} < 0.4\, f_y$ ensures that $\tau_{max} \le 0.45\, f_y$.

LONGITUDINAL SHEAR STRESS IN BEAM

In the Fig. shown below, if $T_2 > T_1$ and $C_2 > C_1$ for, balancing the elements A and B, the direction of



Longitudinal shear force $\Delta H = \frac{VA\bar{y}}{I} \times \Delta x$

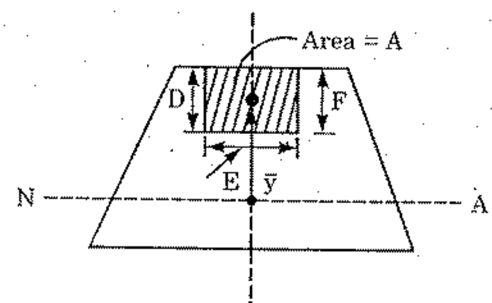
where, Longitudinal shear force/Length = $\frac{VA\bar{y}}{I}$ = Shear flow

I = M.O.I of entire section about N.A.

V = Transverse shear force

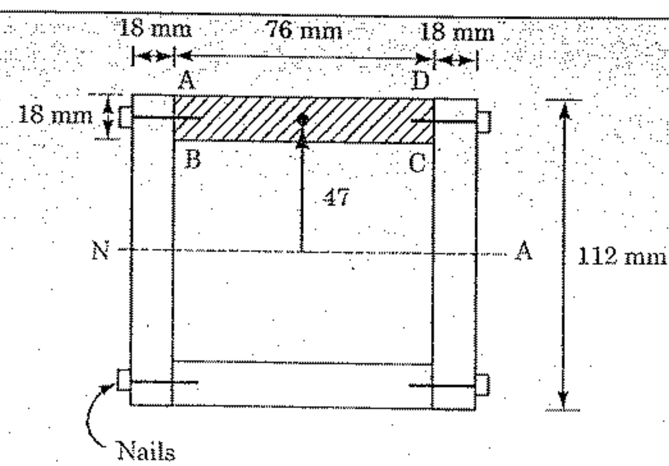
$A\bar{y}$ = Moment of area of shaded portion.

Note:



Shear force per unit length along D, E, F taken together = $\frac{VA\bar{y}}{I}$

Example 1



Spacing between nails = 44 mm

Shear force at the section = 2.5 kN

Sol: Shear force per unit length = $\frac{VA\bar{y}}{I}$ for the shaded area $A\bar{y} = (18 \times 76) \times (47) = 64296\ mm^3$

Moment of inertia of complete section

$$I = \frac{(112)^4 - (76)^4}{12} = 10332 \times 10^3\ mm^4$$

Shear force per unit length along AB and CD taken together (No shear force exist along BC and AD its being a free surface)

$$= \frac{2.5 \times 64296}{10332 \times 10^4} = 15.6\ N/mm$$

Shear force in 44 mm length = 15.6 × 44 = 684.5 N

No. of nails that resist it = 2

$\Rightarrow$ Shear force one each nail = $\frac{684.5}{2} = 342.25$ N

Example 2

For the beam sections shown below determine:

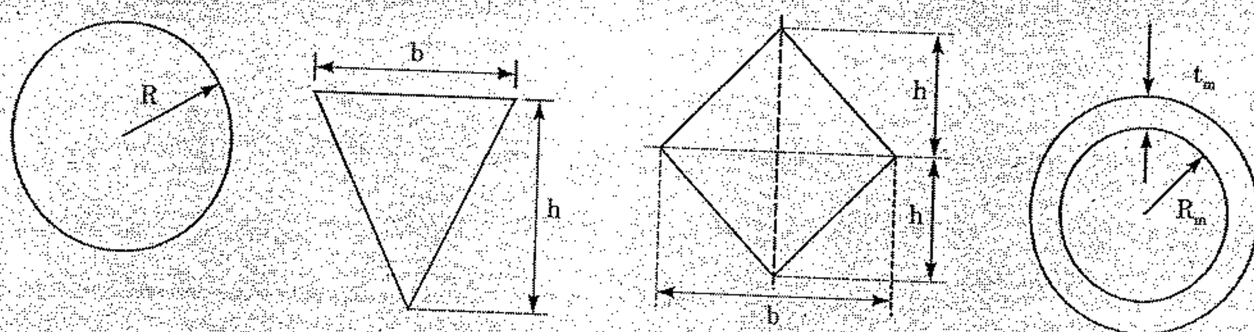
(a) The horizontal line along which shear stress is max.

(b) The constant 'k' in the following expression for max. shearing stress, where k is given by

$$\tau_{max} = k\frac{V}{A}$$

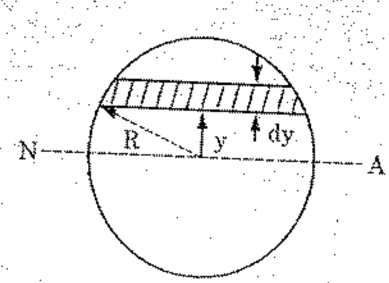
A = Area of x-section of beam

V = Vertical shear force on the section



(i) (ii) (iii) (iv)

Sol: (i)



Shear stress at a distance 'y' from N.A. = $\frac{VA\bar{y}}{Ib}$

$$A\bar{y} = \int_y^R y\,dA = \int_y^R y\left(2\sqrt{R^2 - y^2}\right)dy$$

$$A\bar{y} = \int_y^R \left[(2y)\sqrt{R^2 - y^2}\right]dy$$

Let $R^2 - y^2 = x^2$

$$\Rightarrow -2y\,dy = 2x\,dx$$

at $y = y,\ x = \sqrt{R^2 - y^2}$

at $y = R,\ x = 0$

$$\Rightarrow A\bar{y} = -\int_{\sqrt{R^2-y^2}}^{0} 2x\,dx\,(x) = \frac{-2}{3}x^3\Big|_{\sqrt{R^2-y^2}}^{0}$$

$$\Rightarrow A\bar{y} = \frac{-2}{3}\left[0 - \left(R^2 - y^2\right)^{3/2}\right] = \frac{2}{3}\left(R^2 - y^2\right)^{3/2}$$

$$\boxed{A\bar{y} = \frac{2}{3}(R^2 - y^2)^{3/2}}$$

$$I = \frac{\pi D^4}{64} = \frac{\pi R^4}{4}$$

$$\Rightarrow \text{Shear stress} = \frac{VA\bar{y}}{Ib} = \frac{V \times \frac{2}{3}\left(R^2 - y^2\right)^{3/2}}{\frac{\pi R^4}{4} \times 2\sqrt{R^2 - y^2}}$$

$$\Rightarrow \text{Shear stress} = \frac{4V}{3}\,\frac{R^2 - y^2}{R^2} \times \frac{1}{\pi R^2}$$

$$\Rightarrow \text{Shear stress} = \frac{4}{3}\left(\frac{V}{\pi R^2}\right)\left(1 - \frac{y^2}{R^2}\right)$$

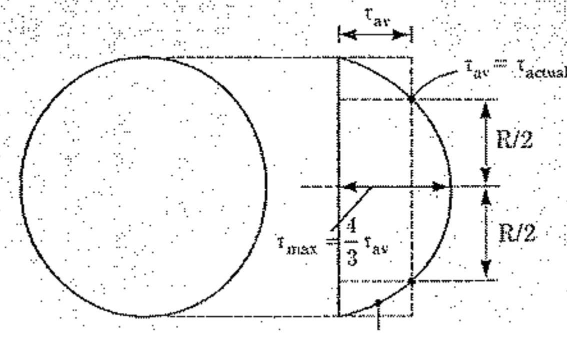
$$\Rightarrow \tau = \frac{4}{3}\left(\frac{V}{A}\right)\left(1 - \frac{y^2}{R^2}\right)$$

$$\Rightarrow \boxed{\tau = \frac{4}{3}\tau_{av}\left(1 - \frac{y^2}{R^2}\right)}$$

(a) Max shear stress occurs at y = 0 i.e. at neutral axis.

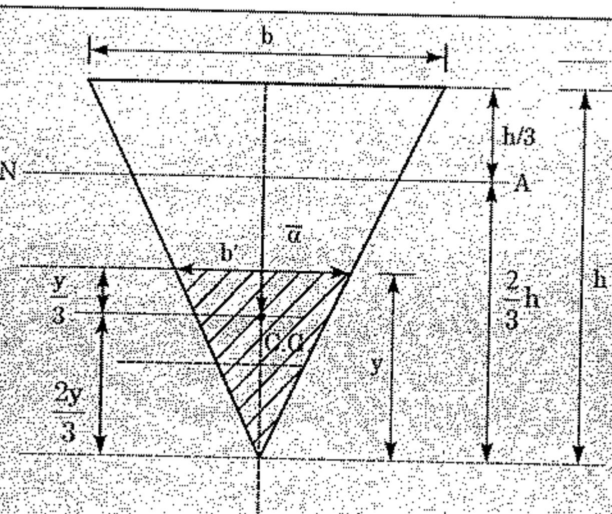
(b) $\boxed{\tau_{max} = \frac{4}{3} \times \tau_{av}}$ i.e. $k = \frac{4}{3}$

Shear Stress Distribution



1. $\tau_{max} = \frac{4}{3}\ \tau_{av.}$
2. τ_{max} occur at y = 0 i.e. at N.A.
3. τ_{av} occur at $y = \frac{R}{2}$ from N.A.

(ii)



$$\text{Shear stress} = \frac{V(A\bar{\alpha})}{Ib'}$$

$$A = \frac{1}{2} y \cdot b' = \frac{b'y}{2}$$

$$\bar{\alpha} = \frac{2}{3}(h) - \frac{2}{3}(y) = \frac{2}{3}(h - y)$$

$$\text{Shear stress} = \frac{V(A\bar{\alpha})}{Ib'} = \frac{V\frac{b'y}{2} \times \frac{2}{3}(h-y)}{I \times b'}$$

$$\boxed{\tau = \frac{Vy(h-y)}{3I}} \Rightarrow \text{Variation is parabolic}$$

For τ to be max $\frac{d\tau}{dy} = 0$

$\Rightarrow$ $h - 2y = 0$ $\Rightarrow$ $y = \frac{h}{2}$

$\Rightarrow$ Shear stress is max at mid depth of triangular section. -- (α)

Variation of Shear Stress

$$\tau = \frac{Vy(h-y)}{3I}$$

at $y = \frac{2}{3}h$ i.e. at N.A.,

$$\tau = \frac{2Vh^2}{27I} = \frac{2}{27}\frac{Vh^2}{\frac{bh^3}{36}} = \frac{72}{27} \times \frac{V}{bh}$$

$$= \frac{8}{3}\frac{V}{bh} = \frac{4}{3}\left(\frac{V}{\frac{bh}{2}}\right)$$

$$\Rightarrow \boxed{\tau_{NA} = \frac{4}{3}\tau_{av}} \text{ ------------------------ } (\beta)$$

$$\tau_{max} = \frac{Vh^2}{12 \times \frac{bh^3}{36}} = 3\frac{V}{bh} = \frac{3}{2}\left[\frac{V}{\left(\frac{bh}{2}\right)}\right]$$

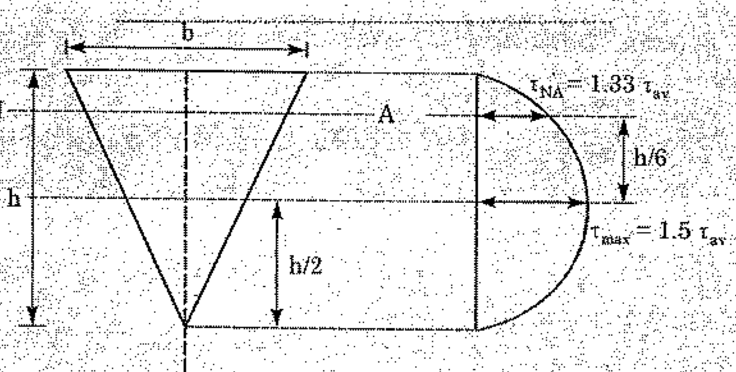
$$\boxed{\tau_{max} = \frac{3}{2}\tau_{av}} \Rightarrow k = \frac{3}{2} \text{ ------------------------ } (\gamma)$$

$$I = \frac{bh^3}{36} \text{ (about N.A.)}$$

$$\Rightarrow \tau_{max} = \frac{V\,y(h-y)}{3I}, \text{ as } \tau_{max} \text{ occurs at } y = \frac{h}{2}$$

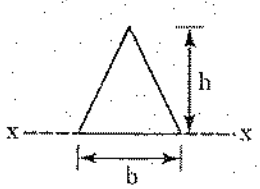
$$\tau_{max} = \frac{V}{3I}\frac{h}{2}\times\left(h - \frac{h}{2}\right)$$

$$\boxed{\tau_{max} = \frac{Vh^2}{12I}} \text{ ------------------------ } (\delta)$$

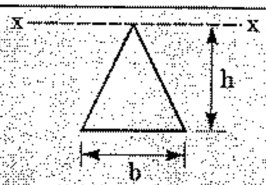


1. $\tau_{max} = 1.5\ \tau_{av.}$
2. $\tau_{N.A.} = \frac{4}{3}\ \tau_{av.}$
3. τ_{max} at $\frac{h}{2}$
4. Distance between N.A. and τ_{max} location = $\frac{h}{6}$

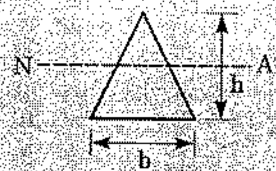
Note:



$$I_{xx} = \frac{bh^3}{12}$$

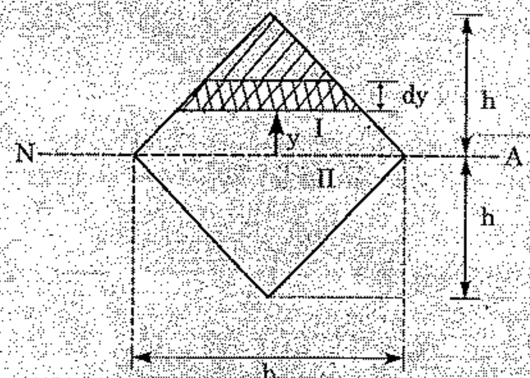


$$I_{xx} = \frac{bh^3}{4}$$



$$I_{NA} = \frac{bh^3}{36}$$

(ii) Let us analyse the 1st half of the section



$$A\bar{y} = \frac{1}{2} \times \left[\frac{b}{h}(h-y)^2\right]\left[y + \frac{1}{3}(h-y)\right]$$

$$= \frac{b}{2h}(h-y)^2\left[\frac{2y+h}{3}\right] = \frac{b(h-y)^2(2y+h)}{6h}$$

$$I = 2\int_0^h \left[\frac{b}{h}(h-y) \times dy\right] y^2$$

$$= \frac{2b}{h}\int_0^h \left(hy^2 - y^3\right) dy = \frac{2b}{h}\left[\frac{hy^3}{3} - \frac{y^4}{4}\right]_0^h$$

$$= \frac{2b}{h}\left[\frac{h^4}{3} - \frac{h^4}{4}\right] = \frac{2bh^4}{12h} = \frac{bh^3}{6}$$

$$I = \frac{bh^3}{6}$$

Shear stress at the level y

$$\tau = \frac{VA\bar{y}}{Ib'} = \frac{V\left(\frac{b(h-y)^2(2y+h)}{6h}\right)}{\left(\frac{bh^3}{6}\right)\left[\frac{b(h-y)}{h}\right]}$$

$$\boxed{\tau = \frac{V}{bh^3}(h-y)(2y+h)}$$

for τ to be max $\frac{d\tau}{dy} = 0$

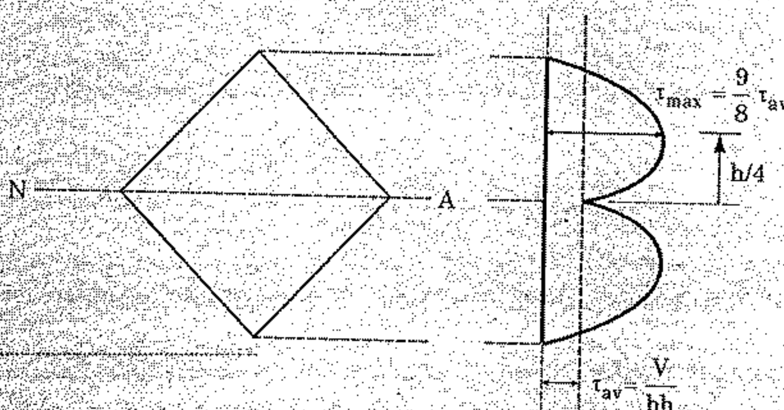
$$\Rightarrow y = \frac{h}{4}$$

$\Rightarrow$ $\boxed{\text{Shear stress is max at } y = \frac{h}{4}}$

$$\tau_{max} = \frac{V}{bh^3}\left(\frac{3h}{4}\right)\left(\frac{3h}{2}\right) = \frac{9}{8}\left(\frac{V}{bh}\right)$$

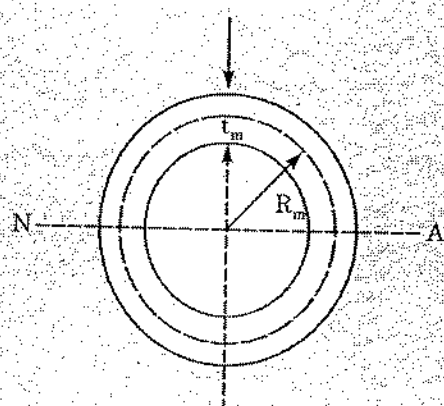
- $\boxed{\tau_{max} = \frac{9}{8}\left(\frac{V}{A}\right) = \frac{9}{8}\tau_{av}} \Rightarrow k = \frac{9}{8}$
- $\boxed{\text{at N.A., } \tau = \frac{V}{bh}}$

Shear Stress Distribution



(iv) For thin walled section

$\boxed{A = 2\pi R_m \times t_m}$



Shear stress will be max at N.A.

$$I = \frac{AR_m^2}{2} = \frac{2\pi t_m R_m^3}{2} = \pi t_m R_m^3 \qquad [\textit{Note: } J = 2I]$$

$\boxed{I = \pi t_m R_m^3}$

$$\bar{y} = \frac{A_1\bar{y}_1 + A_2\bar{y}_2}{A}$$

$$A\bar{y} = \left[\frac{\pi\left(R_m + \frac{t_m}{2}\right)^2}{2} \times \frac{4\left(R_m + \frac{t_m}{2}\right)}{3\pi}\right] - \left[\frac{\pi\left(R_m - \frac{t_m}{2}\right)^2}{2} \times \frac{4}{3\pi}\left(R_m - \frac{t_m}{2}\right)\right]$$

$$A\bar{y} = \frac{2}{3}\left[\left(R_m + \frac{t_m}{2}\right)^3 - \left(R_m - \frac{t_m}{2}\right)^3\right]$$

$$= \frac{2}{3}R_m^3\left[\left(1 + \frac{t_m}{2R_m}\right)^3 - \left(1 - \frac{t_m}{2R_m}\right)^3\right]$$

$$= \frac{2}{3}R_m^3\left[\left(1 + \frac{3t_m}{2R_m} + ...\right) - \left(1 - \frac{3t_m}{2R_m}\right)\right]$$

$$= \frac{2}{3}R_m^3\left[\frac{3t_m}{R_m}\right] = 2t_m R_m^2$$

$$\boxed{A\bar{y} = 2t_m R_m^2}$$

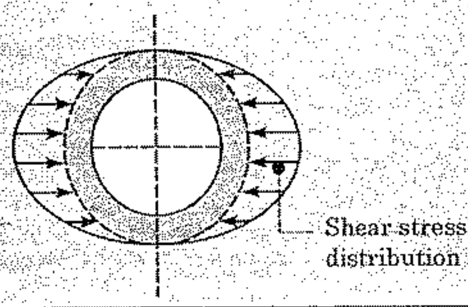
Shear Stress at N.A.

$$\tau_{max} = \frac{VA\bar{y}}{I \times 2t_m} = \frac{V2t_m R_m^2}{\pi t_m R_m^3 \times 2t_m}$$

$$\tau_{max} = \frac{V}{\pi t_m R_m} = \frac{2V}{2\pi t_m R_m} = \frac{2V}{A}$$

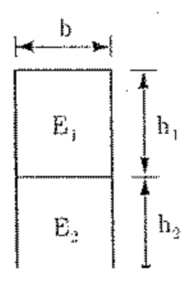
$$\boxed{\tau_{max} = 2\,\tau_{av.}} \Rightarrow K = 2$$

Shear Stress Distribution

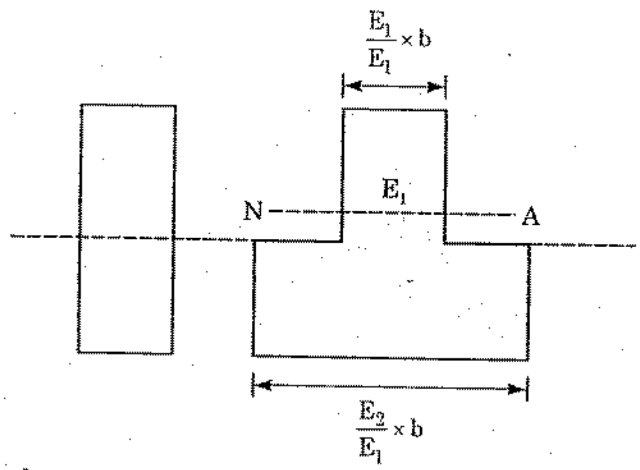


COMPOSITE BEAM

We already know that a beam made of two different material is called composite beam. Steps followed in the calculation of shear stress in composite beam are as follows.



1. Draw the transformed section



2. Find N.A. of transformed section.

3. Find shear force per unit length = shear flow = $\frac{VA\bar{y}}{I} = \alpha$

where

V = Shear force on the section

$A\bar{y}$ = Moment of transformed area above the point at which we have to find shear stress about NA

I = Moment of inertia of transformed section about NA

4. $$\boxed{\text{Shear stress at any point} = \left(\frac{VA\bar{y}}{I}\right) \times \frac{1}{b} = \frac{\alpha}{b}}$$

b = actual width at any section.

Example 3

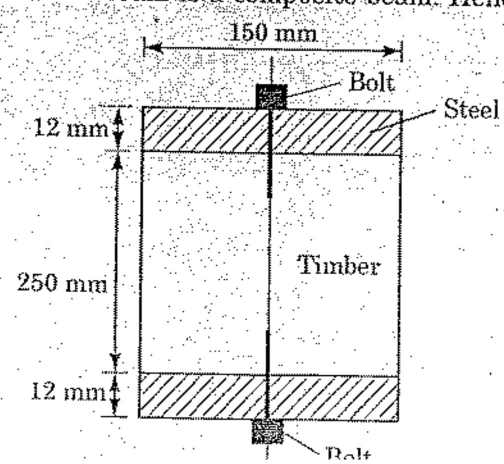
A composite beam is made by attaching the timber and steel portions shown with bolt of 12 mm dia spaced longitudinally every 200 mm.

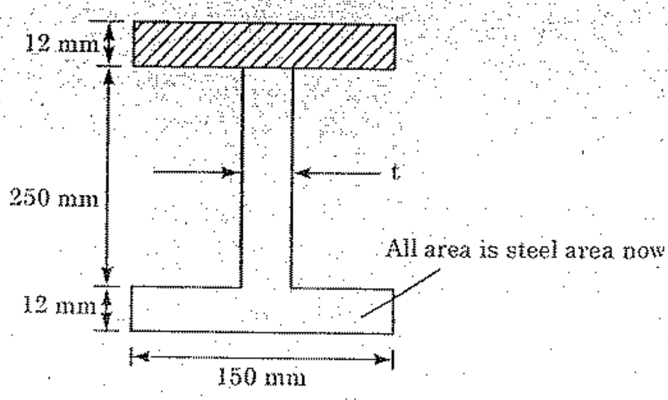
$E_{timber} = 10^4$ N/mm^2

$E_{steel} = 2 \times 10^5$ N/mm^2

Determine the average shear stress in the bolt caused by vertical shearing for of 4 kN.

Sol: The beam is a composite beam. Hence let us transform all area into steel area





$$t = 150 \times \frac{E_t}{E_s} = \frac{150 \times 10^4}{2 \times 10^5} = 7.5 \text{ mm}$$

Moment of inertia of transformed section $= I = \frac{150\,(274)^3}{12} - \frac{(150 - 7.5)(250)^3}{12}$

$$I = \frac{150\,(274)^3}{12} - \frac{(150 - 7.5)(250)^3}{12}$$

$$= 71.589 \times 10^6 \text{ mm}^4$$

We will calculate shear force per unit length at the junction of steel and timber

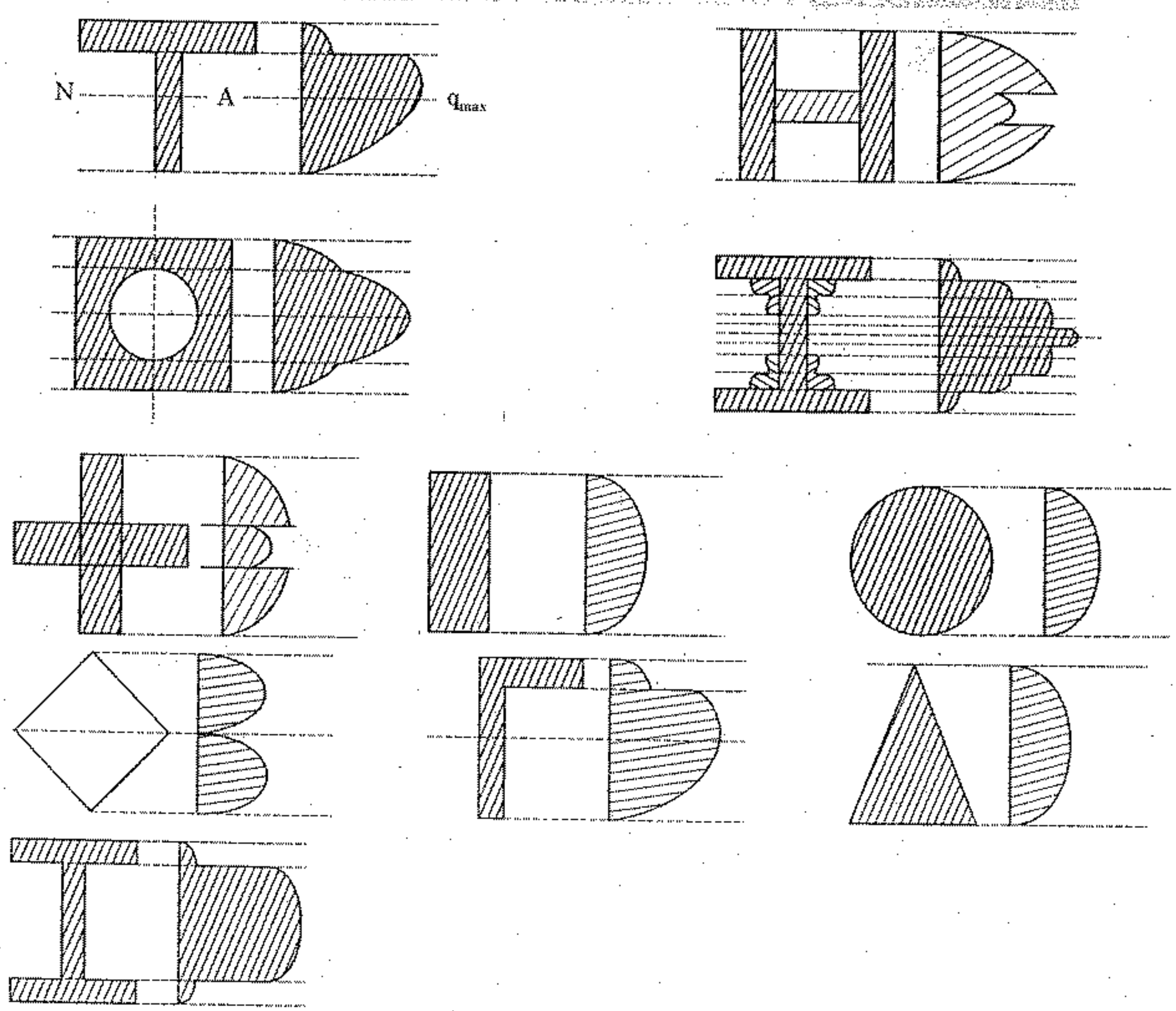
$$\Rightarrow \quad A\bar{y} = (150 \times 12)(125 + 6) = 235.8 \times 10^3 \text{ mm}$$

Shear stress per unit length $= \frac{VA\bar{y}}{I} = \frac{4000 \times 235.8 \times 10^3}{71.589 \times 10^6}$ N/mm = 13.175 N/mm

Shear force is 200 mm length = 13.175 × 200 = 2635.00 N

Shear stress is bolt $= \frac{2655}{\frac{\pi}{4}(12)^2} = 23.31 \text{ N/mm}^2$

TRANSVERSE SHEAR STRESS DISTRIBUTION IN VARIOUS SECTIONS

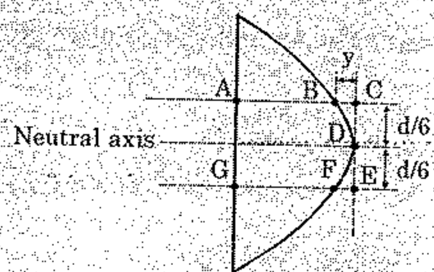


Example 4

A beam of rectangular x-sec is subjected to a vertical shear force 'V' the shear force carried by the upper $\frac{1}{3}$rd of the x-section

(a) Zero (b) $\frac{7V}{27}$ (c) $\frac{8V}{27}$ (d) $\frac{V}{3}$

Sol:



$$x = \frac{VA\bar{y}}{Ib}$$

$$= \frac{V \times \frac{bd}{3} \times \left(\frac{d}{6} + \frac{d}{6}\right)}{\frac{bd^3}{12} \times b}$$

$$= \frac{\frac{Vbd^2}{9}}{\frac{bd^3 \times b}{12}}$$

$$= \frac{12}{9}\frac{V}{bd} = \frac{4}{3}\frac{V}{bd}$$

$$\Rightarrow \quad y = \frac{3}{2}\frac{V}{bd} - \frac{4}{3}\frac{V}{bd}$$

$$y = \frac{1}{6}\frac{V}{bd}$$

Shear Force Carried by Middle 1/3rd of beam (i.e. due to stress distribution ABDFG)

$= \left(\frac{3}{2} \times \frac{V}{bd} \times b \times \frac{d}{3}\right) - \left(\frac{1}{3} \times \frac{1V}{6bd} \times \frac{d}{6}\right) \times 2 \times b$ = Area of rectangular distribution (ACEG) – Area of sprandral distribution (BCDEFD)

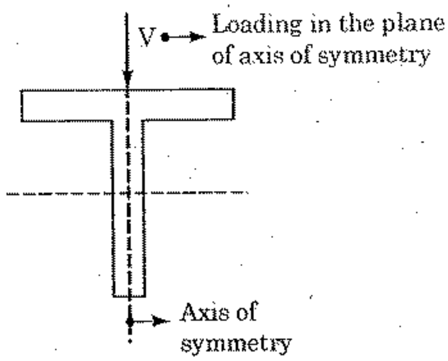
$$= \frac{V}{2} - \frac{V}{54} = \frac{13V}{27}$$

$\Rightarrow$ Shear force carried by upper 1/3rd $= \left(V - \frac{13V}{27}\right) \times \frac{1}{2} = \frac{7V}{27}$

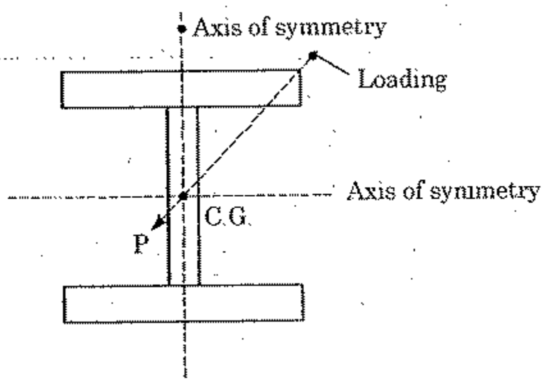
SHEARING STRESS IN THIN WALLED MEMBERS SUBJECTED TO TRANSVERSE SHEAR FORCE

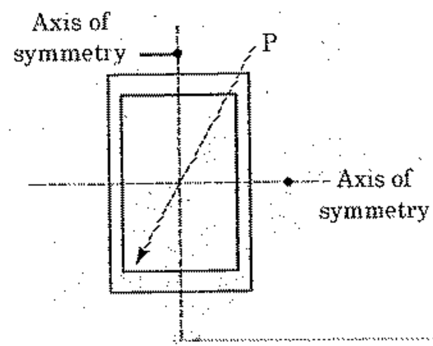
- The formula, shear stress = $\frac{VA\bar{y}}{Ib}$, is applicable to find out resultant shear stress at a point in a x-section only when there is no twisting of the section. Hence we need to find out when will the thin walled sections have no twisting. The following recommendations will help to identify no twisting conditions.

1. When the section is having a plane of symmetry and loading is in that plane, there will be no twisting

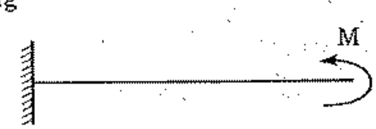


2. When a doubly symmetric section has <u>skew loading</u> passing through the C.G. of the section, it will have no twisting.

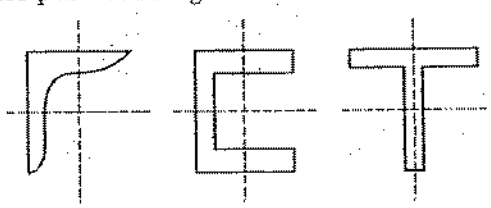




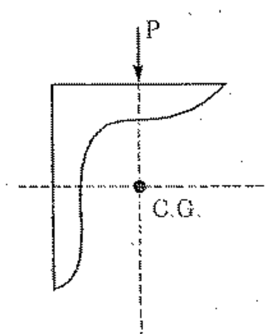
3. A beam of unsymmetrical section or a section having only one plane of symmetry will not twist if the beam is subjected to pure bending



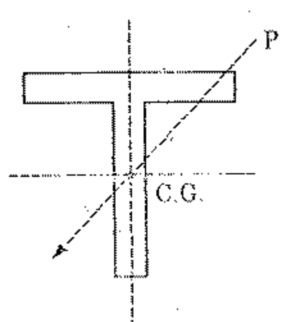
Hence following sections under pure bending will not twist.



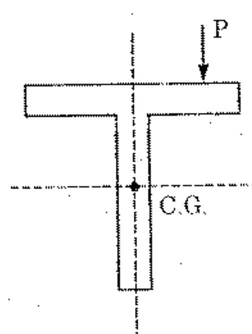
4. If a beam section possess no plane of symmetry or if it possesses a single plane of symmetry and is subjected to a load that is not contained in the plane of symmetry.



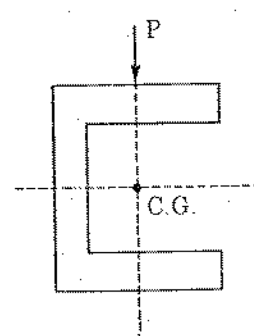
The beam section will twist



The beam section will twist



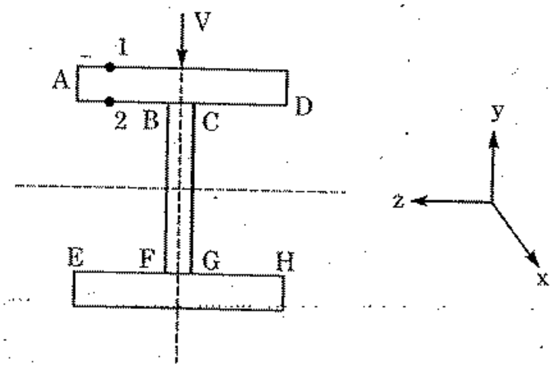
Twisting will occur



Twisting will occur

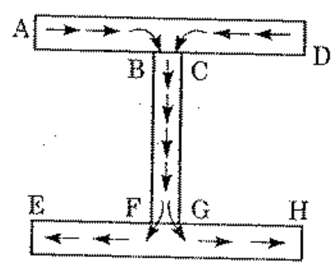
THIN WALLED SECTION WITH LOAD IN THE PLANE OF SYMMETRY [SHEAR FLOW CONCEPT]

- In thin walled section, we use the concept of shear flow. Let us take a doubly symmetric section subjected to a loading as shown in the figure below.



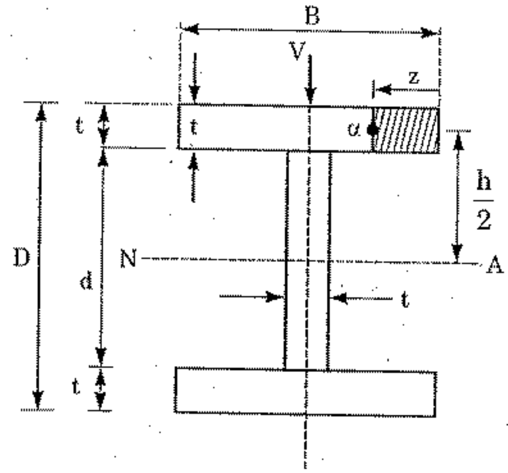
- We have already shown that even in the case of vertical loading (in y-direction) τ_{xz} will exist, although $\bar{\tau}_{xz} = 0$ over the x-section.
- In portion AB and CD, τ_{xz} will be significant but τ_{xy} will be negligible.
- τ_{xy} will be negligible in AB and CD because thickness of ABCD in y-direction is negligible. As point (1) and (2) are on free surface, there will be no shear stress at these points and hence variation in negligible thickness of 1-2 will also be insignificant.
- On similar logic one can say that τ_{xz} in web portion will be negligible. In the web portion, only τ_{xy} will be significant.

  Thus if we show the direction of significant shear we will have a figure as shown below.



- This representation of shear looks like a flow. Hence we say that we have shear flow in this walled sections.
- Shear flow behaves like a flow of liquid in which continuity equation is satisfied. Thus at B and C if shear flow are 'q_0' and 'q_0' respectively from A to B and D to C, then they will add up to give $2q_0$ shear flow in BF at B.
- In thin walled sections, shear flow is always in the direction of tangent to the surface. Thus, in thin circular section direction of shear flow is as shown below.

VARIATION OF SHEAR FLOW



Variation in Horizontal Direction

We know that shear flow is given by

$$q = \frac{VA\bar{y}}{I}$$

where I = moment of inertial of complete section about N.A.

V = Shear force at the section

$A\bar{y}$ = moment of area of shaded portion about N.A.

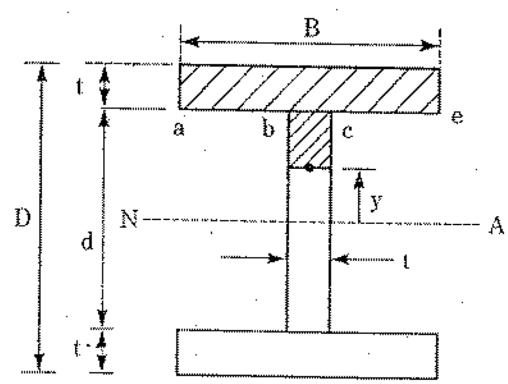
Thus at point α shown in the figure

$$q_f = \frac{VA\bar{y}}{I} = \frac{V(zt)\left(\frac{h}{2}\right)}{I} = \left(\frac{Vht}{2I}\right).z \quad \text{------------------------- (A)}$$

where $\boxed{h = \frac{(D-t)}{2}}$

Thus, shear flow variation in horizontal direction is linear.

Variation in Vertical Direction



As we have already calculated the variation of shear stress in the web of a I-section. We will use the result directly.

Shear flow in web = (Shear stress) x thickness of web

$$\Rightarrow \text{Shear flow in web} = \boxed{q_w = \frac{V(D^2-d^2)}{8I}\times B + \frac{V\left(\frac{d^2}{4}-y^2\right)t}{2I}} \quad \text{------ (B)}$$

Variation of shear flow in vertical direction is parabolic.

⇒ Note that at the junction of flange and web q_0 (i.e. shear flow from one side)

$$= \frac{Vht}{2I}\times\frac{B}{2} = \frac{Vht\,B}{4I} = \frac{V(D-t)\times t_B}{4I}$$

$$= \frac{V\left(D-\frac{D-d}{2}\right)\times t_B}{4I} = \frac{V(D+d)}{8I}\,\frac{(D-d)}{2}\times B$$

$$= \frac{V(D^2-d^2)}{16\,I}B$$

⇒ Shear flow from both side i.e. from A to B and from D to C should add up to give total shear flow at junction of flange and web.

⇒ Total shear flow at junction = $2q_0$ [As obtained from horizontal variation of shear flow) i.e. equation A

$$= \frac{V(D^2-d^2)B}{8I}$$

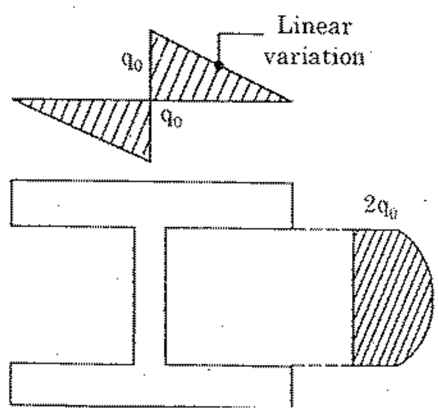
Total shear flow at junction (as obtained from vertical variation of shear flow) is obtained by putting in equation B, $y = \frac{d}{2}$.

$$\Rightarrow q_w = \frac{V(D^2-d^2)}{8I}\times B + \frac{V}{2I}\left(\frac{d^2}{4}-\frac{d^2}{4}\right)\times t$$

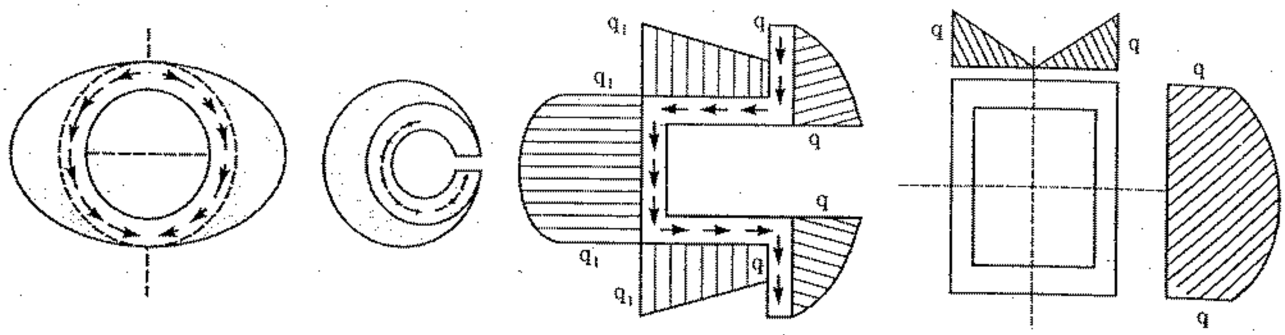
$$q_w = \frac{V(D^2-d^2)}{8I}B$$

Thus $2q_0 = q_w$.

The variation of shear flow can thus be plotted as:



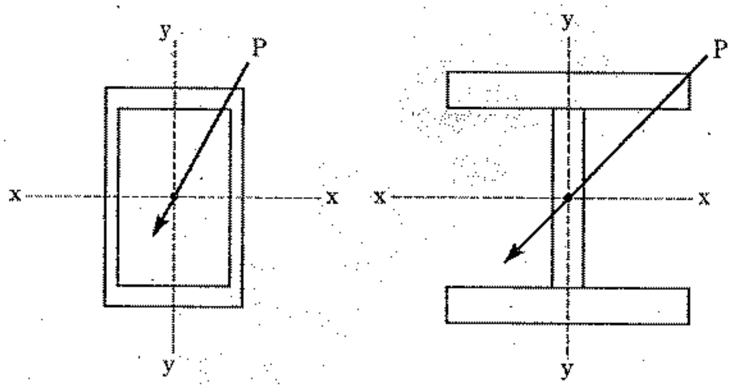
VARIATION OF SHEAR FLOW IN VARIOUS SECTIONS



- Note that variation of shear flow on horizontal leg is linear whereas that in vertical leg is parabolic.
- Once the shear flow variation is obtained, shear stress at any point in the thin walled section can be calculated by dividing shear flow with thickness of the section at the point at which shear stress is to be calculated

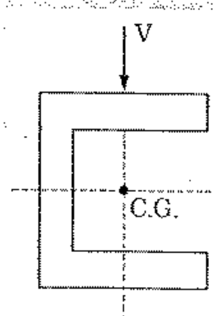
$$\Rightarrow \boxed{\text{Shear stress} = \frac{\text{Shear flow}}{\text{Thickness}}}$$

Shear stress in doubly symmetric thin walled section subjected to skew loading passing through C.G.

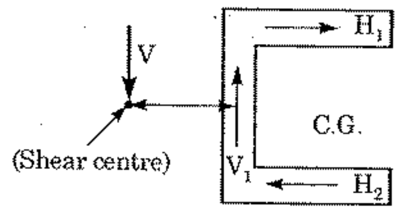


- In this case, load P is resolved in x and y-directions and shear flow due to each component is calculated and algebrically added to get resultant shear flow at any point. From resultant shear flow, shear stress is calculated from the relation, Shear stress = $\frac{\text{Shear flow}}{\text{Thickness}}$

UNSYMMETRIC LOADING OF THIN WALLED MEMBERS: SHEAR CENTRE



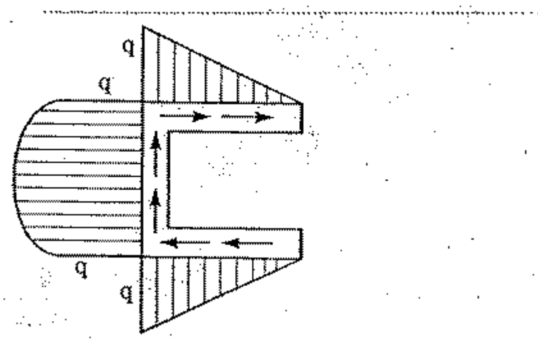
- We know that if a beam section possesses no plane of symmetry or if is possesses a single plane of symmetry and is subjected to a load that is not contained in the plane of symmetry, the beam will twist and bend.
- However, if the loading passes through a point called shear centre, the beam will not twist. It will only bend.
- Thus shear centre is a point through which if transverse bending load passes, the beam will have no twisting. It is also the point through which resultant of shearing force on the section passes.



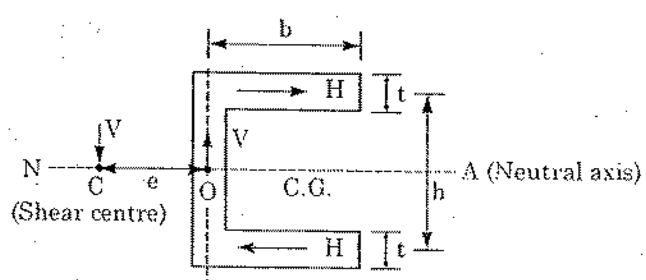
- Thus if V is the resultant of shearing forces (H_1, H_2 and V_1) on the section as shown in the figure below, then moment about any point due to forces V_1, H_1, H_2 and V must be equal to zero. This concept is used to locate shear centre.

Example:

Shear Centre of a Channel Section:



- If the transverse loading is such that twisting does not occur than shear flow formula $q = \frac{VA\bar{y}}{I}$ can be applied and the variation of shear flow will be as shown in the figure above.
- Let the transvers shear force passes through shear centre



Hence net moment about point O = 0

$\Rightarrow$ $Ve = H \times h$

$\Rightarrow$ $\boxed{e = \frac{Hh}{V}}$

$$H = \left(\frac{q_{av}}{t}\right) \times bt = q_{av} b = \frac{1}{2}\left[\frac{V(bt)\frac{h}{2}}{I}\right] \times b$$

$\Rightarrow$ $H = \frac{Vhtb^2}{4I}$

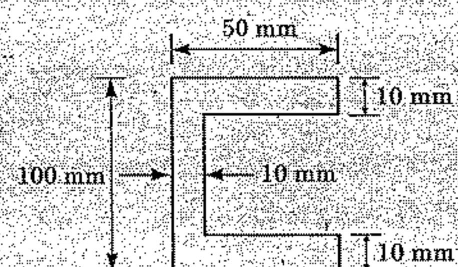
$\Rightarrow$ $e = \frac{Vhtb^2}{4I} \times \frac{h}{V} = \frac{b^2h^2t}{4I}$

$\Rightarrow$ $\boxed{e = \frac{b^2h^2t}{4I}}$ = Location of shear centre

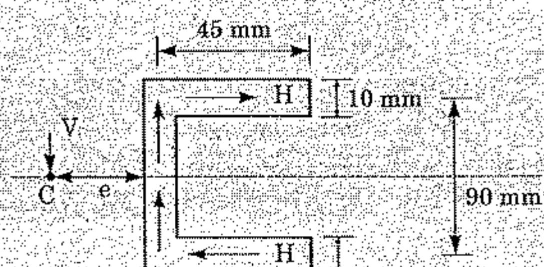
I = moment of inertia of the section about N.A.

Example 5

Draw shear flow diagram and locate shear centre for the channel section as shown below.



Sol:



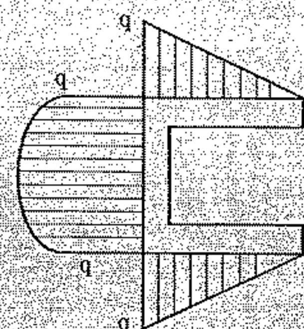
$$e = \frac{b^2h^2t}{4I}$$

$$I = \frac{10(100)^3}{12} + 2\left[\frac{40 \times (10)^3}{12} + 40 \times 10 \times (45)^2\right]$$

$$I = 2.46 \times 10^6 \text{ mm}^4$$

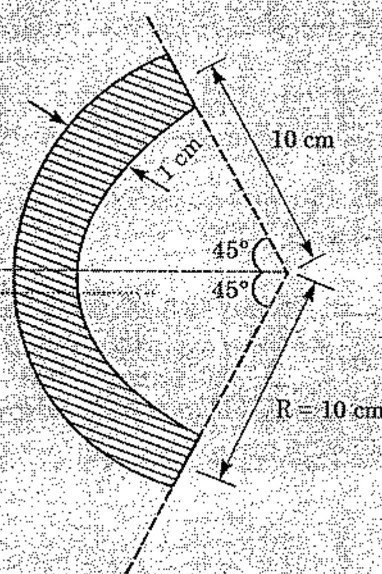
$$e = \frac{(45)^2(90)^2 \times 10}{4 \times 2.46 \times 10^6} = 16.67 \text{ mm}$$

Shear flow diagram is as follows:

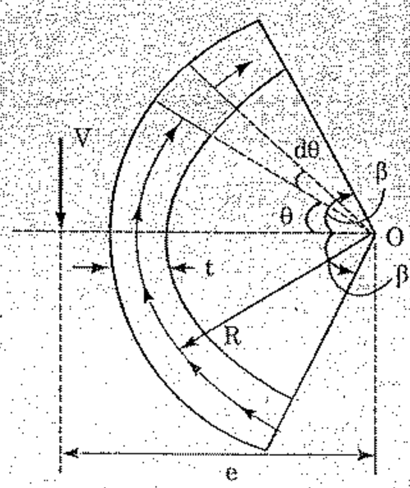


Example 6

Locate shear centre.



Sol:



Shaded area = Rdθ × t

Net moment about O = 0

$$\Rightarrow \int_{-\beta}^{\beta}\left(\frac{q_\theta}{t}\times t\times Rd\theta\right)\times R - Ve = 0, \text{ where } q_\theta = \frac{VA\bar{y}}{I}$$

$$A\bar{y} = \int ydA$$

$$A\bar{y} = \int_{\theta}^{\beta} R\sin\theta\,(Rd\theta)\times t$$

$$A\bar{y} = R^2t\int_{\theta}^{\beta}\sin\theta\, d\theta = -R^2t\,(\cos\beta - \cos\theta)$$

$$\Rightarrow \boxed{A\bar{y} = R^2t\,(\cos\theta - \cos\beta)} \quad \text{------------ (i)}$$

$$I = \int y^2dA = 2\int_0^{\beta} R^2\sin\theta\,(Rd\theta)\,t$$

$$I = 2R^3t\int_0^{\beta}\sin^2\theta.d\theta = R^3t\int_0^{\beta}2\sin^2\theta d\theta$$

$$I = R^3t\int_0^{\beta}\left[1-\cos 2\theta\right]d\theta$$

$$I = R^3t\left[\theta - \frac{\sin 2\theta}{2}\bigg|_0^{\beta}\right]$$

$$I = R^3t\left[\beta - \frac{\sin 2\beta}{2}\right]$$

$$\boxed{I = R^3t\left[\beta - \sin\beta\cos\beta\right]} \quad \text{------------ (ii)}$$

$$\Rightarrow q_\theta = \frac{VA\bar{y}}{I} = \frac{VR^2t\,(\cos\theta - \cos\beta)}{R^3t\,(\beta - \sin\theta\cos\beta)} = \frac{V\,(\cos\theta - \cos\beta)}{R\,(\beta - \sin\theta\cos\theta)}$$

$$Ve = \frac{\int_{-\beta}^{\beta} q_\theta R^2 d\theta}{V}$$

$$Ve = \int_{-\beta}^{\beta}\frac{V\,(\cos\theta - \cos\beta)}{R\,(\beta - \sin\beta\,\cos\beta)}\times\frac{R^2d\theta}{V}$$

$$\left(\frac{R}{\beta - \sin\beta\cos\beta}\right)\left[\left(\sin\theta - \theta\cos\beta\right)\Big|_{-\beta}^{\beta}\right]$$

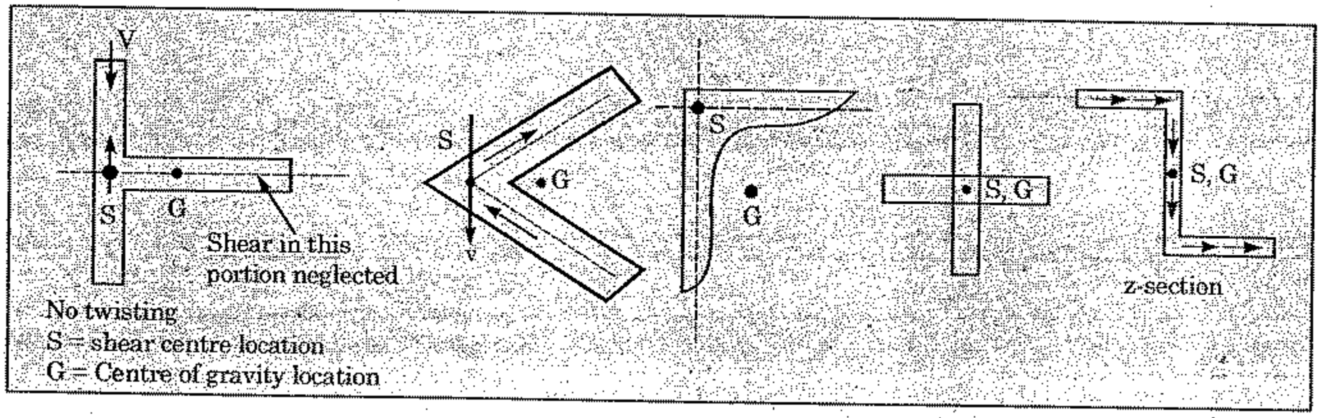
$$\frac{R}{\beta - \sin\beta\cos\beta}\left[\left(\sin\beta - \beta\cos\beta\right) - \left(-\sin\beta + \beta\cos\beta\right)\right]$$

$$\boxed{e = \left(\frac{2R}{\beta - \sin\beta\cos\beta}\right)\left(\sin\beta - \beta\cos\beta\right)} \quad \text{------------ (iii)}$$

Putting $\beta = \frac{\pi}{4}$

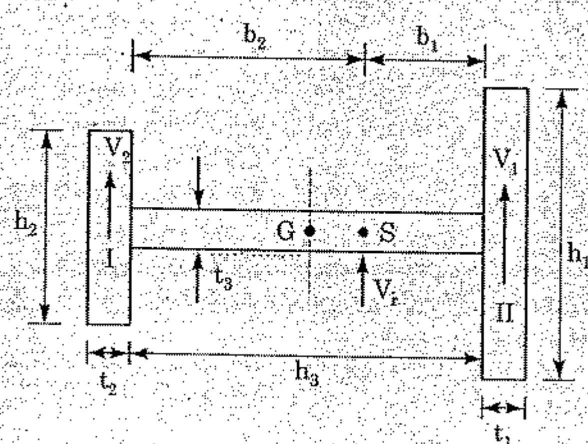
$$e = \frac{2R}{\left(\frac{\pi}{4} - \frac{1}{2}\right)}\left(\frac{1}{\sqrt{2}} - \frac{\pi}{4\sqrt{2}}\right)$$

Shear centre of section consisting of two intersecting narrow rectangles always lies at the intersection of centerlines of the two rectangles.



Example 7

Assuming the bending resistance of web to be negligible, find shear centre for the section shown below. The beam is a cantilever beam.



Sol: Total shear should be resisted only in the two flanges as shown.

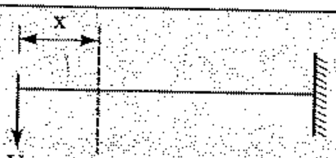
Assuming S = shear center location

$\Rightarrow \quad V_2 b_2 = V_1 b_1$ .... (i)

Assuming that two flanges bend as though they were separate beams that have identical radius of curvature.

$$\Rightarrow \quad \frac{M_1}{I_1} = \frac{M_2}{I_2} = \frac{E}{R} \quad \text{............ (ii)}$$

but $M_1 = V_1 x$, $M_2 = V_2 x$



V_1 $\quad M_1 = V_1 x$

$$\Rightarrow \frac{V_1 x}{I_1} = \frac{V_2 x}{I_2}$$

$$\Rightarrow \frac{V_1}{I_1} = \frac{V_2}{I_2} \quad \text{......(iii)}$$

But moment about S = 0

$$\Rightarrow V_2 b_2 = V_1 b_1$$

$$\Rightarrow \frac{I_1}{I_2} = \frac{b_2}{b_1} \quad \text{......(iv)}$$

also, $b_1 + b_2 = h_3$ ......(v)

from (iv) and (v) b_1 and b_2 can be calculated.

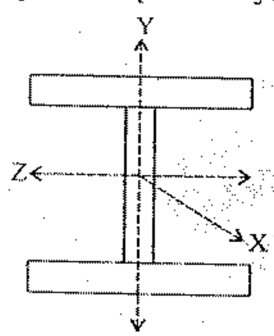
Note: *Shear centre lies between centroid of section and the centroid of the flange having larger moment of inertia.*

OBJECTIVE QUESTIONS

1. Transverse shear applied to a beam normally results in
   (a) Normal stress only
   (b) Shear stress only
   (c) Normal and shear stress
   (d) Axial stress only

2. The formula $\frac{VA\bar{Y}}{Ib}$ = shear stress (symbols have usual meaning) theoritically applies to
   (a) Prismatic beams
   (b) When plane of loading is in the plane of symmetry of the section
   (c) Even when there is torsion in the section
   (d) In a section subjected to pure bending only.
   Of these statements which of the following is correct :
   (a) 1 and 4 only (b) 1, 2 and 3
   (c) 1, 2 and 4 (d) 1 and 2 only

3. When transverse shear is acting along vertical plane of symmetry, which of the following are correct



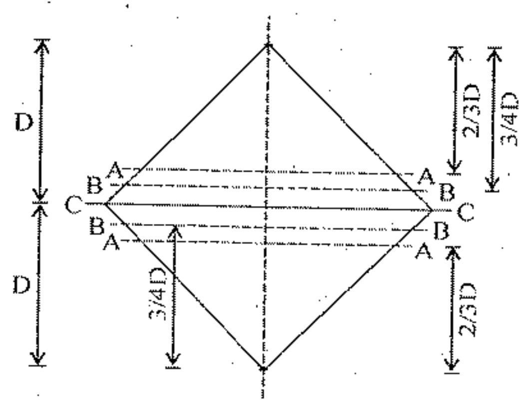
   (1) $\Sigma F_x = 0$
   (2) $\Sigma M_y = 0$
   (3) $\tau_{xz} = 0$ at all points in the X-section
   (4) $\tau_{xy} = 0$ at all points in the X section
   (a) 1 and 2 (b) 1, 2 and 3
   (c) 1, 2 and 4 (d) 1, 2, 3 and 4

4. A beam of square cross-section (B × B) is used as a beam with one diagonal horizontal. The location of the maximum shear stress from the neutral axis will be at distance of
   (a) zero (b) $\frac{B}{4}$
   (c) $\frac{B}{4\sqrt{2}}$ (d) $\frac{B}{8}$

5. In a beam of solid circular cross-section, what is the ratio of the maximum shear stress to the average shear stress?
   (a) $\frac{3}{4}$ (b) $\frac{4}{3}$

(c) $\frac{3}{2}$ (d) $\frac{2}{3}$

6. A beam of square cross-section is placed horizontally with one diagonal horizontal as shown in the figure below. It is subjected to a vertical shear force acting along the depth of the cross section. Maximum shear stress across the depth of cross section occurs at a depth 'x' from the top. What is the value of x?



(a) x = 0 (b) x = (2/3)D
(c) x = (3/4)D (d) x = D

7. The shear stress at the neutral axis in a beam of triangular section with a base of 40 mm and height 20 mm, subjected to a shear force of 3 kN is
(a) 3 MPa (b) 6 MPa
(c) 10 MPa (d) 20 MPa

8. A timber beam is simply supported at the ends and carries a concentrated load at mid-span. The maximum longitudinal stress 'f' is 12 N/mm^2 and the maximum shear stress 'q' is 1.2 N/mm^2. The ratio of span to depth would be
(a) 10 (b) 6
(c) 5 (d) 4

9. A rectangular beam 10 cm wide, is subjected to a maximum shear force of 50000 N, the corresponding maximum shear stress being 3 N/mm^2. The depth of the beam is
(a) 25 cm (b) 22 cm
(c) 16.67 cm (d) 30 cm

10. A beam of rectangular section 100 mm × 300 mm carries certain loads such that the bending moment at a section A is M and at another section B it is (M + C). The distance between the sections A and B is 0.5 m and there are no external loads acting between the two sections. If the value of C is 10,000 N-m, then the maximum shear stress is
(a) 1.5 MN/m^2 (b) 1.0 MN/m^2
(c) 0.5 MN/m^2 (d) 0.25 MN/m^2

11. **Assertion (A):** A beam of circular section is stronger in shear compared to one of rectangular section having same cross-section area.

Reason (R): For equal cross-sectional areas and shearing force, the maximum intensity of shear stress in the section of a beam of circular cross-section is smaller than that for the rectangular section.

Of these statements

(a) both A and R are true and R is the correct explanation of A
(b) both A and R are true but R is not a correct explanation of A
(c) A is true but R is false
(d) A is false but R is true

12. Consider the following statements :
1. In a beam, the maximum shear stress occurs at the neutral axis of the beam cross-section.
2. The maximum shear stress in a beam of circular cross-section is 50% more than the average shear stress.
3. The maximum shear stress in a beam of triangular cross-section, with its vertex upwards occurs at b/6 above the neutral axis.

Which of these statements are correct ?
(a) 1, 2 and 3 (b) 2 and 3 only
(c) 1 and 2 only (d) 1 and 3 only

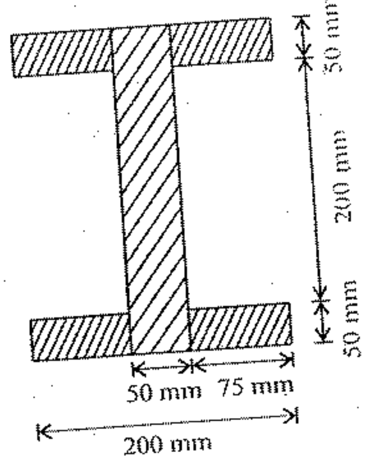
13. A timber beam is 100 mm wide and 150 mm deep. The beam is simply supported and carries a central concentrated load W. If the maximum stress in shear is 2 N/mm^2, what would be the corresponding load W on the beam ?
(a) 20 kN (b) 30 kN
(c) 40 kN (d) 25 kN

14. If a circular shaft is subjected to a torque T and a bending moment M, the ratio of the maximum shear stress to the maximum bending stress is given by
(a) $\frac{2M}{T}$ (b) $\frac{T}{2M}$
(c) $\frac{2T}{M}$ (d) $\frac{M}{2T}$

15. Consider the following statements:
Assertion (A): A rectangular element is subjected to pure shear. This will result in cracks along one diagonal and crushing along the other diagonal.
Reason (R): Pure shear on a rectangular element results in tension along one diagonal and compression along the other diagonal.
Of these statements
(a) both A and R are true and R is the correct explanation of A
(b) both A and R are true but R is not a correct explanation of A
(c) A is true but R is false
(d) A is false but R is true

16. If a beam of rectangular cross-section is subjected to a vertical shear force V, the shear force carried by the upper one-third of the cross-section is
(a) zero (b) $\frac{7V}{27}$
(c) $\frac{8V}{27}$ (d) $\frac{V}{3}$

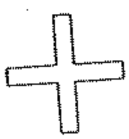
17. I-section of a beam is formed by gluing wooden planks as shown in the figure below. If this beam transmits a constant vertical shear force of 3000 N, the glue at any of the four joints will be subjected to a shear force (in kN per meter length) of

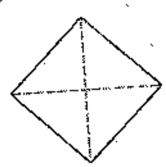


(a) 3.0
(b) 4.0
(c) 8.0
(d) 10.7

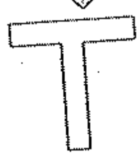
18. The shear stress distribution shown in figure-I represents a beam with cross-section


Figure-I

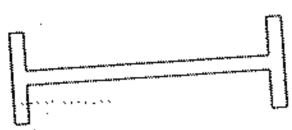
(a)


(b)


(c)

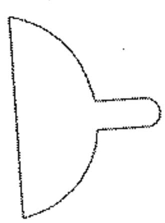

(d)


19. A simply supported I beam with its web horizontal is shown in the given figure. It is subjected to a vertical load.



The shape of the shear stress distribution in the cross-section of the beam under the load would be

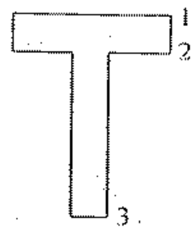
(a)

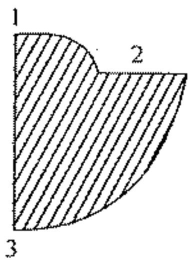

(b)


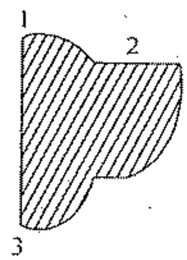
(c)

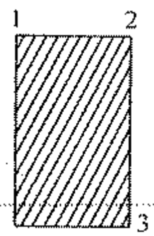

(d)

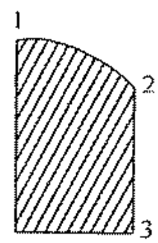

20. Which one of the following diagrams indicates the shear stress distribution for the beam as shown in the figure below?



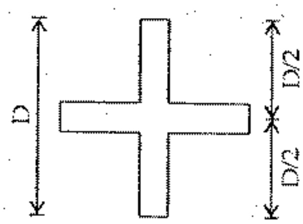
(a)


(b)


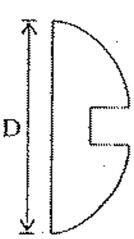
(c)


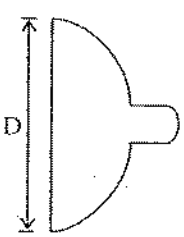
(d)


21. The cross-section of a beam in bending is as shown in the figure below. It is subjected to a shear force acting in the plane of cross-section.

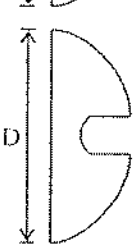


Which among the following figure shows the correct shear stress distribution across the depth of the cross-section of the beam?

(a) D


(b) D


(c) D

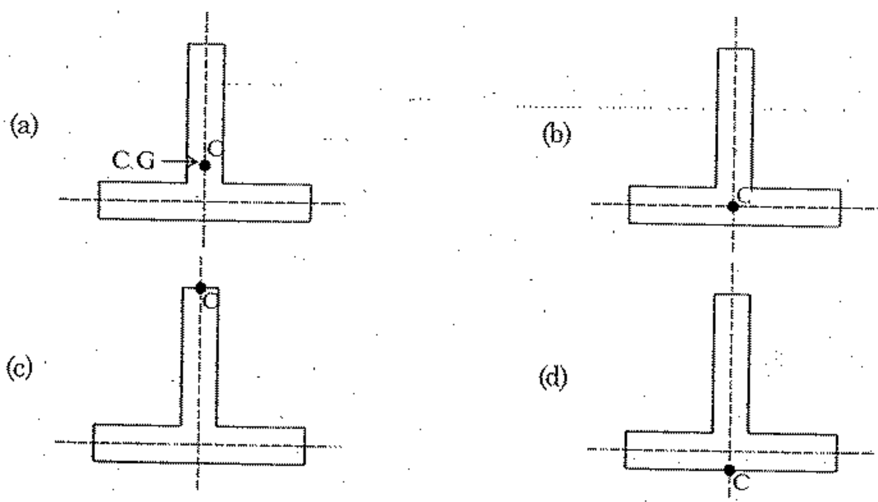

(d) D


22. Consider the following statements:
    1. If a beam has two axes of symmetry even then shear centre does not coincide with the centroid.
    2. For a section having one axis of symmetry, the shear centre does not coincide with the centroid but lies on the axis of symmetry.
    3. If a load passes through the shear centre, then there will be only bending in the cross-section and no twisting.

    Which of these statements are correct?

    (a) 1, 2 and 3 (b) 1 and 2
    (c) 2 and 3 (d) 1 and 3

23. In a thin-wall T-section, the shear centre C is located at the point shown in



24. Match **List-I** with **List-II** and select the correct answer using the codes given below the lists:

List-I	**List-II**
A. Shear centre	**1.** Tension
B. Principal plane	**2.** Slope
C. Fixed end	**3.** Shear stress
D. Middle third rule	**4.** Twisting

Codes:

	A	B	C	D
(a)	4	3	2	1
(b)	3	1	4	2
(c)	4	1	2	3
(d)	4	2	3	1

25. Consider the following statements :

    A beam of channel cross section with vertical web loaded with a concentrated load at mid-span in a plane perpendicular to the plane of symmetry passing through the centroid is subjected to
    1. bending moment
    2. twisting moment
    3. shear force
    4. axial thrust

    Which of these statements are correct?

    (a) 2, 3 and 4 (b) 1, 2 and 3
    (c) 1 and 2 (d) 1 and 3

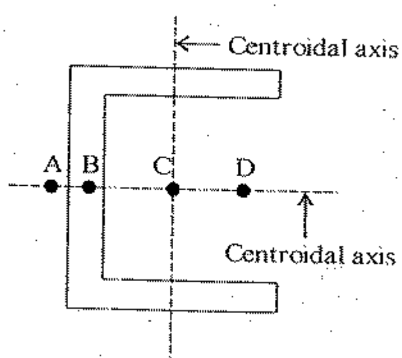
26. The shear centre of a section is defined as the point
   (a) through which the load must be applied to produce zero twisting moment on the section
   (b) at which the shear force is zero
   (c) at which the shear force is maximum
   (d) at which the shear force is minimum

27. Given that for a channel section, the width of flange = b, the depth of the web between centres of flanges = h, the thickness of flange = t, the moment of inertia of the channel about the axis of bending = I, the distance of the shear centre outside the channel section from the mid-thickness of the web is

   (a) $\frac{th^2 b^2}{I}$ (b) $\frac{t^2 h^2 b}{4I}$

   (c) $\frac{t h^2 b^2}{4I}$ (d) $\frac{t^2 h b^2}{4I}$

28. In the symmetrical channel section shown in the figure below, which point is likely to be the shear centre?



   (a) A (b) B
   (c) C (d) D

29. Match **List-I** with **List-II** and select the correct answer using the codes given below the lists:

List-I
- A. Impulse
- B. Torsion
- C. Plane of loading
- D. Instantaneous centre of rotation

List-II
1. Shear centre
2. Plane motion
3. Modulus of rigidity
4. Time effect of a force

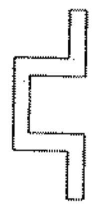
Codes:

	A	B	C	D
(a)	4	3	2	1
(b)	4	3	1	2
(c)	3	4	1	2
(d)	2	4	1	3

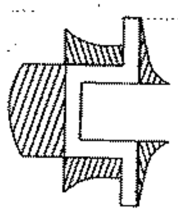
30. If an applied load passes through the shear centre of the section of the beam, then the beam will have

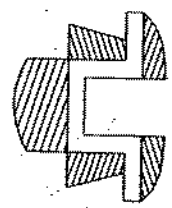
(b) neither bending nor twisting
(c) no bending
(d) no twisting

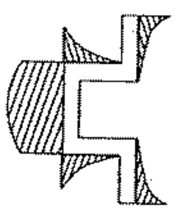
31. If $\frac{VA\bar{Y}}{I}$ = shear flow, the shear flow diagram for a section shown below, which is loaded vertically such that no twisting occurs

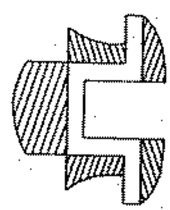


is given by

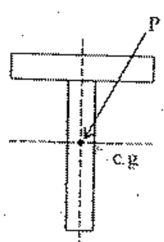
(a) 

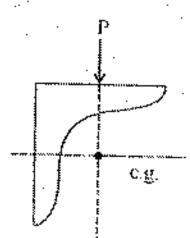
(b) 

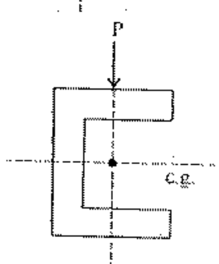
(c) 

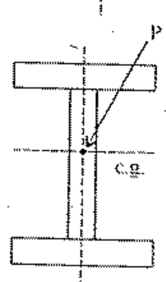
(d) 

32. Which of the following loading condition will not lead to twisting of the section

1. 

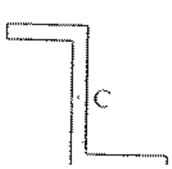
2. 

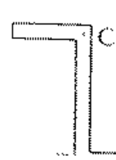
3. 

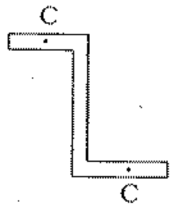
4. 

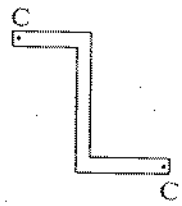
(a) 1 and 4 (b) 2 and 3
(c) 2, 3 and 4 (d) 4 only

33. Location of shear centre in the following diagram is

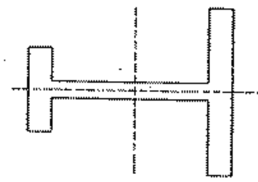
(a) 

(b) 

(c) 

(d) 

34. Location of shear centre in the following diagram is



(a) At the centroid of the section

(b) Between centroid of the section and the centroid of flange having larger moment of the inertia

(c) Between centroid of the section and the centroid of flange having smaller moment of inertia

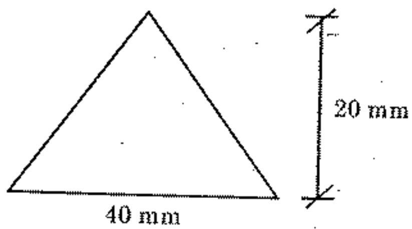
(d) Midway between the two flanges in the web

ANSWERS

1. (c)	10. (b)	19. (d)	28. (a)
2. (d)	11. (a)	20. (a)	29. (b)
3. (a)	12. (d)	21. (c)	30. (d)
4. (c)	13. (c)	22. (c)	31. (b)
5. (b)	14. (b)	23. (b)	32. (d)
6. (c)	15. (a)	24. (a)	33. (a)
7. (c)	16. (b)	25. (b)	34. (b)
8. (c)	17. (b)	26. (a)	
9. (a)	18. (a)	27. (c)	

SOLUTION...

7. (c)

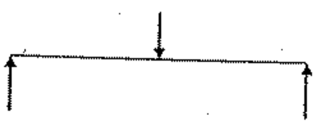


$$\tau_{NA} = \frac{4}{3}\tau_{mean}$$

$$\tau_{mean} = \frac{V}{\left(\frac{bh}{2}\right)} = \frac{3\times10^3}{\frac{40\times20}{2}}$$

$$\tau_{NA} = 10\,\text{MPa}$$

8. (c)



$$M = \frac{WL}{4}\ ; \quad SF = \frac{W}{2}$$

$$f_{max} = 12\ \text{MPa} \quad ; \frac{l}{d} = ?$$

$$q_{max} = 1.2\ \text{Mpa}$$

$$\frac{f_{max}}{q_{max}} = \frac{\frac{Wl/4}{bd^2/6}}{\frac{3}{2}\times\frac{W}{2bd}} = \frac{12}{1.2} = 10$$

$$\frac{\frac{L}{(d/6)}}{3} = 10$$

$$\frac{L}{d} = 5$$

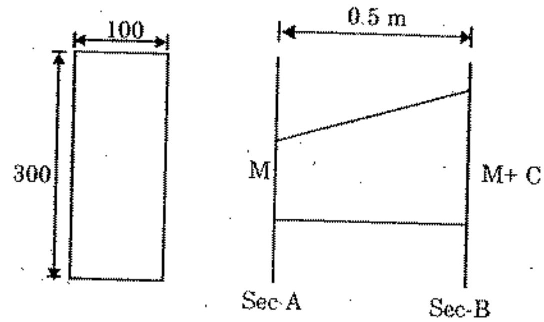
9. (a)

$$\tau_{max} = \frac{3}{2}\left(\frac{V}{bd}\right)$$

$$3 = \frac{3}{2}\times\frac{5\times10^4}{100\times d}$$

$$d = 250\text{ mm} = 25\text{ cm}$$

10. (b)



BMD

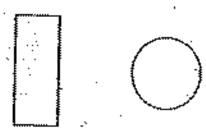
$$\frac{(M+C)-M}{0.5} = SF = 2C$$

$$\tau_{max} = \frac{3}{2}\left(\frac{2C}{bd}\right) = \frac{3\times10^4}{0.1\times0.3}$$

$$= 10^6\ \text{N/m}^2$$

$$= 1\ \text{MN/m}^2$$

11. (a)

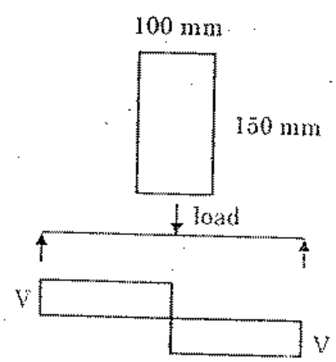


$$\tau_{max} = \frac{3}{2}\left(\frac{V}{\text{area}}\right) = 1.5$$

$$\tau_{max} = \frac{4}{3}\left(\frac{V}{\text{Area}}\right) = 1.33$$

12. (d) Statement 1 is valid for cross sections having symmetry about horizontal axis.

13. (c)



$$\tau = \frac{3}{2}\left(\frac{V}{\ }\right)$$

$$2 = \frac{3}{2}\times\frac{V}{100\times150}$$

max SF, V = 20 kN

$\therefore$ load W = 20 × 2 = 40 kN

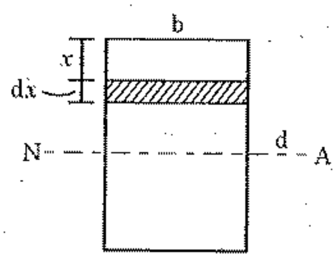
14. (b) Max. Bending Stress,

$$f = \frac{M\times d/2}{\left(\frac{\pi d^4}{64}\right)} = \frac{32M}{\pi d^3}$$

Max. Shear Stress, $\tau = \frac{T\times d/2}{\left(\frac{\pi d^4}{32}\right)} = \frac{16T}{\pi d^3}$

$$\therefore \quad \frac{\tau}{f} = \frac{T}{2M}$$

16. (b)



Shear stress at depth x

$$\tau_x = \frac{VQ}{Ib}$$

$$\tau_x = \frac{Vbx(d/2 - x/2)}{\frac{bd^3}{12}\times b}$$

$$\tau_x = \frac{12Vx(d/2 - x/2)}{bd^3}$$

$$\tau_x = \frac{6Vx(d-x)}{bd^3}$$

Shear force resisted by dx part

$$F_x = \tau_x \cdot b \cdot dx$$

$$F_x = \frac{6Vx(d-x)}{bd^3}\times b\times dx$$

$$F_x = \frac{6V}{d^3}x(d-x)\,dx$$

Shear force resisted by upper one third part

$$F = \int_0^{d/3}\frac{6V}{d^3}(xd - x^2)\,dx$$

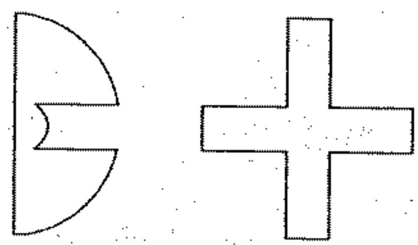
$$F = \frac{6V}{d^3}\left\{\frac{x^2 d}{2} - \frac{x^3}{3}\right\}_0^{d/3}$$

$$F = \frac{6V}{d^3}\left\{\frac{d^3}{18} - \frac{d^3}{81}\right\}$$

$$F = \frac{6V}{d^3}\times\frac{7d^3}{162}$$

$$\boxed{F = \frac{7V}{27}}$$

18. (a)

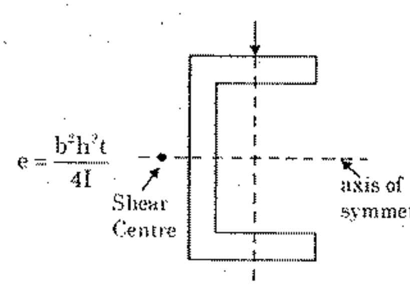


Hint :

1. Shear stress decreases, if the width resisting it increases.
2. Profile of stress distribution $\propto \frac{I}{\text{area}}$ profile of c/s area .
   (Only variation is to be looked at)

22. (c) There are properties/ rules related to shear centre

25. (b)

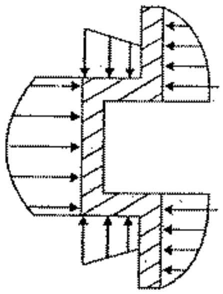


For channel, shear centre lies away from face of channel.

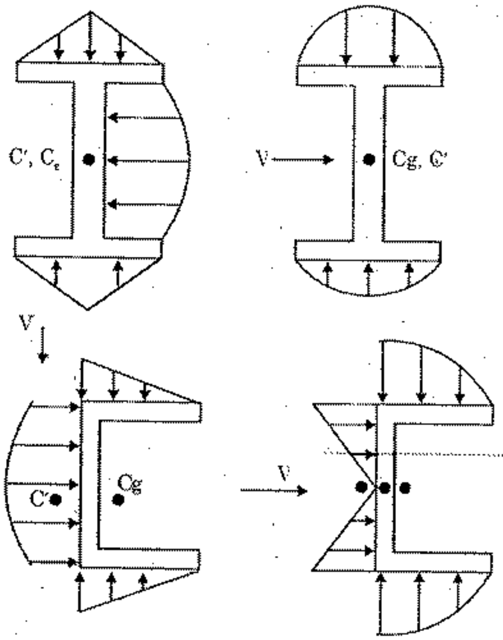
$\therefore$ Shear, BM & Twisting moment occurs.

29. (b) Impulse → Large force applied in short time ⇒ time effect of force.

31. (b) Shear Flow diagram



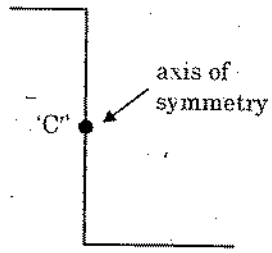
Vertical and Horizontal Loadings



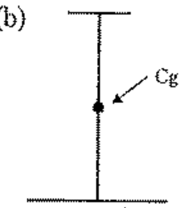
shear flow diagram
(refer internet)

32. (d) Locate shear centre 'C' & for no twisting 'C' ' and Loading should coincide.

33. (a)



34. (b)



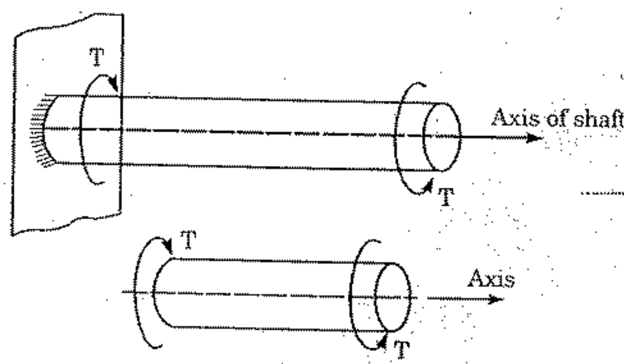
'C'' lies between CG and larger Flang's CG portion.

8

Torsion of Circular Shaft

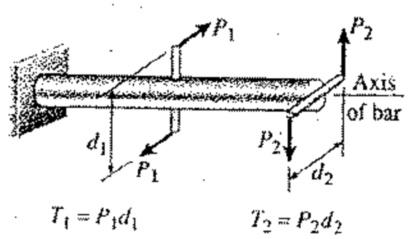
INTRODUCTION

Torsion refers to the twisting of a straight bar when it is loaded by torques that tend to produce rotation about the longitudinal axis of bar.



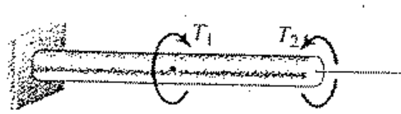
PURE TORSION

- A member is said to be in pure torsion when its cross sections are subjected to only torsional moments and not accompanied by axial forces or bending moment.



(a)

(b)



- Torsion produces shearing stress in the section. This can be shown by following figure.

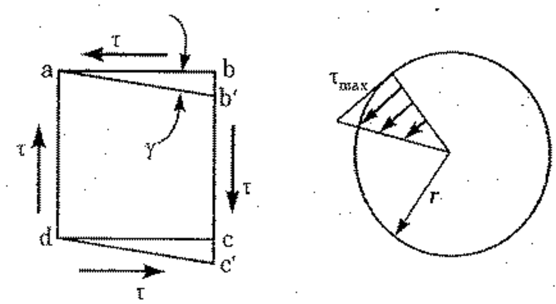
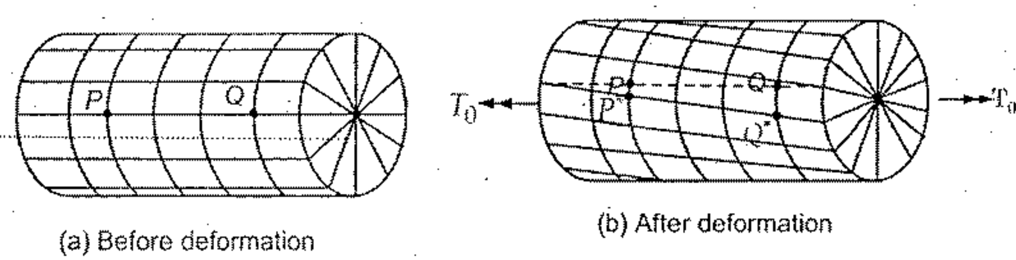


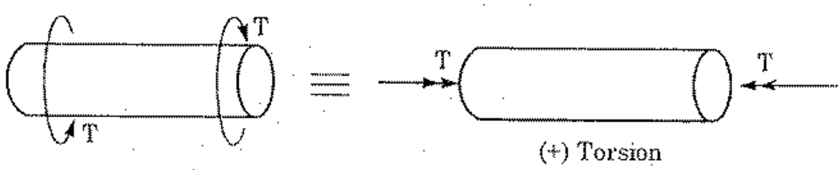
Fig. Shear stresses in a circular bar in torsion.

DEFORMATION UNDER PURE TORSION

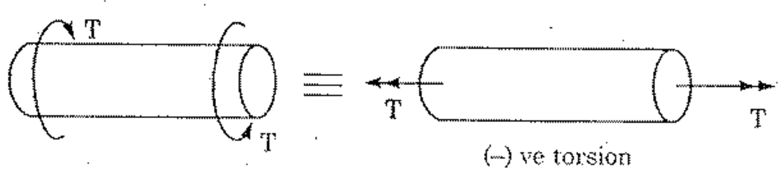


SIGN CONVENTION

- Direction of *Torsion Vector* is the direction obtained by righthand thumb rule.



- When the direction of torsion vector points towards the section it is taken as (+) ve. Similarly

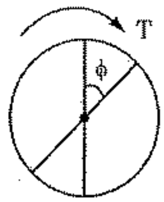


- When the direction of torsion vector points away from the section it is taken as (–) ve.

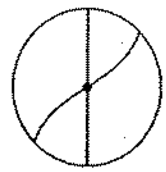
BASIC ASSUMPTION IN DERIVING TORSION FORMULA

1. Circular section remains circular.
2. Plane Section remains plane and do not warp.

Note: Non-circular section generally warps on twisting.



Straight line through centre remains straight after twisting

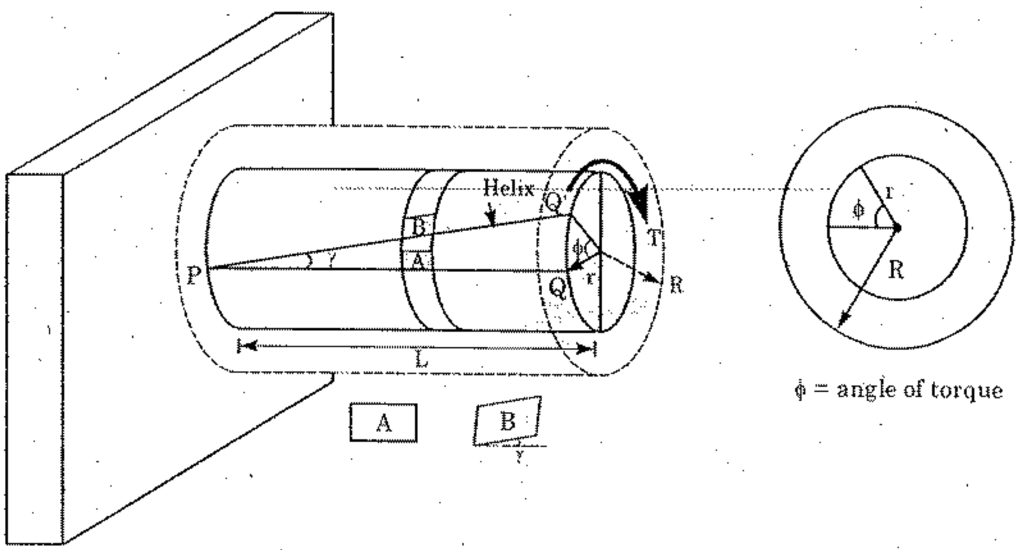


Warping

ϕ = angle of twist.

3. Shaft is loaded by twisting couples in planes that are $\perp^r$ to the axis of shaft.
4. Stresses do not exceed the proportional limit.

TORSION FORMULA



ϕ = angle of torque

From the figure it is clear that the element A before torsion will take a shape as shown in by element B after torsion. As torsion produces shearing strain, shearing strain as shown in element B is γ. Hence

$$\gamma L = r\phi$$

$$\Rightarrow \gamma = r\left(\frac{\phi}{L}\right)$$

⇒ Strain variation is linear over the x-section.

But τ = shear stress = $G\gamma = \frac{G\phi}{L}r$ [From Hooke's law]

⇒
- Shear stress variation is linear over the x-section.
- This linear variation of stress is a consequence of Hooke's law.
- If stress-strain variation is non linear, the stresses will vary non-linearly.

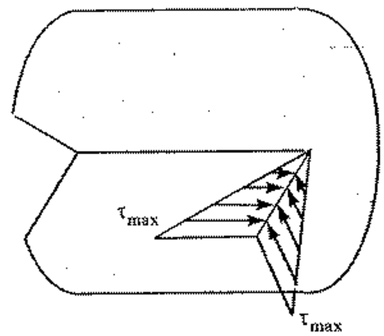
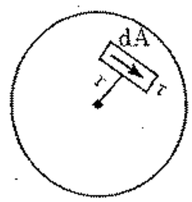


Fig. Longitudinal and transverse shear stresses in a circular bar subjected to torsion.

$$\Rightarrow \quad \boxed{\frac{\tau}{r} = \frac{G\phi}{L}} \quad \text{-------- (i)}$$

Also



$$\int_A (\tau \cdot dA) r = T = \text{Torque}$$

$$\Rightarrow \quad \int_A \left(\frac{G\phi}{L} r\right) r dA = \frac{G\phi}{L} \int_A r^2 dA = T$$

$$\Rightarrow \quad \boxed{T = \frac{G\phi}{L} J}$$

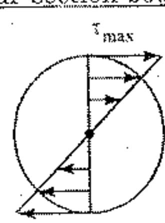
where J is the polar moment of inertia of the section.

$$\Rightarrow \quad \boxed{\frac{T}{J} = \frac{G\phi}{L}}$$

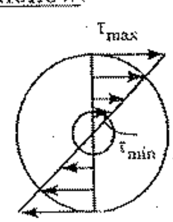
Thus complete torsion formula is

$$\boxed{\frac{T}{J} = \frac{\tau}{r} = \frac{G\phi}{L}}$$

- This formula holds for circular section both solid and hollow.



Variation of torsional shear stress (Solid Section)

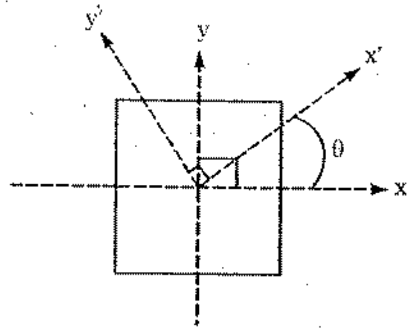


Variation of torsional shear stress (Hollow Section)

Note:

- When section passes perpendicular to the axis of shaft, and an stress element is selected on this section, then only pure shear will act on the element.

- By using stress transformation equation, principal stress can be found out.



- $$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$
  $$\tau_{x'y'} = \frac{-(\sigma_x - \sigma_y)}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$
  [Stress transformation equation]

- At 45° inclined plane i.e. for $\theta = 45°$
  $\sigma_{x'} = 0 + 0 + \tau_{xy}\sin 90° = \tau_{xy}$
  $\sigma_{y'} = 0 + 0 + \tau_{xy}\sin 2(\theta + 90°) = \tau_{xy}\sin(180 + 2\theta) = -\tau_{xy}$
  $\tau_{x'y'} = 0 + \tau_{xy}\cos 90° = 0$
  $\Rightarrow$ Max principal stress $(\sigma_1) = \tau_{xy}$ (tensile) and min principal stress $(\sigma_2) = -\tau_{xy}$ (compressive)
- Therefore, a rectangular element with sides at 45° to the axis of the shaft will be subjected to tensile and compressive stresses.
- If a torsion bar is made of a material that is weaker in tension than in shear, failure will occur in tension along a helix inclined at 45° to the axis. This is the condition for brittle material.

However, for ductile material, failure is due to shear.
- Thus, Ductile material under torsion will fail on section subjected to largest shear stress i.e. on section $\perp^r$ to the axis of shaft.

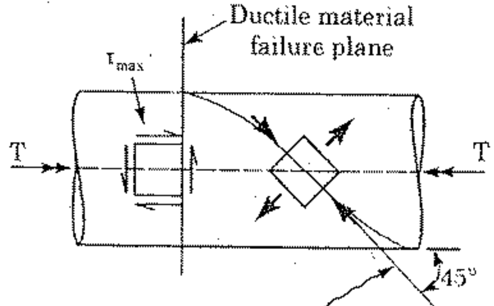


Fig. Potential torsional failure surfaces in ductile and brittle materials.

Max Normal Strain:

$$\varepsilon_{max} = \frac{\sigma_1}{\ } - \frac{\mu\sigma_2}{\ }$$

$$\varepsilon_{max} = \frac{\tau_{xy}}{E} - \frac{\mu(-\tau_{xy})}{E} = \frac{\tau_{xy}}{E}(1+\mu)$$

$$\Rightarrow \quad \boxed{\varepsilon_{max} = \frac{\tau_{xy}(1+\mu)}{E}}$$

also,

$$\varepsilon_{max} = \frac{\varepsilon_x + \varepsilon_y}{2} + \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

as $\varepsilon_x = 0$ and $\varepsilon_y = 0$

$$\Rightarrow \quad \boxed{\varepsilon_{max} = \frac{\gamma_{xy}}{2}}$$

⇒ Under pure torsion, max normal strain $= \frac{1}{2} \times$ shear strain

This relation can also be used to prove that $G = \frac{E}{2(1+\mu)}$.

Note:

$$\varepsilon_{max} = \frac{\tau_{xy}(1+\mu)}{E} = \frac{\gamma_{xy}}{2}$$

$$\Rightarrow \quad \frac{\tau_{xy}}{\gamma_{xy}} = \frac{E}{2(1+\mu)}$$

$$\Rightarrow \quad G = \frac{E}{2+(1+\mu)} = \text{Modulus of rigidity}$$

DETERMINATION OF G USING TORSION TEST

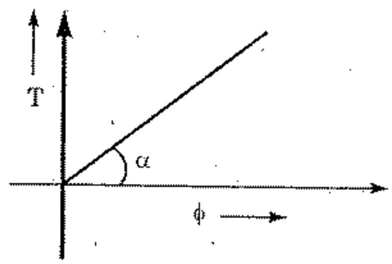
- From torsion formula, we know that

$$\frac{T}{J} = \frac{\tau}{r} = \frac{G\phi}{L}$$

$$\Rightarrow \quad \boxed{\phi = \frac{TL}{GJ}}, \text{ with in proportional limit}$$

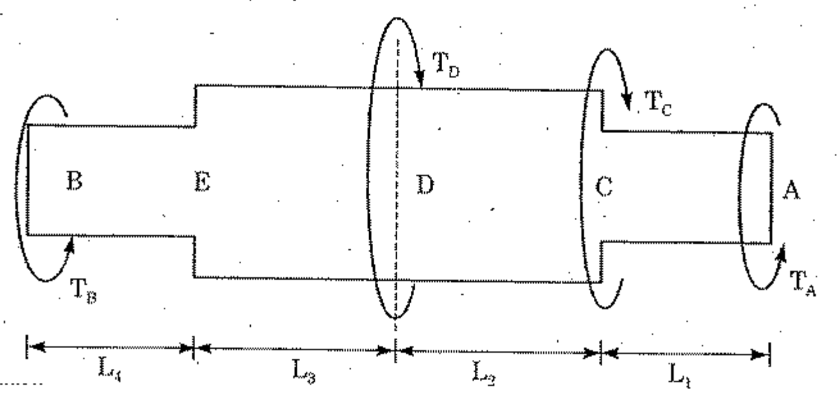
- In a torsion testing machine, by applying T and measuring ϕ on a specimen of known length and dia we can obtain $(T - \phi)$ plot.
- Slope of this line is $\frac{T}{\phi} = \frac{GJ}{L}$

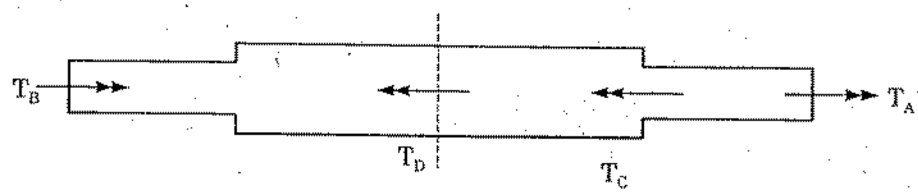
  Hence $G = \left(\frac{T}{\phi}\right)\frac{L}{J} = \frac{(\tan\alpha)\,L}{J}$
- Thus as length of bar, dia of bar and slope of $(T - \phi)$ plot is known, G can be calculated.

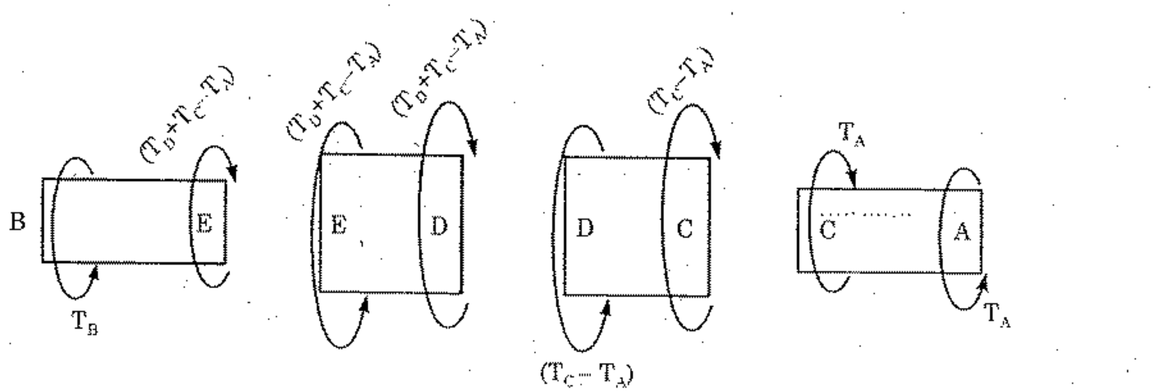


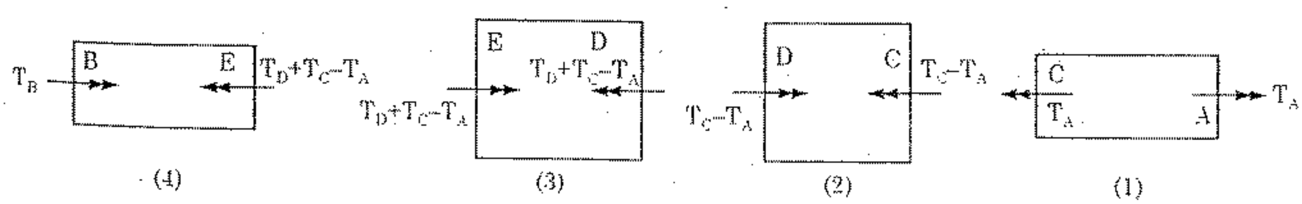
ANGLE OF TWIST FOR SERIES AND PARALLEL CONNECTION

(i) Series Connection









Note: *Although no torque is acting at section 'E' still section has to be cut at E, because x-section changes at E.*

$$\boxed{\phi_{A/B} = \phi_1 + \phi_2 + \phi_3 + \phi_4}$$

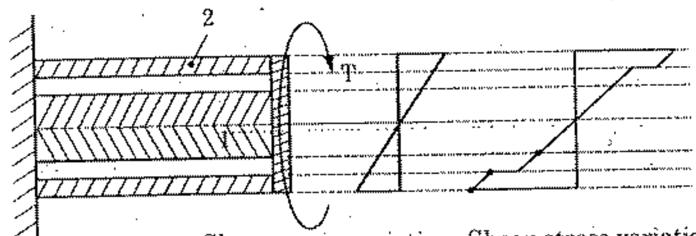
$$= -\frac{T_A L_1}{G_1 J_1} + \frac{(T_C - T_A)L_2}{G_2 J_2} + \frac{(T_D + T_C - T_A)L_3}{G_3 J_3} + \frac{T_B L_4}{G_4 J_4}$$

here, $\phi_{A/B} = \phi_A - \phi_B$

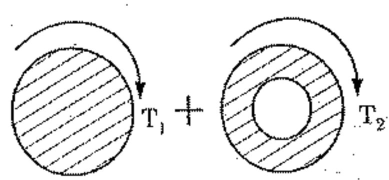
- if end B is fixed then $\phi_B = 0$

$\Rightarrow \quad \phi_{A/B} = \phi_A - 0 = \phi_A$

(ii) Parallel connection



Shear strain variation Shear stress variation



Let torque resisted by member 1 be T_1 and that resisted by member 2 be T_2.

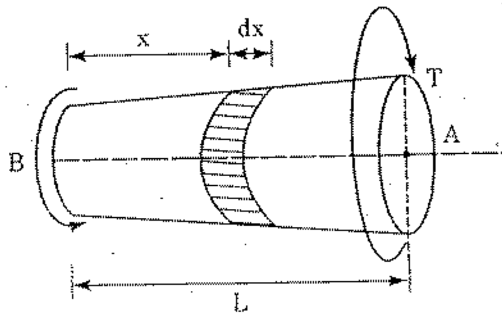
$$\Rightarrow \quad T = T_1 + T_2 \quad \text{.......... (i)}$$

Also, $\phi_1 = \phi_2$ (in parallel connection)

$$\Rightarrow \quad \frac{T_1 L}{G_1 J_1} = \frac{T_2 L}{G_2 J_2} \quad \text{.......... (ii)}$$

By solving (i) and (ii) T_1 and T_2 can be calculated.

ANGLE OF TWIST FOR VARIABLE X-SECTION

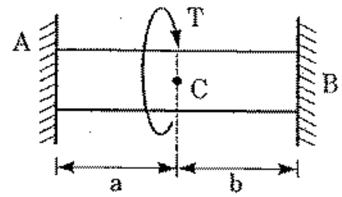


Note:

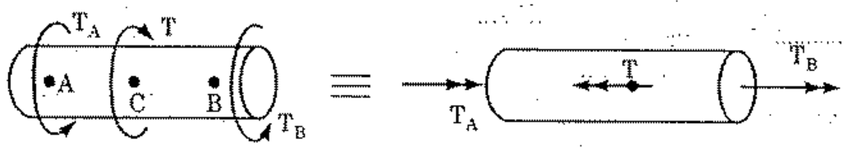
$$d\phi = \frac{T.dx}{GJ}$$

$$\boxed{\phi_{AB} = \int^{L} \frac{T dx}{GJ}}$$

STATICALLY INDETERMINATE PROBLEMS



- A torsional shaft AB is subjected to torsion T at point C and ends A and B of the shaft are fixed. We have to find the torque transmitted to end A and B.
- This is a case of statically indeterminate problem of single degree of redundancy.
- Let the torque transmitted to end A and B be T_A and T_B as shown in the following figure.



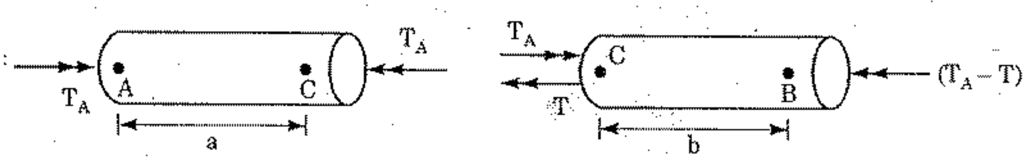
$$\Rightarrow \quad T = T_A + T_B \quad \text{.......... (i)}$$

- We use a compatibility equation in this case like

$$\phi_B = \phi_A = 0$$

$$\Rightarrow \quad \phi_{BA} = 0$$

- The free body diagram for the above loading is as shown below.



$$\phi_{BA} = \phi_{BC} + \phi_{CA}$$

$$\phi_{BA} = \frac{(T_A - T)\,b}{GJ} + \frac{T_A a}{GJ}$$

But, $\phi_{BA} = 0$

$$\Rightarrow \quad (T_A - T)b + T_A.a = 0 \quad \text{.......... (ii)}$$

From (i) and (ii)

$$T_A b - Tb + T_A a = 0$$

$$T_A(a + b) + T_B(a + b) = T(a + b)$$

$$-Tb - T_B\,(a + b) = -T(a + b)$$

$$\Rightarrow \quad -T_B\,(a + b) = -Ta$$

$$\Rightarrow \quad \boxed{T_B = \frac{Ta}{a + b}}$$

Similarly

$$T_A = T - \frac{Ta}{a + b} = \frac{Tb}{a + b}$$

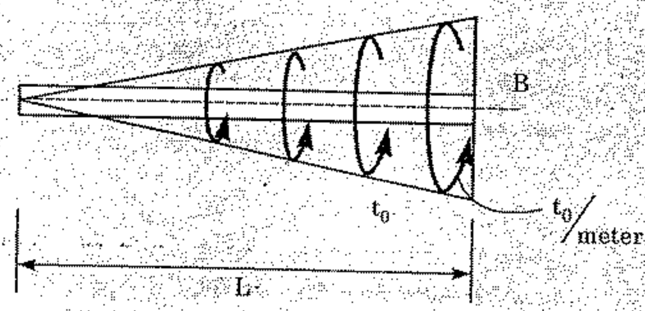
$$\Rightarrow \quad \boxed{T_A = \frac{Tb}{a + b}}$$

Example 1

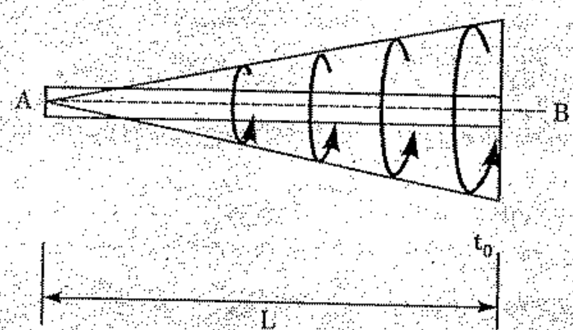
An elastic circular bar AB of constant GJ (i.e. Torsional rigidity) is subjected to a uniformly varying torque t_x as shown in figure below. Determine the rotation of the bar along its length and the reactions at ends A and B for two cases:

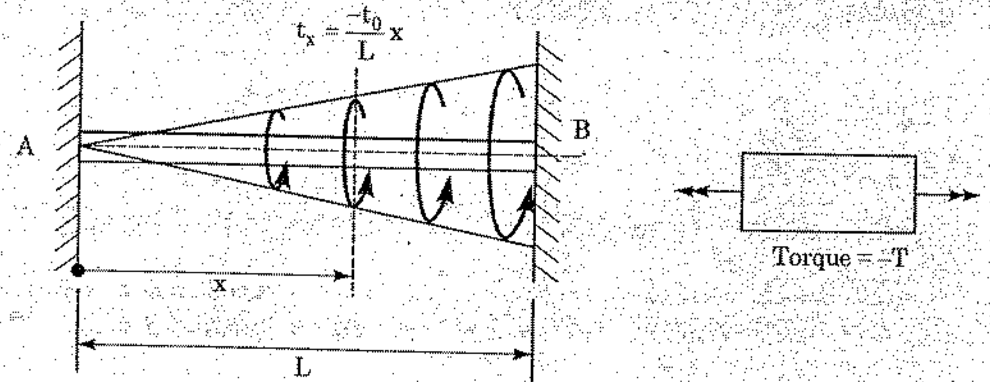
(a) Assume end A is free and B is fixed.

(b) Both ends A and B are fixed.



Sol: (a) When end A is free and end B is fixed.





We know that $d\phi = \dfrac{Tdx}{GJ}$

$$\Rightarrow \frac{d\phi}{dx} = \frac{T}{GJ}$$

$$\boxed{GJ\frac{d^2\phi}{dx^2} = \frac{dT}{dx} = \text{rate of change of torque} = \text{torque intensity}}$$

In this problem

$$GJ\frac{d^2\phi}{dx^2} = \frac{-t_0 x}{L} \quad [(-) \text{ because as per our sign convention, T is } (-) \text{ ve}].$$

On integration

$$\Rightarrow \quad GJ\frac{d\phi}{dx} = \frac{-t_0x^2}{2L} + C_1 = T$$

On further integration

$$\boxed{GJ\,\phi = \frac{-t_0x^3}{6L} + C_1x + C_2} \quad \text{------------ (A)}$$

- When end A is free and B is fixed, the boundary condition becomes
  at x = 0, T = 0 $\Rightarrow$ $C_1 = 0$

  at x = L, $\phi = 0$ $\Rightarrow$ $C_2 = \frac{t_0L^3}{6L} = \frac{t_0L^2}{6}$

$$\Rightarrow \quad \boxed{GJ\phi = \frac{t_0L^2}{6} - \frac{t_0x^3}{6L}}$$

- **When both end are fixed the boundary conditions are**
  at x = 0, $\phi = 0$
  at x = L, $\phi = 0$

$\Rightarrow$ From expression (A), $C_2 = 0$, $C_1 = \frac{t_0L}{6}$

$$\Rightarrow \quad \boxed{GJ\phi = \frac{-t_0x^3}{6L} + \frac{t_0Lx}{6}}$$

Thus at A, $T = \frac{t_0L}{6}$

at B, $T = \frac{t_0L}{6} - \frac{t_0L}{3}$

$$\boxed{T = \frac{-t_0L}{2}}$$

TORSIONAL STRAIN ENERGY

$$\boxed{U = \frac{T\phi}{2}} = \text{work done}$$

$$\Rightarrow \quad \boxed{U = \int_0^L \frac{T^2dx}{2GJ}} = \text{Torsional strain energy}$$

if T and J are constant i.e. for prismatic bar under constant torque $\boxed{U = \frac{T^2L}{2GJ}}$

Note:

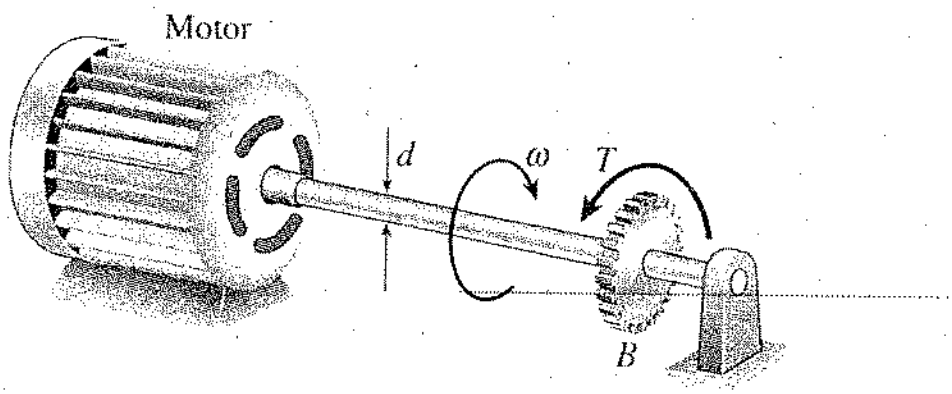
Strain energy density $= \frac{1}{2}\tau \times \gamma$ i.e. strain energy per unit volume

GJ = Torsional rigidity

$\frac{GJ}{L}$ = Torsional Stiffness

POWER TRANSMISSION BY SHAFT

Members subjected to torque are widely used as rotating shaft for transmitting power from one device or machine to another. The power is transmitted through the rotary motion of the shaft and the amount of power transmitted depends upon the magnitude of torque and speed of rotation.



Power = Torque × Speed of rotation

$\Rightarrow$ $\boxed{P = T \times w}$

where P = Power transmitted

w = Rotational speed in rad/sec

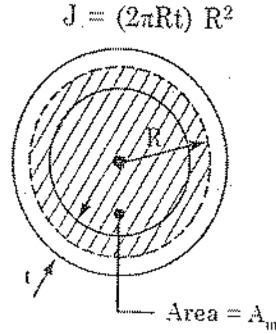
w = 2πf, where f = frequency in Hz

$w = \frac{2\pi N}{60}$, where N = speed in rpm

THIN WALLED HOLLOW SHAFT

For thin walled section, max shear stress is taken corresponding to mean radius.

$$J = (2\pi R t)\, R^2$$



$$J = 2\pi R^3 t$$

$$\tau_{max} = \frac{TR}{J} = \frac{T}{2\pi R^2 t}$$

$$\left(\tau_{max}.t\right) = \frac{T}{2\pi R^2}$$

$$\Rightarrow \boxed{\tau \cdot t = \frac{T}{2.A_m}} \quad \text{-------------------- (i)}$$

- We know that $\tau \times t$ = shear flow

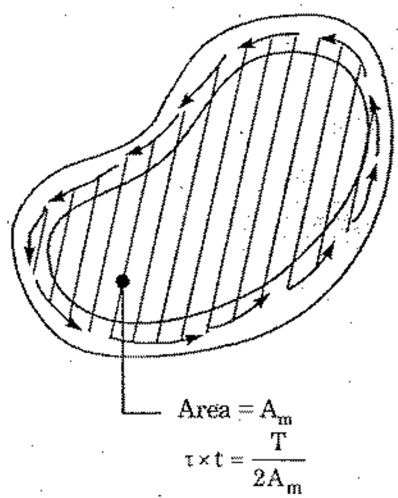
$$\Rightarrow \boxed{\frac{T}{2A_m} = q = \text{Shear flow (constant)}}$$

where A_m = area under the mean circle.

- Although above formula has been derived for circular hollow section, but the same is valid for all types of thin walled hollow section.
- Shear stress at any point in the thin walled section can be found out by dividing shear flow with thickness of section

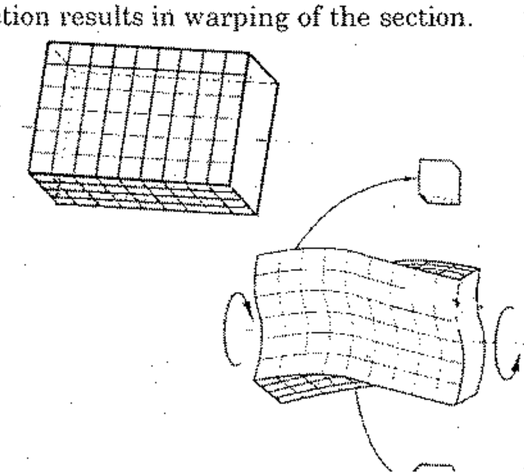
$$\Rightarrow \tau = \left(\frac{T}{2A_m}\right)\frac{1}{t}$$

thus thicker section will have smaller shear stress as compared to thinner section.



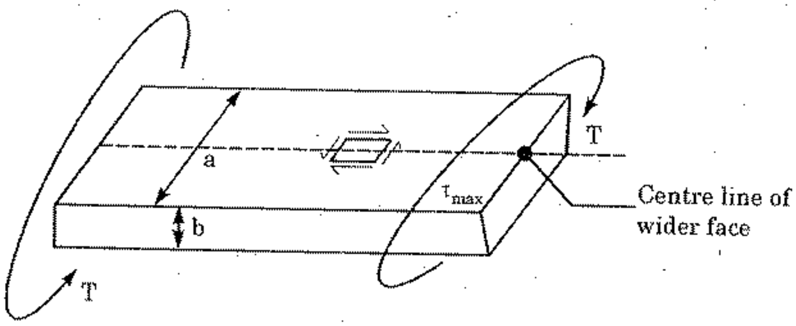
TORSION OF NON-CIRCULAR MEMBERS

- Torsion in non-circular section results in warping of the section.



For rectangular section

- Max shear stress occurs along the centre line of wider face.
- Shear stress at corners = 0

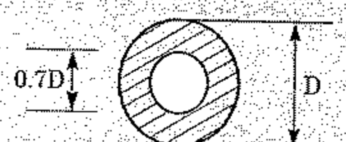


- $\boxed{\tau_{max} = \dfrac{T}{c_1 ab^2}}$
- $\boxed{\phi = \dfrac{TL}{c_2 ab^3 G}}$

where $\boxed{a > b}$ and c_1 and c_2 depends on $\dfrac{a}{b}$ values

Example 2

The internal dia of steel shaft = 70% of the external dia. The shaft is to transmit 3500 kW at 200 rpm. If τ_{max} = 50 MPa, calculate the dia of shaft. Find also the maximum twist of the shaft, when it is stressed to the max permissible value. The length of shaft = 4 m, G = 80 MPa.



Sol:

$$\text{Power } P = 3500 \text{ kW}$$
$$N = 200 \text{ rpm}$$
$$\tau_{max} = 50 \text{ MPa}$$
$$d = ?$$
$$L = 4 \text{ m}$$
$$G = 80 \text{ MPa}$$
$$\phi_{max} = ?$$

$$P = \frac{2\pi N}{60} \times T$$

$$T = \frac{60P}{2\pi N} = \frac{60 \times 3500 \times 10^3}{2 \times \pi \times 200}$$

$$T = 167.197 \text{ kNm}$$

Also from torsion formula

$$\frac{T}{J} = \frac{\tau}{r} = \frac{G\phi}{L}$$

$$\Rightarrow \frac{T}{\frac{\pi(D^4 - d^4)}{32}} = \frac{\tau_{max}}{\frac{D}{2}}$$

$$\Rightarrow \frac{32T}{\pi D^4 (1 - (0.7)^4)} = \frac{2\tau_{max}}{D}$$

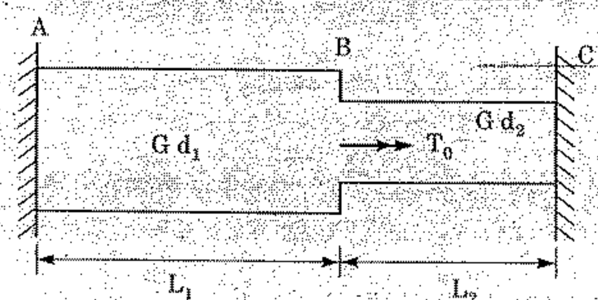
$$\Rightarrow D^3 = \frac{16T}{\tau_{max}\pi(0.7599)}$$

$$\Rightarrow D = 281.99 \text{ mm} = \text{external dia}$$

Also $\phi = \frac{\tau L}{Gr} = \frac{50 \times 4000 \times 2}{80 \times 281.99} = 17.73 \text{ rad}$

Example 3

Two circular shaft AB and BC of the same material but different dia are welded together at point B as shown in figure. End 'A' and 'C' are fixed. An external torque with magnitude T_0 is applied to the shaft at point 'B'. Find the torque exerted on the ends of the shaft A and C.



Sol:

$$\phi_{A/C} = 0 = \frac{T_C L_2}{GJ_2} - \frac{(T_0 - T_C) L_1}{GJ_1} = 0$$

$$\Rightarrow \frac{T_C L_2}{GJ_2} + \frac{T_C L_1}{GJ_1} = \frac{T_0 L_1}{GJ_1}$$

$$T_C = \left(\frac{T_0 L_1}{GJ_1}\right) \Big/ \left(\frac{L_2}{GJ_2} + \frac{L_1}{GJ_1}\right)$$

$$T_C = \frac{\frac{T_0 L_1}{J_1}}{\frac{L_2}{J_2} + \frac{L_1}{J_1}}$$

$$\Rightarrow \boxed{T_C = \frac{T_0 L_1 J_2}{L_2 J_1 + L_1 J_2}}$$

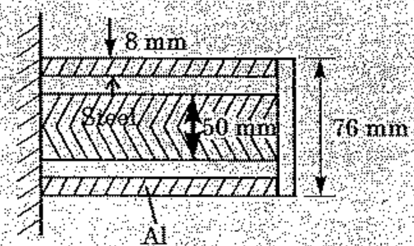
$$\boxed{T_A = \frac{T_0 L_2 J_1}{L_2 J_1 + L_1 J_2}}$$

Example 4

A steel shaft and an aluminium tube are connected to a fixed support and to a rigid disk as shown below. Determine the max torque T_0 that can be applied to the disk if the allowable stress are 120 MPa in steel and 70 MPa in Aluminium.

$G_{steel} = 77$ GPa

$G_{Al} = 27$ GPa



Sol:

$$T_0 = T_{Al} + T_{steel}$$

$$\phi_{Al} = \phi_{st}$$

$$\Rightarrow \frac{T_{Al}}{G_{Al}J_{Al}} = \frac{T_{st}}{G_{st}J_{st}} \quad (\because \text{Length is same})$$

$$J_{Al} = \frac{\pi}{32}\left(d_1^4 - d_2^4\right) = \frac{\pi}{2}\left(r_1^4 - r_2^4\right)$$

$$= \frac{\pi}{2}\left[(38)^4 - (30)^4\right]$$

$$\boxed{J_{Al} = 2.003 \times 10^{-6}\,m^4}$$

$$\boxed{J_{st} = \frac{\pi}{32}(50)^4 = 0.614 \times 10^{-6}\,m^4}$$

$$\frac{T_{Al}}{T_{st}} = \frac{G_{Al}}{G_{st}}\frac{J_{Al}}{J_{st}} = \frac{27}{77} \times \frac{2.003}{0.614}$$

$$\Rightarrow \frac{T_{Al}}{T_{st}} = 1.144$$

$$T_{Al} + T_{st} = T_0$$

$$\Rightarrow T_{st} \times 1.144 + T_{st} = T_0$$

$$\Rightarrow \boxed{T_{st} = 0.467\,T_0} \quad \boxed{T_{Al} = 0.533\,T_0}$$

- When max permissible stress is reached in steel

$$\frac{T_{st}R_{st}}{\tau_{st}} = \tau_{st}$$

$$\Rightarrow \frac{0.467T_0 \times 25}{0.614 \times 10^6} = 120$$

$$\Rightarrow T_0 = 6.31 \text{ kNm}$$

- when max permissible stress is reached in Aluminium

$$\frac{T_{Al} \cdot R_{Al}}{J_{Al}} = \tau_{Al}$$

$$\Rightarrow \quad \frac{0.533T_0 \times 38}{2.003 \times 10^6} = 70$$

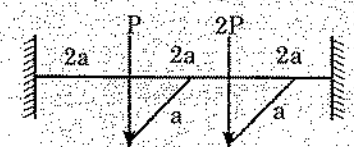
$$\boxed{T_0 = 6.92\ \text{kNm}}$$

$\Rightarrow$ Max torque that can be safely applied is

$$\Rightarrow \quad \boxed{T_0 = 6.31\ \text{kNm}}$$

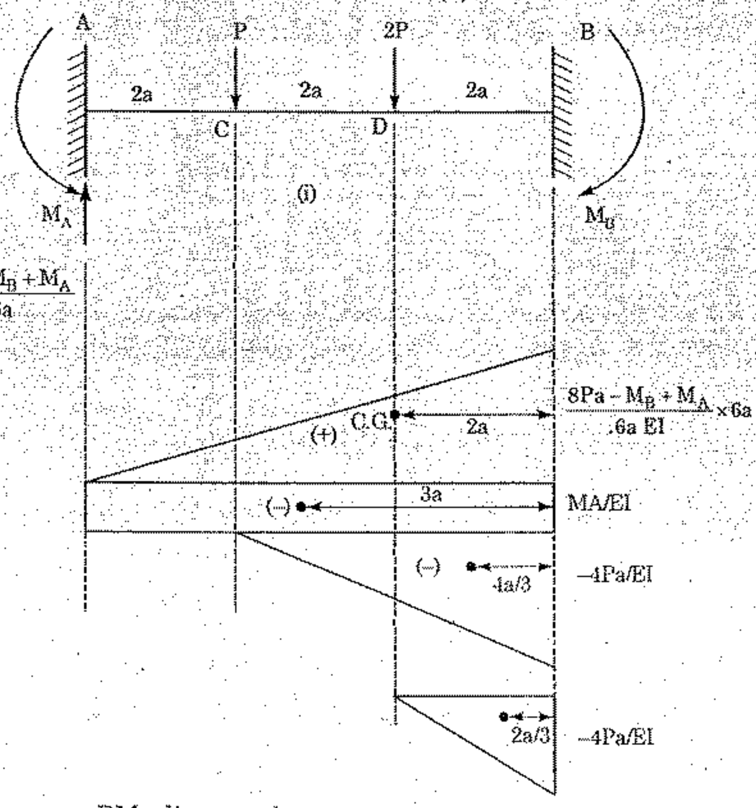
Example 5

A horizontal beam of length 6a has built in ends. The beam has two horizontal normally projecting cantilevers of length 'a' at one third points which carry vertical loads of magnitudes 'P' and '2P' one at each end. Draw bending moment, SF and Torsion diagram.



Sol:

The above loading is equivalent



$$\frac{2P}{3} + \frac{2P}{3} - \frac{M_B - M_A}{6a} = \frac{8Pa - M_B + M_A}{6a}$$

By area moment theorem

$\theta_{B/A} = 0$

$$6a \times \left(\frac{8\,Pa - M_B + M_a}{2EI}\right) - \frac{M_A \times 6a}{EI} - \frac{4Pa \times 4a}{2EI} - \frac{4Pa \times 2a}{2EI} = 0$$

$$48Pa - 6M_B + 6M_A - 12M_A - 16Pa - 8Pa = 0$$

$$\Rightarrow \boxed{24Pa - 6M_B - 6M_A = 0} \quad \text{------ (i)}$$

Again by area moment theorem

$t_{B/A} = 0$

$$\frac{(8Pa - M_B + M_A) \times 6a}{2EI} \times 2a - \frac{M_A \times 6a}{EI} \times 3a - \frac{4Pa \times 4a}{2EI} \times \frac{4a}{3} - \frac{4Pa \times 2a}{2EI} \times \frac{2a}{3} = 0$$

$$48Pa - 6M_B + 6M_A - 18M_A - \frac{32Pa}{3} - \frac{8Pa}{3} = 0$$

$$\left(48Pa - \frac{40Pa}{3}\right) - 6M_B - 12M_A = 0$$

$$\frac{104Pa}{3} - 6M_B - 12M_A = 0$$

$$104Pa - 18M_B - 36M_A = 0$$

$$\Rightarrow \boxed{52Pa - 9M_B - 18M_A = 0} \quad \text{------ (ii)}$$

From (i) and (ii)

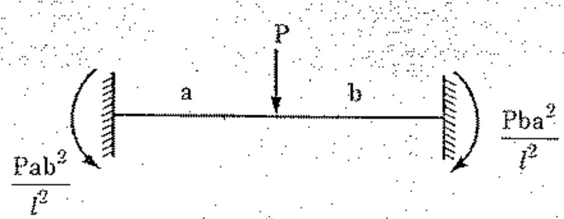
$$-20Pa + 9M_B = 0$$

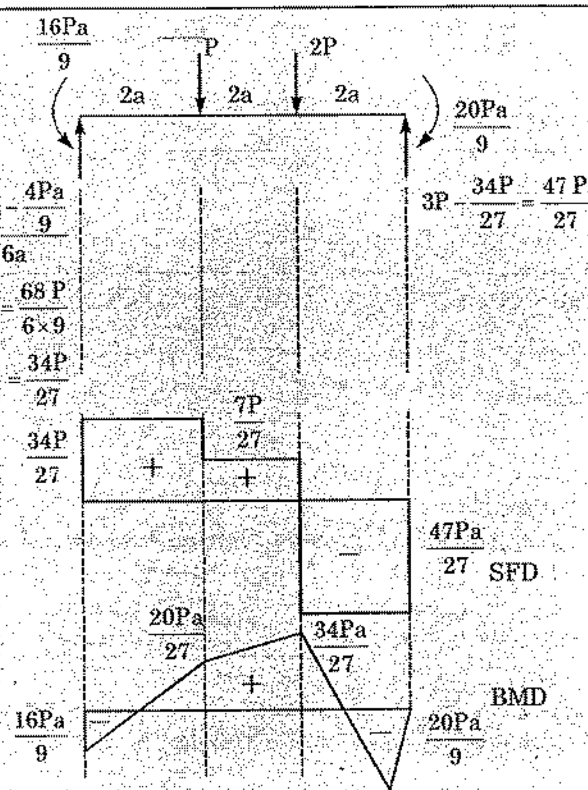
$$\boxed{M_B = \frac{20Pa}{9}}$$

$$M_A = \frac{24Pa - 6 \times \frac{20\,Pa}{9}}{6} = \frac{16Pa}{9}$$

$$\boxed{M_A = \frac{16\,Pa}{9}}$$

Alternatively, we can use standard results like

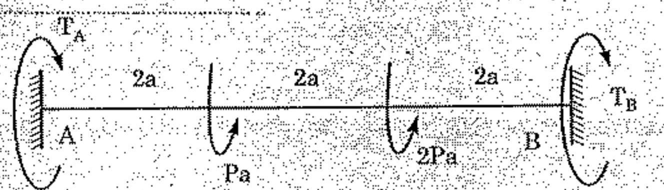




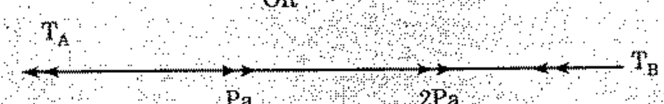
$$M_2 = \frac{8Pa - \frac{20Pa}{9} + \frac{16Pa}{9}}{6a} = \frac{8Pa - \frac{4Pa}{9}}{6a}$$

$$= \frac{(72-4)P}{6\times 9} = \frac{68P}{6\times 9}$$

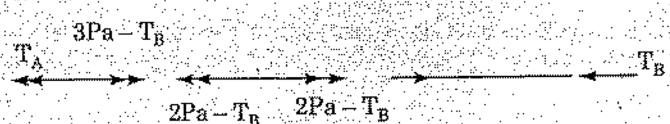
$$= \frac{34P}{3\times 9} = \frac{34P}{27}$$



OR



$$\boxed{T_A + T_B = 3Pa} \quad \text{(i)}$$



$\phi_{BA} = 0$

$$\Rightarrow \frac{T_B(2a)}{GJ} - \frac{(2Pa - 2T_B)(2a)}{GJ} - \frac{(3Pa - T_B)(2a)}{GJ} = 0$$

$$T_B - 2Pa + T_B - 3Pa + T_B = 0$$

$$3T_B - 5Pa = 0$$

$$\Rightarrow \boxed{T_B = \frac{6Pa}{3}}$$

$$\Rightarrow \boxed{T_A = \frac{4Pa}{3}}$$

Example 6

Find the max torque that can be applied safely to a shaft of 300 mm dia. The permissible angle of twist is 1.5° in a length of 7.5m and the shearing stress is not to exceed 42 N/mm^2.
G = 84.4 kN/mm^2

Sol:

$$\phi_{max} = 1.5° = \frac{1.5\times\pi}{180}\text{ radian}$$
$$= 0.026167 \text{ radian}$$

Dia = D = 300 mm

L = 7.5 m

τ_{max} = 42 N/mm^2

G = 84.4 kN/mm^2

From torsion formula

$$\frac{T}{J} = \frac{\tau}{r} = \frac{G\phi}{L}$$

$$\Rightarrow \tau = \frac{Gr\phi}{L}$$

$$(\tau_{max}) = \frac{GR\phi_{max}}{L}$$

$$(\tau_{max}) = \frac{84.4\times0.15\times0.02617}{7.5}\times1000 = 44.17 \text{ N/mm}^2 > 42 \text{ N/mm}^2$$

$$\Rightarrow \tau_{max\ permissible} \text{ governs.}$$

$$\Rightarrow T = \frac{\tau_{max}\times J}{R}$$

$$= \frac{\tau_{max}\dfrac{\pi(D^4)}{32}}{\dfrac{D}{2}}$$

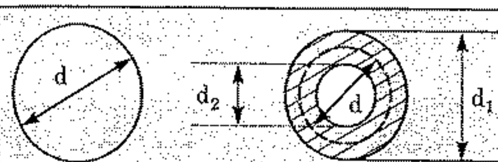
$$= \frac{\tau_{max}\pi D^3}{16}$$

$$= \frac{42 \text{ N/mm}^2\times\pi\times(300)^3 \text{ mm}^3}{16}$$

$$\Rightarrow \boxed{T_{max} = 222.548 \text{ kNm}}$$

Example 7

A solid circular shaft of dia 'd' has the same weight as a hollow circular shaft of mean dia 'd'. Assuming the same maximum shear stress in both the cases, determine the ratio of torques transmitted by the two shafts. Aslo compare the angles of twist per unit length in the two shafts. From these values draw your conclusion regarding the strength and stiffness of the two shaft.



They have same weight

$\Rightarrow$ Area of x-section will be same for the two shaft

$$\Rightarrow \frac{\pi d^2}{4} = \frac{\pi(d_1^2 - d_2^2)}{4}$$

$$\Rightarrow d^2 = (d_1 + d_2)(d_1 - d_2) \quad \text{------------ (i)}$$

But

$$d = \frac{d_1 + d_2}{2} \quad \text{------------ (ii)}$$

$$\Rightarrow \frac{(d_1 + d_2)^2}{4} = (d_1 + d_2)(d_1 - d_2)$$

$$\Rightarrow \frac{d_1 + d_2}{4} = (d_1 - d_2)$$

$$\Rightarrow d_1 + d_2 = 4d_1 - 4d_2$$

$$\Rightarrow 3d_1 = 5d_2$$

$$\Rightarrow d = \frac{d_1 + \frac{3d_1}{5}}{2}$$

$$\frac{10d}{8} = d_1$$

$$\boxed{d_1 = \frac{5d}{4}}$$

$$d_2 = \frac{3}{5} \times d_1 = \frac{3}{5} \times \frac{5}{4} d = \frac{3}{4} d$$

$$\boxed{d_2 = \frac{3d}{4}}$$

Assuming same max stress in both the cases

$$\frac{T}{\tau} = \frac{\tau}{R} = \frac{G\phi}{L}$$

$$T = \frac{\tau J}{R}$$

$$\frac{T_h}{T_s} = \frac{\tau_h \cdot J_h \Big/ \frac{d_1}{2}}{\tau_s \cdot J_s \Big/ \frac{d}{2}} = \frac{J_h \times d}{\frac{5d}{4} \times J_s} \quad (\because \tau_h = \tau_s)$$

$$\frac{T_h}{T_s} = \frac{4J_h}{5J_s} = \frac{4}{5}\left(\frac{d_1^4 - d_2^4}{d^4}\right)$$

$$= \frac{4}{5}\left(\left(\frac{5}{4}\right)^4 - \left(\frac{3}{4}\right)^4\right)$$

$$\frac{T_h}{T_s} = \frac{4}{5}\left[\frac{625-81}{256}\right] = 1.7$$

$$\boxed{\frac{T_h}{T_s} = 1.7} \quad \text{------ (A)}$$

Angle of twist

$$\frac{G\phi}{L} = \frac{\tau}{R}$$

$$\phi = \frac{\tau L}{GR}$$

$$\frac{\phi_h/L_h}{\phi_s/L_s} = \left(\frac{\tau}{GR}\right)_h \Big/ \left(\frac{\tau}{GR}\right)_s$$

$$= \frac{R_s}{R_h} = \frac{d/2}{d_1/2} = \frac{d}{d_1} = \frac{d}{\frac{5d}{4}} = 0.8$$

$$\boxed{\frac{\phi_h/L_h}{\phi_s/L_s} = 0.8} \quad \text{------ (B)}$$

From A we conclude that:

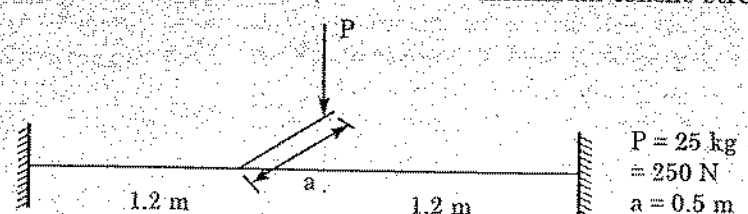
- For the same weight, strength of hollow circular shaft is greater than that of solid circular shaft.

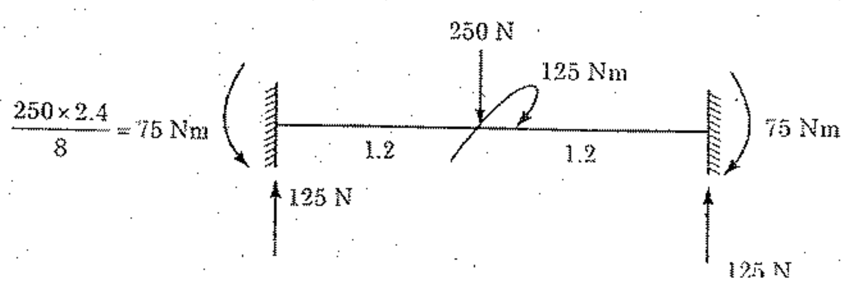
  $$\text{Stiffness} = \frac{\text{Rigidity}}{\text{Length}} = \frac{GJ}{L} = \frac{T}{\phi}$$

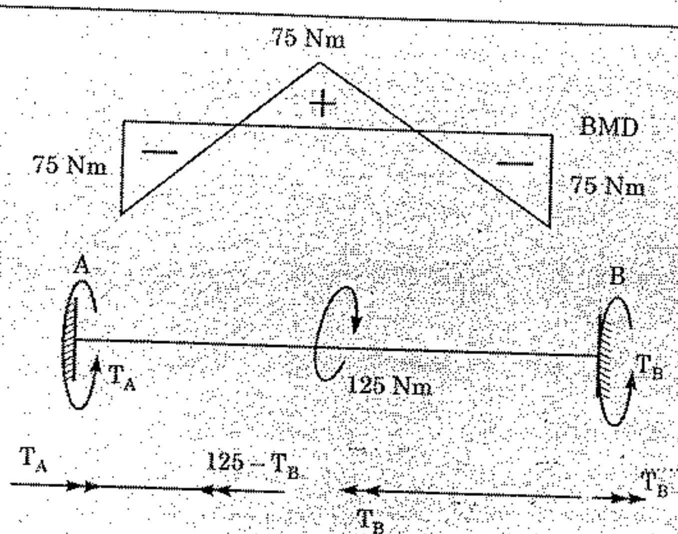
  Hence for same length and weight, Stiffeness of hollow shaft is greater than solid shaft.

Example 8

A horizontal steel bar 4 cm dia solid section is 2.4 m long and is rigidly held at both ends so that no angular rotation occur either axially or circumferentially. If a bracket at the centre of span support a vertical load of 25 kg at a horizontal lever arm of 50 cm, what is the maximum tensile stress in the bar.







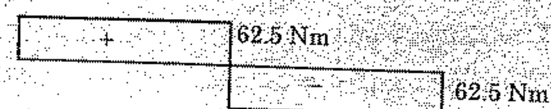
$\phi_{AB} = 0$

$$\Rightarrow -\frac{T_B \times 1.5}{GJ} + \frac{(125 - T_B)\times 1.2}{GJ} = 0$$

$$-T_B + 125 - T_B = 0$$

$$T_B = \frac{125}{2} = 62.5 \text{ Nm}$$

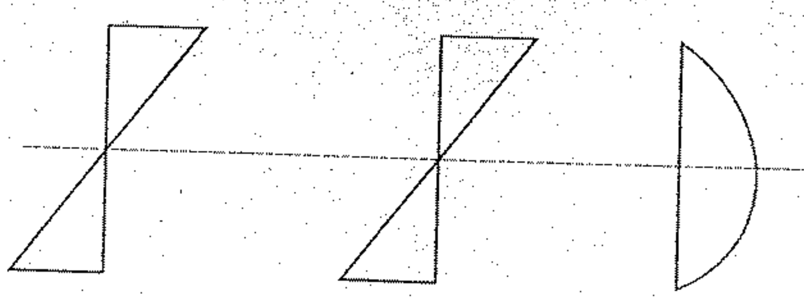
- Maximum shear stress due to torsion is



Torsion diagram

$$\frac{Tr}{J} = \frac{62.5 \text{ Nm} \times (2\times10^{-2}\text{m})}{\frac{\pi}{32}(4\times10^{-2})^4 \text{ m}^4}$$

$$= 4.976 \text{ N/mm}^2$$



Bending stress variation

Torsional shear

Transverse shear

- Max shear stress due to transverse shear at extreme fibre is equal to zero.
- Bending normal stress is maximum at the extreme fibre.

Hence at extreme fibre

$$\text{Max Normal Stress} = \frac{My}{I} = \frac{75\times(2\times10^{-2})}{\frac{\pi}{64}(4\times10^{-2})^4}\ \text{N/m}^2$$

$$= \frac{4.976\times2}{62.5}\times75$$

$$= 11.942\ \text{N/mm}^2$$

$$\text{Principal stress} = \frac{\sigma_x+\sigma_y}{2}\pm\sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau^2}$$

$$= \frac{11.942}{2}\pm\sqrt{\left(\frac{11.942}{2}\right)^2+(4.976)^2}$$

$$= 5.971 \pm 7.773$$

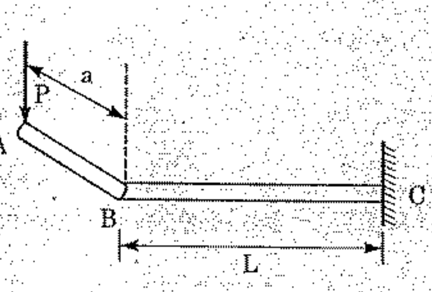
$$= 13.744\ \text{N/mm}^2,\quad -1.802\ \text{N/mm}^2$$

$\Rightarrow$ Max Tensile stress in bar = 13.744 N/mm².

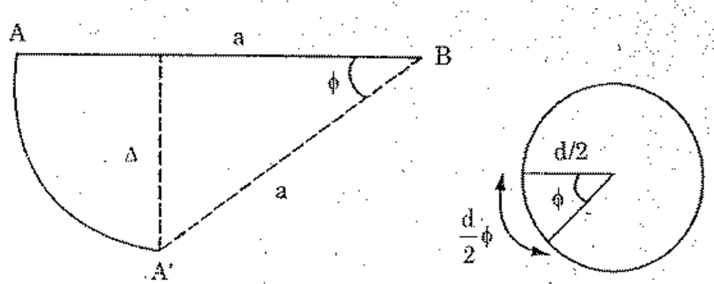
Example 9

A solid cylindrical rod is attached to the rigid lever AB and to the fixed support at 'C'. The vertical force 'P' applied at 'A' causes a small displacement 'Δ' at point A. Show that corresponding maximum shearing stress in the rod is

$$\tau = \frac{Gd}{2La}\Delta,\ \text{where d is the dia of rod, G = modulus of rigidity}$$



Sol:



$$\sin\phi = \frac{\Delta}{a}$$

for small ϕ, $\frac{\Delta}{a} = \phi$

$$\frac{d\phi}{2L} = \gamma = \text{shearing strain}$$

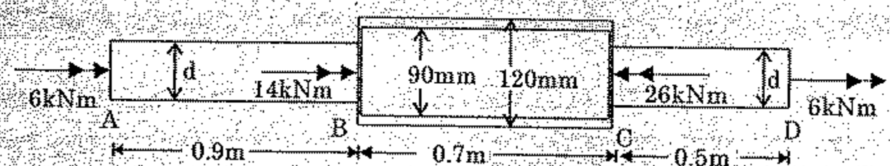
$$\tau = G\gamma$$

$$\Rightarrow \quad \tau = \frac{G\phi d}{2L}$$

$$\Rightarrow \quad \tau = \frac{Gd\Delta}{2La}$$

Example 10

Shaft BC is hollow with inner and outer diameters of 90 mm and 120 mm, respectively. Shafts AB and CD are solid and of diameter *d*. For the loading shown, determine (*a*) the maximum and minimum shearing stress in shaft BC, (b) the required diameter *d* of shafts AB and CD if the allowable shearing stress in these shafts is 65 MPa.



Sol. Equations of Statics: Denoting by T_{AB} the torque in shaft AB, we pass a section through shaft AB and, for the free body shown, we write

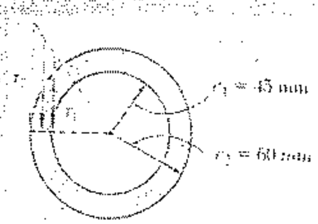
$\Sigma M_x = 0$, $\quad (6\ \text{kN . m}) - T_{AB} = 0 \quad T_{AB} = 6\ \text{kN . m}$

We now pass a section through shaft BC and, for the free body shown, we have

$\Sigma M_x = 0$, $\quad (6\ \text{kN . m}) + (14\ \text{kN .m}) - T_{BC} = 0 \quad T_{BC} = 20\ \text{kN . m}$

(a) Shaft BC: For this hollow shaft we have

$$J = \frac{\pi}{2}\left(c_2^4 - c_1^4\right) = \frac{\pi}{2}\left[(0.060)^4 - (0.045)^4\right] = 13.92\times10^{-6}\,m^4$$



Maximum Shearing Stress: On the outer surface, we have

$$\tau_{max} = \tau_2 = \frac{T_{BC}c_2}{J} = \frac{(20\ \text{kN}\cdot\text{m})(0.060\ \text{m})}{13.92\times10^{-6}\,\text{m}^4}$$

$$\tau_{max} = 86.2\ \text{MPa}$$

Minimum Shearing Stress: We write that the stresses are proportional to the distance from the axis of the shaft.

$$\frac{\tau_{min}}{\tau_{max}} = \frac{c_1}{c_2} \qquad \frac{\tau_{min}}{86.2\ \text{MPa}} = \frac{45\ \text{mm}}{60\ \text{mm}}$$

$$\tau_{min} = 64.7\ \text{MPa}$$

(b) Shafts AB and CD: We note that in both of these shafts the magnitude of the torque is T = 6 kN . m and $\tau_{all} = 65$ MPa. Denoting by c the radius of the shafts, we write

$$\tau = \frac{TC}{J} \qquad 65\ \text{MPa} = \frac{(6\ \text{kN}\cdot\text{m})c}{\frac{\pi}{2}c^4}$$

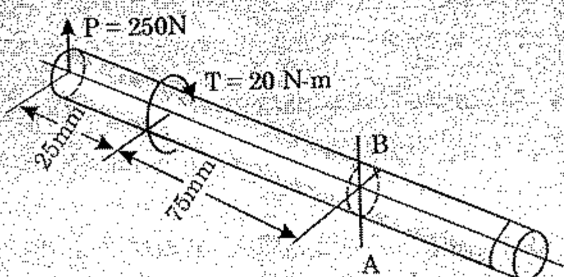
$$c^3 = 58.8 \times 10^{-6}\text{m}^3 \qquad c = 38.9 \times 10^{-3}\ \text{m}$$

$$d = 2c = 2(38.9\ \text{mm})$$

$$d = 77.8\ \text{mm}$$

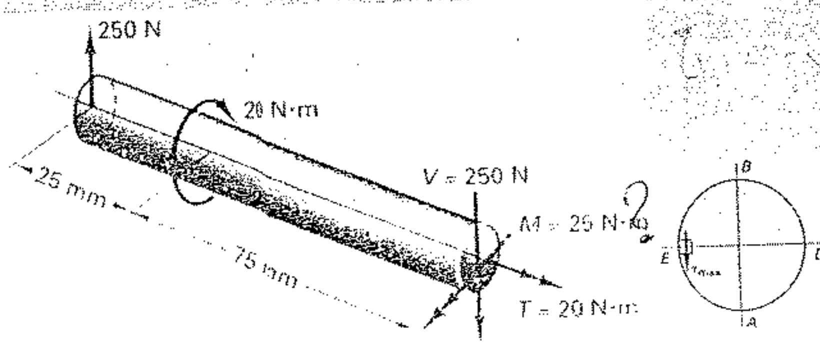
Example 11

Find the maximum shear stress due to the applied forces in plane A–B of the 10-mm-diameter high-strength steel shaft shown in Fig.(a).



(a)

Sol. The system of forces at the cut necessary to keep this segment in equilibrium consists of a torque, T = 20 N.m, a shear, V = 250 N, and a bending moment, M = 25 N.m.



(b)

$$A\bar{y} = \frac{\pi c^2}{2}\frac{4c}{3\pi} = \frac{2c^2}{3}$$, c = Radius of shaft

Moreover, since $t = 2c$ and $I = \pi c^4/4$,

the maximum direct shear stress at E is

$$\tau_{max} = \frac{VQ}{It} = \frac{V}{2c}\frac{2c^3}{3}\frac{4}{\pi c^4} = \frac{4V}{3\pi c^2} = \frac{4V}{3A}$$

$$I_p = \frac{\pi d^4}{32} = \frac{\pi \times 10^4}{32} = 982\ \text{mm}^4$$

$$I = \frac{I_p}{2} = 491\ \text{mm}^4$$

$$A = \frac{1}{4}\pi d^2 = 78.5\ \text{mm}^2$$

Maximum shear stress due to torsional moment at E is

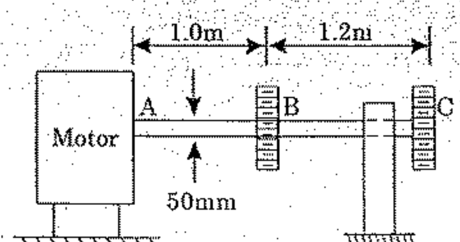
$$(\tau_{max})_{torsion} = \frac{T_C}{I_p} = \frac{20 \times 10^3 \times 5}{982} = 102\ \text{MPa}$$

$$(\tau_{max})_{direct} = \frac{VQ}{It} = \frac{4V}{3A} = \frac{4 \times 250}{3 \times 78.5} = 4\ \text{MPa}$$

Total maximum shear stress = 102 + 4 = 106 MPa

Example 12

A steel shaft ABC (Fig. a) of 50mm diameter is driven at A by a motor that transmits 50 kW to the shaft at 10 Hz. The gears at B and C remove 30 kW and 20 kW, respectively. Compute the maximum shear stress τ in the shaft and the angle of twist ϕ between the ends A and C. G = 80 GPa



(a)

Sol. The power transmitted between A and B is 50 kW, hence the torque is

$$T = \frac{P}{2\pi f} = \frac{50\text{kW}}{2\pi(10\text{Hz})} = 796\ \text{N.m}$$

(b)

This torque is assumed to have the direction shown in Fig (b). The shear stress and angle of twist for part AB of the shaft are

$$\tau_{ab} = \frac{16T}{\pi d^3} = \frac{16(796\text{ N.m})}{\pi(50\text{mm})^3} = 32.4\text{ MPa}$$

$$\phi_{ab} = \frac{TL}{GI_p} = \frac{(796\text{ N.m})(1.0\text{m})}{(80\text{ GPa})(\pi/32)(50\text{mm})^4} = 0.0162$$

For the other part BC of the shaft, the power transmitted is 20 kW; hence,

$$T = \frac{P}{2\pi f} = \frac{20\text{kW}}{2\pi(10\text{ Hz})} = 318\text{N.m}$$

The corresponding shear stress and angle of twist are

$$\tau_{bc} = \frac{16T}{\pi d^3} = \frac{16(318\text{ N.m})}{\pi(50\text{ mm})^3} = 13.0\text{ MPa}$$

$$\phi_{bc} = \frac{TL}{GI_p} = \frac{(318\text{ N.m})(1.2\text{m})}{(80\text{ GPa})(\pi/32)(50\text{ mm})^4} = 0.0078\text{ rad}$$

Thus, the maximum shear stress is $\tau = 32.4$ MPa, which occurs in part AB. Also the total angle of twist is

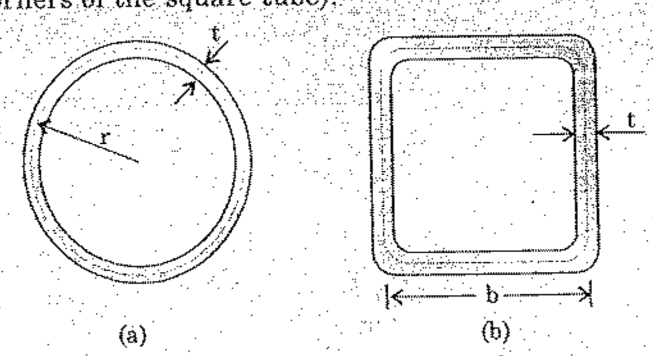
$$\phi = \phi_{ab} + \phi_{bc} = 0.0240\text{ rad}$$

both parts of the shaft twist in the same direction (Fig. b).

Example 13

A circular tube and a square tube (Fig.) are constructed of the same material and subjected to the same torque. Both tubes have the same length, same wall thickness, and same cross-sectional area.

What are the ratios of their shear stresses and angles of twist? Disregard the effects of stress concentrations at the corners of the square tube).



Comparison of circular and square tubes

Sol. Circular tube: For the circular tube, the area A_{m1} enclosed by the median line of the cross section is

$$A_{m1} = \pi r^2$$

Where r is the radius to the median line. Also, the torsion constant (Equation) and cross-sectional area are

$$J_1 = 2pr^3t \qquad a_1 = 2\pi rt$$

Square tube: For the square tube, the cross-sectional area is

$$A_2 = 4bt$$

where b is the length of one side, measured along the median line. In as much as the areas of the tubes are the same, we obtains $b = \pi r/2$. Also, the torsion constant and area enclosed by the median line of the cross section are

$$J_2 = b^3t = \frac{\pi^3 r^3 t}{8} \qquad A_{m2} = b^2 = \frac{\pi^2 r^2}{4}$$

Ratios: The ratio τ_1/τ_2 of the shear stress in the circular tube to the shear stress in the square tube is

$$\frac{\tau_1}{\tau_2} = \frac{A_{m2}}{A_{m1}} = \frac{\pi^2 r^2/4}{\pi r^2} = \frac{\pi}{4} = 0.79$$

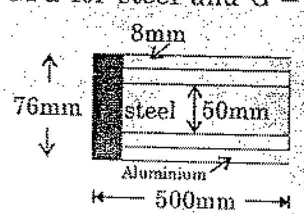
The ratio of the angles of twist is

$$\frac{\phi_1}{\phi_2} = \frac{J_2}{J_1} = \frac{\pi^3 r^3 t/8}{2\pi r^3 t} = \frac{\pi^2}{16} = 0.62$$

These results show that the circular tube not only has a 20% lower shear stress than does the square tube but also a greater stiffness against rotation.

Example 14

A steel shaft and in aluminum tube are connected to a fixed support and to a rigid disk as shown in the cross section. Knowing that the Initial stresses are zero, determine the maximum torque T_0 that can be applied to the disk if the allowable stresses are 120 MPa in the steel shaft and 70 MPa in the aluminum tube. Use G = 77 GPa for steel and G = 27 GPa for aluminum



Sol. T_1 = the torque exerted on the tube.

T_2 = the torque exerted on the shaft

T_0 = torque exerted by the rigid disk

Deformations: Since both the tube and the shaft are connected to the rigid disk, we have

$$\phi_1 = \phi_2 \qquad \frac{T_1L_1}{J_1G_1} = \frac{T_2L_2}{J_2G_2}$$

$$\frac{T_1(0.5\text{m})}{(2.003\times10^{-6}\text{m}^4)(27\text{ GPa})} = \frac{T_2(0.5\text{m})}{(0.614\times10^{-6}\text{m}^4)(77\text{ GPa})}$$

$T_2 = 0.874T_1$ ....(2)

Shearing Stresses: We assume that the requirement $\tau_{alum} \le 70$ MPa is critical. For the aluminum tube, we have

$$T_1 = \frac{\tau_{alum}J_1}{c_1} = \frac{(70\text{ MPa})(2.003\times10^{-6}\text{m}^4)}{0.038\text{ m}} = 3690\text{ N.m}$$

Using equation (2), we compute the corresponding value T_2 and then find the maximum shearing stress in the steel shaft.

$T_2 = 0.874\ T_1 = 0.874(3690) = 3225$ N.m

$$\tau_{steel} = \frac{T_2c_2}{J_2} = \frac{(3225\text{N . m})(0.025\text{m})}{0.614\times10^{-6}\text{ m}^4} = 131.3\text{ MPa}$$

We note that the allowable steel stress of 120 MPa is exceeded; our assumption was wrong. Thus the maximum torque T_0 will be obtained by making $\tau_{steel} = 120$ MPa. We first determine the torque T_2.

$$T_2 = \frac{\tau_{steel}J_2}{c_2} = \frac{(120\text{ MPa})(0.614\times10^{-6}\text{m}^4)}{0.025\text{ m}} = 2950\text{ N.m}$$

From Equation (2), we have

2950 N . m = 0.874 T_1; T_1 = 3375 N.m

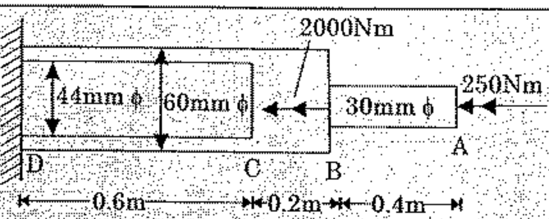
Using Equation (1), we obtain the maximum permissible torque

$T_0 = T_1 + T_2$ = 3375 N.m + 2950 N.m

T_0 = 6.325 kN

Example 15

The horizontal shaft AD is attached to a fixed base at D and is subjected to the torques shown. A 44-mm-diameter hole has been drilled into portion CD of the shaft. Knowing that the entire shaft is made of steel for which G = 77 GPa, determine the angle of twist at end A.



Sol. Statics: Passing a section through the shaft between A and B and using the free body shown, we find

$\Sigma M_x = 0 \qquad (250\text{ N . m}) - T_{AB} = 0 \quad T_{AB} = 250\text{N . m}$

Passing now a section between B and C, we have

$\Sigma M_x = 0$, $(250\text{ N . m}) + (2000\text{ N . m}) - T_{BC} = 0 \qquad T_{BC} = 2250\text{ N . m}$

Since no torque is applied at C,

$T_{CD} = T_{BC} = 2250\text{ N . m}$

Polar Moments of Inertia

$$J_{AB} = \frac{\pi}{2}c^4 = \frac{\pi}{2}(0.015\text{ m})^4 = 0.0795 \times 10^{-6}\text{ m}^4$$

$$J_{BC} = \frac{\pi}{2}c^4 = \frac{\pi}{2}(0.030\text{ m})^4 = 1.272 \times 10^{-6}\text{ m}^4$$

$$J_{CD} = \frac{\pi}{2}\left(c_2^4 - c_1^4\right) = \frac{\pi}{2}\left[(0.030\text{ m})^4 - (0.022\text{ m})^4\right] = 0.904 \times 10^{-6}\text{ m}^4$$

Angle of Twist:

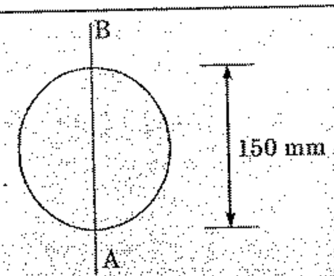
$$\phi_A = \sum_i \frac{T_i L_i}{J_i G} = \frac{1}{G}\left(\frac{T_{AB}L_{AB}}{J_{AB}} + \frac{T_{BC}L_{BC}}{J_{BC}} + \frac{T_{CD}L_{CD}}{J_{CD}}\right)$$

$$\phi_A = \frac{1}{77\text{ GPa}}\left[\frac{(250\text{ N}\cdot\text{m})(0.4\text{ m})}{0.0795 \times 10^{-6}\text{ m}^4} + \frac{(2205)(0.2)}{1.272 \times 10^{-6}} + \frac{(2250(0.6))}{0.904 \times 10^{-6}}\right]$$

$$\phi_A = (0.0403\text{ rad})\frac{360^\circ}{2\pi\text{ rad}} = 2.31^\circ$$

Example 17

A solid shaft of 150 mm diameter is transmitting a torque of 20 kN-m. At the same time it is subjected to a bending moment of 12 kN-m and axial thrust of 200 kN. Find the maximum and minimum principal stresses developed at extreme fibres along vertical axis A, B as shown in Fig.



Sol. Bending stress at A, $f_A = \frac{M}{I}y = \frac{M}{\frac{\pi}{64}d^4} \times \frac{d}{2} = \frac{32M}{\pi d^3}$

d = dia of shaft

M = Applied Bending moment

$= \frac{32 \times 12 \times 10^6}{\pi \times 150^3} = 36.23 \text{ N/mm}^2$

$\therefore$ Bending stress at B, $f_B = -36.22 \text{ N/mm}^2$

Axial thrust = 200 kN

Axial stress due to thrust $= \frac{200 \times 10^3}{\frac{\pi}{4} \times 150^2} = 11.32 \text{ N/mm}^2$

$\therefore$ At A $p_x = 36.22 - 11.32 = 24.9 \text{ N/mm}^2$ (Tensile)

At B, $p_x = -36.22 - 11.318 = -47.54 \text{ N/mm}^2$ (Compressive)

and $p_y = 0$ at A as well as at B (Direct shear stress)

Torsional Shearing stress at A and B are given by

$q = \frac{T}{J}R = \frac{T}{\frac{\pi}{32}d^4} \times \frac{d}{2} = \frac{16T}{\pi d^3}$

$= \frac{16 \times 20 \times 10^6}{\pi \times 150^3} = 30.19 \text{ N/mm}^2$

At A: $p_1 = \frac{24.90 + 0}{2} + \sqrt{\left(\frac{24.90 - 0}{2}\right)^2 + 30.19^2}$

$= 12.45 + 32.66 = 45.11 \text{ N/mm}^2$ **Ans**

$p_2 = 12.45 - 32.66 = -20.21 \text{ N/mm}^2$

$\Rightarrow$ Maximum Principal stress at A = 45.11 N/mm² (Tensile)

Minimum Principal stress at A = – 20.21 N/mm² (Compressive)

= 20.21 **Ans.**

$$q_{max} = \sqrt{\left(\frac{24.902-0}{2}\right)^2 + 30.18^2} = 32.648 \text{ N/mm}^2$$

At B: $$p_1 = \frac{p_x + p_y}{2} + \sqrt{\left(\frac{p_x - p_y}{2}\right)^2 + q^2}$$

$$= -\frac{47.54}{2} + \sqrt{\left(\frac{-47.54}{2}\right)^2 + 30.19^2}$$

$p_1 = -23.77 + 38.42 = 14.65$ N/mm²

$p_2 = -23.77 - 38.42 = -62.19$ N/mm²

Maximum Principal stress at B = −62.19 N/mm² (Compressive)

Minimum Principal stress at B = 14.65 N/mm² (Tensile)

$q_{max} = 38.414$ N/mm²

OBJECTIVE QUESTIONS

1. Consider the following statements:
   If a solid circular shaft and a hollow circular shaft have the same torsional strength, then
   1. the weight of the hollow shaft will be greater than that of the solid shaft
   2. the external diameter of the hollow shaft will be greater than that of the solid shaft
   3. the stiffness of the hollow shaft will be equal to that of the solid shaft

   Which of these statements is/are correct?
   (a) 1, 2 and 3 (b) 2 and 3
   (c) 1 alone (d) 1 and 2

2. A shaft turns at 150 rpm under a torque of 1500 Nm. Power transmitted is
   (a) 15π kW (b) 10π kW
   (c) 7.5π kW (d) 5π kW

3. A hollow steel shaft of external diameter 100 mm and internal diameter 50 mm is to be replaced by a solid alloy shaft. Assuming the same value of polar modulus for both, the diameter of the solid alloy shaft will be
   (a) $10 \times \sqrt[3]{9375}$ mm (b) $10 \times \sqrt[2]{9375}$ mm
   (c) $10 \times \sqrt[3]{\left(\frac{9375}{10}\right)}$ mm (d) $\sqrt[3]{9375}$ mm

4. Consider a circular shaft of radius 'R' having the maximum shear stress 'f_s' developed by an applied torque.

   Assertion (A): The shear stress 'q' at a point on the section having coordinate (0, y) is $\frac{f_s y}{R}$

   Reason (R): In the shaft, the shear stress 'q' at a point of coordinate (x, y) is $\frac{f_s}{R}\sqrt{x^2 + y^2}$

   Of these statements
   (a) both A and R are true and R is the correct explanation of A
   (b) both A and R are true but R is not a correct explanation of A
   (c) A is true but R is false
   (d) A is false but R is true

5. A solid shaft has diameter 80 mm. It is subjected to a torque of 4 kN·m. The maximum shear stress induced in the shaft would be
   (a) $\frac{75}{\pi}$ N/mm² (b) $\frac{250}{\pi}$ N/mm²
   (c) $\frac{125}{\pi}$ N/mm² (d) $\frac{150}{\pi}$ N/mm²

6. Two steel shafts A and B are used for transmitting power. The ratio of revolutions of shafts i.e., $\frac{N_A}{N_B} = 2$. The ratio of torques on shafts i.e., $\frac{T_A}{T_B} = \frac{1}{2}$. The ratio of the horse power transmitted by the shafts i.e. $\frac{P_A}{P_B}$ would be

(a) 1/2 (b) 1/4
(c) 1 (d) 2

7. A bar AB of diameter 40 mm and 4 m long is rigidly fixed at its ends. A torque of 600 Nm is applied at a section of the bar, 1 m from end A. The fixing couples T_A and T_B at the supports A and B, respectively, are
(a) 450 Nm and 150 N-m
(b) 200 N-m and 400 N-m
(c) 300 N-m and 150 N-m
(d) 300 N-m and 100 N-m

DIRECTIONS :
The following items consists of two statements; one labelled as **'Assertion (A)'** and the other as **'Reason (R)'**. You are to examine these two statements carefully and select the answers to these items using the codes given below:

Codes:
(a) Both A and R are true and R is the correct explanation of A
(b) Both A and R are true but R is not a correct explanation of A
(c) A is true but R is false
(d) A is false but R is true

8. **Assertion (A):** The failure surface of a mild steel torsion specimen subject to a torque about its axis is along a surface perpendicular to its axis.
Reason (R): Mild steel is relatively weaker in shear than in tension and the plane of maximum shear is perpendicular to its axis.

9. Match **List-I** with **List-II** and select the correct answer using the codes given below the lists:

List-I
A. Torque-twist relationship for a circular shaft
B. Strain energy of elastic torsion
C. Circumferential shear stress
D. Maximum shearing stress due to combined torsion and direct stress

List-II
1. $\frac{1}{2}\sqrt{\sigma^2+4\lambda^2}$
2. $\frac{Gr\theta}{L}$
3. $\left(\frac{GJ}{2L}\right)\theta^2$
4. $\frac{GJ}{L}\theta$

Codes:

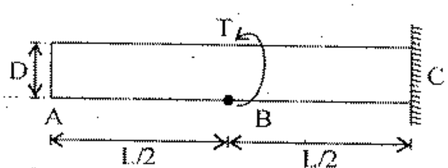
	A	B	C	D
(a)	2	3	4	1
(b)	4	1	2	3
(c)	2	1	4	3
(d)	4	3	2	1

10. Two shafts having same length and material are joined in series and subjected to a torque of 10 kN-

(a) 16 : 1 (b) 2 : 1
(c) 1 : 2 (d) 1 : 16

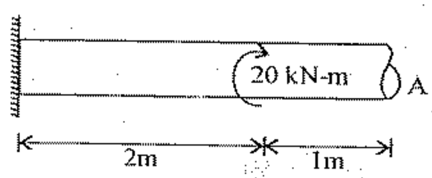
11. When a cantilever shaft of brittle material is subjected to a clockwise twisting moment at the free end, the possible crack propagation will be
(a) 45° clockwise with respect to the axis of shaft
(b) 45° anticlockwise with respect to the axis of shaft
(c) perpendicular to the axis
(d) parallel to the axis

12. A bar of circular cross-section of diameter D is subjected to a torque T at B as shown in the figure given below.



What is the angle of twist at A?
(a) Same as that at B (b) Zero
(c) Twice as that at B (d) Half as that at B

13. A circular shaft of diameter 30 mm having shear modulus G = 80 GPa is subjected to moment as shown below.



What is the maximum shear stresses developed at periphery of shaft at A?
(a) 20.6 MPa (b) 15.3 MPa
(c) 7.4 MPa (d) Zero

14. What would be the shape of the failure surface of a standard cast iron specimen subjected to torque ?
(a) Cup and cone shape at the centre
(b) Plane surface perpendicular to the axis of the specimen
(c) Pyramid type wedge-shaped surface perpendicular to the axis of the specimen.
(d) Helicoidal surface at 45° to the axis of the specimen.

15. A solid shaft has length and diameter 'L_s' and D respectively. A hollow shaft of length L_h, external diameter D, and internal diameter 'd' respectively. Both are of the same material. The ratio of torsional stiffness of hollow shaft to that of solid shaft is

(a) $\left[1+\left(\frac{d}{D}\right)^4\right]\cdot\frac{L_s}{L_h}$ (b) $\left[1-\left(\frac{d}{D}\right)^4\right]\cdot\frac{L_h}{L_s}$

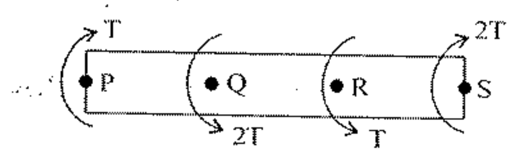
(c) $\left[1-\left(\frac{d}{D}\right)^4\right]\cdot\frac{L_s}{L_h}$ (d) $\left[1-\left(\frac{D}{d}\right)^4\right]\cdot\frac{L_s}{L_h}$

16. **Assertion (A):** A hollow circular shaft has more power transmitting capacity than a solid shaft of same material and same weight per unit length.
Reason (R): In a circular shaft, shear stress developed at a point due to torsion is proportional to

Of these statements

(a) both A and R are true and R is the correct explanation of A
(b) both A and R are true but R is not a correct explanation of A
(c) A is true but R is false
(d) A is false but R is true

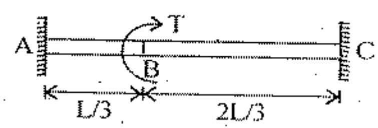
17. A shaft PQRS is subjected to torques at P, Q, R, S as shown in the given figure.



The maximum torque for the shaft section occurs between

(a) P and Q (b) P and R
(c) Q and R (d) R and S

18. A circular shaft of length L is held at two ends without rotation. A twisting moment T is applied at a distance L/3 from left support as shown in the given figure. The twisting moment in the portion AB will be



(a) T (b) T/3
(c) T/2 (d) 2T/3

19. A circular shaft to length L, a uniform cross-sectional area A and modulus of rigidity G is subjected to a twisting moment that produces maximum shear stress τ in the shaft. Strain energy in the shaft is given by the expression $\frac{\tau^2 AL}{KG}$ where K is equal to

(a) 2 (b) 4
(c) 8 (d) 16

DIRECTIONS:

The following items consists of two statements; one labelled as **'Assertion (A)'** and the other as **'Reason (R)'**. You are to examine these two statements carefully and select the answers to these items using the codes given below:

Codes:

(a) Both A and R are true and R is the correct explanation of A
(b) Both A and R are true but R is not a correct explanation of A
(c) A is true but R is false
(d) A is false but R is true

20. **Assertion (A):** To a cantilever beam of circular cross-section, if a moment is applied with its axis perpendicular to the axis of beam, no shear stress will be induced in the beam.

Reason (R): To the above beam, if moment is applied with its axis along the axis of the beam, no bending stress will be induced in the beam.

21. A circular solid shaft of span L = 5 m is fixed at one end and free at the other end. A twisting moment

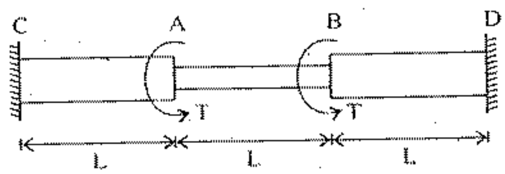
Following statements are made for this shaft:

1. The maximum rotation is 0.01 rad.
2. The torsional strain energy is 1kN-m.

With reference to the above statements, which of the following applies?

(a) Both statements are true
(b) Statement 1 is true but 2 is false
(c) Statement 2 is true but 1 is false
(d) Both the statements are false

22. A circular shaft shown in the figure is subjected to torsion T at two points A and B. The torsional rigidity of portions CA and BD is GJ_1 and that of portion AB is GJ_2. The rotations of shaft at points A and B are θ_1 and θ_2. The rotation θ_1 is



(a) $\frac{TL}{GJ_1 + GJ_2}$ (b) $\frac{TL}{GJ_1}$

(c) $\frac{TL}{GJ_2}$ (d) $\frac{TL}{GJ_1 - GJ_2}$

23. A long shaft of diameter d is subjected to twisting moment T at its ends. The maximum normal stress acting at its cross-section is equal to

(a) zero (b) $\frac{16T}{\pi d^3}$

(c) $\frac{32T}{\pi d^3}$ (d) $\frac{64T}{\pi d^3}$

24. The maximum and minimum shear stresses in a hollow circular shaft of outer diameter 20 mm and thickness 2 mm, subjected to a torque of 92.7 N-m will be

(a) 59 MPa and 47.2 MPa
(b) 100 MPa and 80 MPa
(c) 118 MPa and 160 MPa
(d) 200 MPa and 160 MPa

25. If the diameter of a shaft subjected to torque alone is doubled, then the horse power P can be increased to

(a) 16P (b) 8P
(c) 4P (d) 2P

26. Two shafts of solid circular cross-section are identical except for their diameter 'd_1' and 'd_2'. They are subjected to the same torque T. The ratio of the strain energies stored U_1/U_2 will be

(a) $\left(\frac{d_1}{d_2}\right)^4$ (b) $\left(\frac{d_1}{d_2}\right)^2$

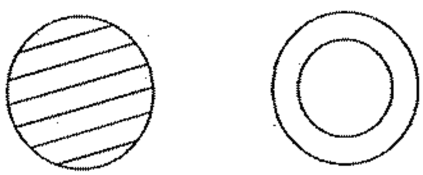
(c) $\left(\frac{d_2}{d_1}\right)^2$ (d) $\left(\frac{d_2}{d_1}\right)^4$

ANSWERS

(b)	8. (a)	15. (c)	22. (b)
(c)	9. (d)	16. (b)	23. (a)
(c)	10. (d)	17. (d)	24. (b)
(a)	11. (a)	18. (d)	25. (b)
(c)	12. (a)	19. (b)	26. (d)
(c)	13. (d)	20. (b)	
(a)	14. (d)	21. (b)	

SOLUTION...

1. (d)



D′ D, d

$$\frac{\tau}{r} = \frac{T}{J} = \frac{G\theta}{L}$$

For same torsion strength

z_p is same

$$z_p = \frac{I_p}{D/2}$$

$$\text{Stiffness} = \frac{GJ}{L}$$

$$I_p\,(\text{Hollow}) = \frac{\pi(D_e^4 - D_i^4)}{32}$$

$$I_p\,(\text{solid}) = \frac{\pi D^4}{32}$$

As I_p is different for hollow and solid circular shaft

Stiffness will be different so statement 3 is wrong.

Given

$$\Rightarrow \frac{\pi\left(D_e^4 - D_i^4\right)}{32(D_e/2)} = \frac{\pi D^3}{16}$$

$$\frac{D_e^4 - D_i^4}{D_e} = D^3$$

$$D_e^3 - \frac{D_i^4}{D_e} = D^3$$

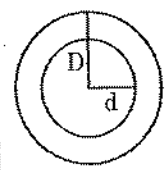
So $D_e > D$ (statement 2 is correct)

Also the weight of hollow shaft will be less than that of solid shaft.

So ans will be (d)

2. (c) $P = \frac{2\pi NT}{60} = \frac{2\pi \times 150 \times 1500}{60} = 7.5\pi$ kW

3. (c) Same Z_p



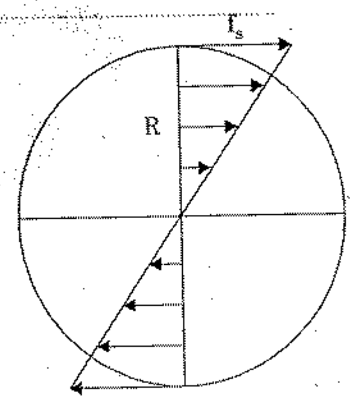
$$Z = \frac{\pi\left(D^4 - d^4\right)}{16D} \quad \left[Z = \frac{\pi D^3}{16}\right]$$

$$\frac{100^4 - 50^4}{100} = D'^3$$

$$10^2(10^4 - 5^4) = D'^3$$

$$D' = 10 \times \left(\sqrt[3]{\frac{9375}{10}}\right)$$

4. (a)



we know $\frac{\tau}{r} = \frac{\tau_{max}}{R}$ ......(i)

Here at any point y, τ will be same

$$f_s = \tau_{max}$$

So $q = \frac{f_s y}{R}$ at (0, y) from (i)

also at point r for (x, 4)

will be $r = \sqrt{x^2 + y^2}$

So $q = \frac{f_s\sqrt{x^2 + y^2}}{R}$ from (i)

5. (c) $\tau_{max} = \frac{16T}{\pi d^3} = \frac{16 \times 4 \times 10^6 \text{N·mm}}{\pi \times 8 \times 8 \times 8 \times 10^3}$

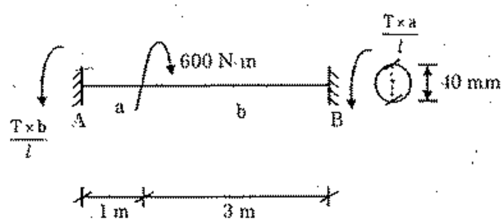
$$= \frac{10^3}{8\pi} \text{ N/mm}^2$$

$$= \frac{125}{\pi} \text{ N/mm}^2$$

6. (c) $P = \frac{2\pi NT}{60}$ W

$$\frac{P_A}{P_B} = \frac{N_A}{N_B} \times \frac{T_A}{T_B} = 2 \times \frac{1}{2} = 1$$

7. (a)



$$T_B = 600 \times \frac{1}{4} = 150 \text{ N-m}$$

$$T_A = 600 \times \frac{3}{4}$$

$$T_A = 450 \text{ N-m}$$

9. (d) $\frac{\tau}{r} = \frac{T}{J} = \frac{G\theta}{L}$

$$T = \frac{GJ}{L}.\theta$$

(i) Torsional Strain Energy $= \int_0^L \frac{T^2 dx}{2GJ}$

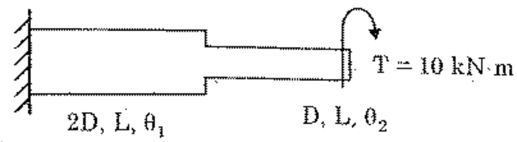
$$U = \frac{T^2 L}{2GJ}.\theta^2$$

$$U = \frac{\left(\frac{GJ}{L}\right)^2 \theta^2 . L}{2GJ}$$

$$U = \left(\frac{GJ}{2L}\right)\theta^2$$

a (i) Circumferential shear stress $\tau = \frac{G.r.\theta}{L}$

10. (a)



Series $\theta = \theta_1 + \theta_2$

$$T_1 = T_2$$

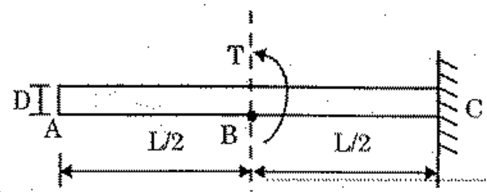
$$\frac{\tau}{r} = \frac{T}{J} = \frac{G\theta}{L}$$

$$\frac{G\theta_1 J_1}{L} = \frac{G.\theta_2.J_2}{L}$$

$$\theta_1 \times \frac{\pi(2D)^4}{32} = \theta_2 \frac{\pi D^4}{32}$$

$$\frac{\theta_1}{\theta_2} = \frac{1}{2^4} = \frac{1}{16}$$

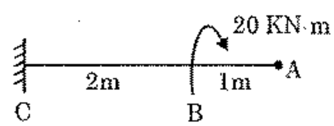
12. (a)



There is no loading over AB portion and it is free to rotate (no restraint to rotate like support)

Hence what ever rotation point 'B' has, (due to loading) the same will be reflected at A.

13. (d)



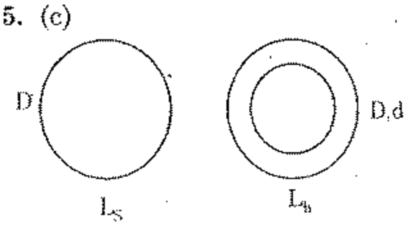
The effect of Torque T = 20 kN-m will be present only over BC portion.

'A' is free and hence rotates freely.

The stress at A = 0

However, 'θ' at point A = 'θ' at point B.

15. (c)

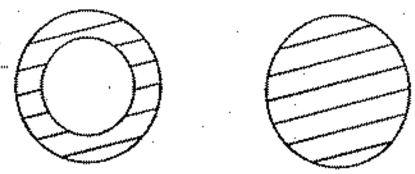


$K_{solid} : K_{hollow}$

$$K = \frac{T}{\theta} = \frac{GJ}{L} = \frac{\frac{\pi D^4}{32}}{L_S} : \frac{\frac{\pi(D^4 - d^4)}{32}}{L_h}$$

$$\frac{K_h}{K_s} = \frac{D^4 - d^4}{D^4}\frac{L_s}{L_h} = \left[1-\left(\frac{d}{D}\right)^4\right]\frac{L_s}{L_h}$$

16. (b)



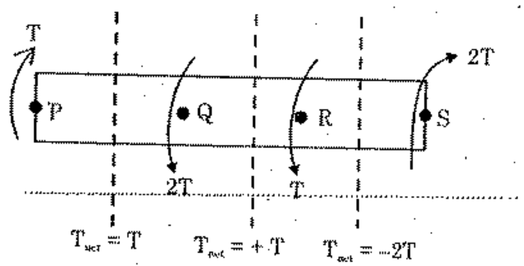
$P = \dfrac{2\pi NT}{60}$ [Same material, same weight/length]

$\dfrac{T}{J} = \dfrac{G\theta}{L}$ [This law is based on the reason explained in (R)].

$\therefore$ Power hollow transmitted > Power solid

$$T_h > T_s$$

17. (d)



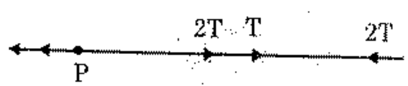
Take any one direction as +ve and other as –ve

Cut the section and find the net torque acting at that section.

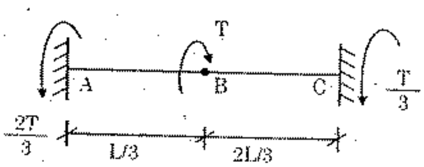
Hence, [Max. Torque (–2T) occurs b/w R & S]

Approach 2

Using right hand thumb rule.



18. (d)



$$T_{AB} = \frac{2T}{3}$$

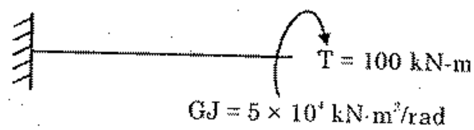
19. (b) L, G, τ $U = \dfrac{1}{2}\tau\times\theta$

Strain Energy $U = \dfrac{T^2L}{2JG}$ [For Solid Shaft]

Using $T = \dfrac{\tau_{max}J}{R}$

$$= \frac{\tau^2_{max}}{4G}\times \text{Vol. of shaft} = \frac{\tau^2}{4G}\times(AL)$$

21. (b)



$$L = 5 \text{ m}$$

$$T = 100 \text{ kN-m}$$

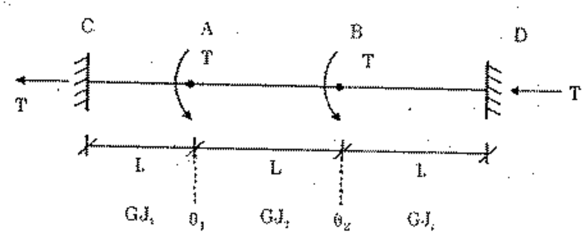
$$\frac{\tau}{r} = \frac{T}{J} = \frac{G\theta}{L}$$

$$GJ = 5\times10^4 \text{ kN-m}^2/\text{rad}$$

$$\theta = \frac{TL}{GJ} = \frac{100\times5}{5\times10^4} = 10^{-2}\text{rad.}$$

$$U = \frac{T^2L}{2GJ} = \frac{10^4\times5}{2\times5\times10^4} = 0.5 \text{ kN-m}$$

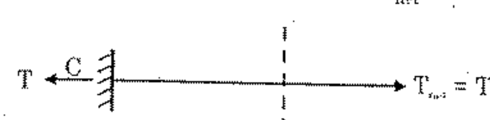
22. (b)



By symmetry, $T_D = T_C = T$

$$\therefore \quad \theta_1 = \frac{TL}{GJ_1}$$

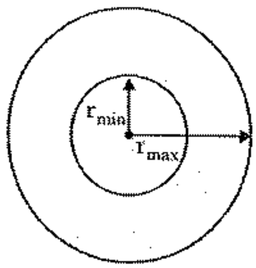
[Cut a section between CA; $T_{net} = T$



23. (a) Torsional stresses will be along the C/S.

$\therefore$ There will be zero normal stresses, if torque 'T' is acting.

24. (b)



$$\frac{\tau}{r} = \frac{T}{J} = \frac{G\theta}{L}$$

$$J = \frac{\pi\left(D^4 - d^4\right)}{32}$$

$$= \frac{\pi}{32}(20^4 - 16^4)$$

$$= 9273.98 \text{ mm}^4$$

$$r_{max} = 10 \text{ mm} \; ; \; r_{min} = 8\text{mm}$$

$$\tau_{max} = \frac{92.7\times10^3}{9273.98}\times\frac{20}{2} = 100 \text{ MPa}$$

$$\tau_{min} = \frac{92.7\times10^3}{9273.98}\times\frac{16}{2} = 80 \text{ MPa}$$

25. (b)

$$P = \frac{2\pi NT}{60}$$

$$P \propto T \propto d^3$$

$$T \propto Z_p \quad \& \quad T \propto d^3$$

$$\frac{\tau}{r} = \frac{T}{J} = \frac{G\theta}{L}$$

if d will be doubled, P will be 8 times

26. (d)

$$U = \frac{T^2L}{2GJ}$$

T, G, L same

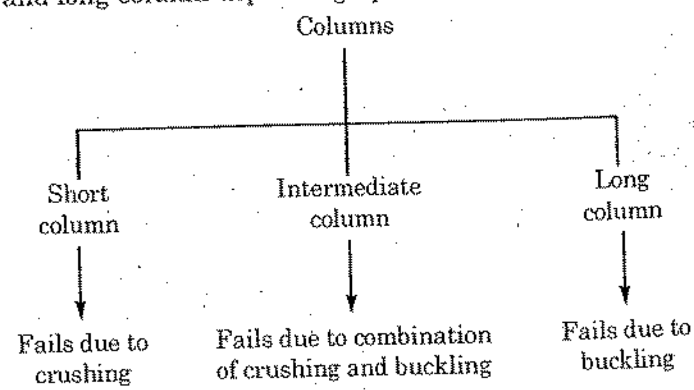
$$\frac{U_1}{U_2} = \frac{J_2}{J_1} = \frac{d_2^4}{d_1^4} = \left(\frac{d_2}{d_4}\right)^4$$

9

Columns

INTRODUCTION

Columns are vertical members carrying vertical loading and moments. It can classified as short column, intermediate column and long column depending upon the mode of failure in the columns.



Normally when length unsupported < 10 × least lateral dimension, the column is called short column

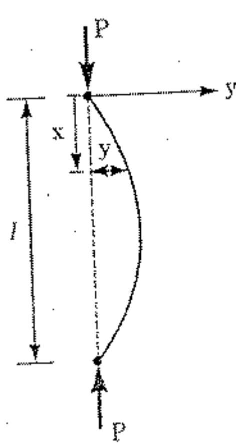
EULER'S THEORY: LONG COLUMN

(a) Pin connected columns (when both the ends are hinged)

$$EI\frac{d^2y}{dx^2} = M = -Py$$

$$\frac{d^2y}{dx^2} = -\frac{Py}{EI}$$

$$\frac{d^2y}{dx^2} + \frac{Py}{EI} = 0$$



Solution of this differential equation is $y = A\sin\sqrt{\frac{P}{EI}}\,x + B\cos\left(\sqrt{\frac{P}{EI}}\,x\right)$

at $x = 0,\ y = 0$
at $x = l,\ y = 0$

$\Rightarrow B = 0$

$\Rightarrow A\sin\left(\sqrt{\frac{P}{EI}}\,l\right) = 0$

'A' cannot be zero, As otherwise all points will have zero deflection

$\Rightarrow \sin\left(\sqrt{\frac{P}{EI}}\,l\right) = 0$

$\Rightarrow \sqrt{\frac{P}{EI}}\,l = n\pi$

$\Rightarrow \frac{Pl^2}{EI} = n^2\pi^2$

$$P = \frac{n^2\pi^2 EI}{l^2}$$

Hence min load for buckling is $\frac{\pi^2 EI}{l^2}$

This load is called critical buckling load P_{cr}. Hence for no buckling $P < P_{cr}$

$\Rightarrow P_{cr} = \frac{\pi^2 EI}{l^2}$ ....

$$P_{cr} = \frac{\pi^2 EI_{min}}{l^2}$$

$$= \frac{\pi^2 E(Ar_{min}^2)}{l^2}$$

As buckling will take place about the axis about which I is I_{min}, hence

$\Rightarrow \frac{P_{cr}}{A} = \frac{\pi^2 E}{\left(\frac{l}{r_{min}}\right)^2}$

As will be explained later 'l' is called effective length (l_{eff}).

$$\frac{P_{cr}}{A} = \frac{\pi^2 E}{\left(\frac{l_{eff}}{r_{min}}\right)^2}$$

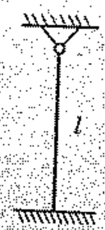
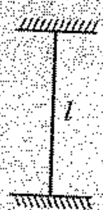
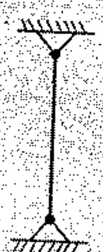
Hence critical buckling stress

$$\boxed{p_{cr} = \frac{\pi^2 E}{\lambda^2}}$$

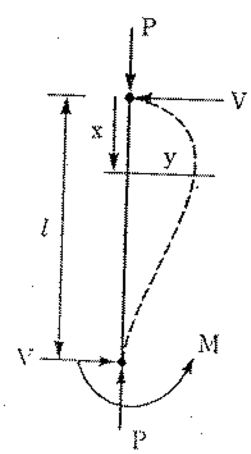
where λ = Slanderness ratio = $\frac{l_{eff}}{r_{min}}$

This critical buckling stress is called Euler's stress

$$\boxed{\text{Euler's stress} = \frac{\pi^2 E}{\lambda^2}}$$

Description	Critical buckling stress	Effective length (l_{eff})	Effective length as per IS code
	$\dfrac{\pi^2 E}{\left[\dfrac{\left(\frac{l}{\sqrt{2}}\right)}{r_{min}}\right]^2}$	$l_{eff} = \dfrac{l}{\sqrt{2}}$	0.8 l
	$\dfrac{\pi^2 E}{\left(\dfrac{\frac{l}{2}}{r_{min}}\right)^2}$	$l_{eff} = \dfrac{l}{2}$	0.65 l
	$\dfrac{\pi^2 E}{\left(\dfrac{2l}{r_{min}}\right)^2}$	$l_{eff} = 2l$	$2l$
	$\dfrac{\pi^2 E}{\left(\dfrac{l}{r_{min}}\right)^2}$	$l_{eff} = l$	l

(b) One end fixed other hinged



$$M = -Py - Vx$$

$$\frac{d^2y}{dx^2} = -\frac{Py}{EI} - \frac{V}{EI}x$$

$$\frac{d^2y}{dx^2} + \frac{Py}{EI} = -\frac{Vx}{EI}$$

Solution of this differential eq. is

$$y = A\sin\left(\sqrt{\frac{P}{EI}}\,x\right) + B\cos\left(\sqrt{\frac{P}{EI}}\,x\right) - \frac{V}{P}x$$

at $x = 0$, $y = 0$ $\Rightarrow$ $B = 0$ (i)

at $x = l$, $y = 0$ $\Rightarrow$ $\boxed{A\sin\left(\sqrt{\frac{P}{EI}}\,l\right) - \frac{Vl}{P} = 0}$ .... (ii)

$$\frac{dy}{dx} = A\sqrt{\frac{P}{EI}}\cos\left(\sqrt{\frac{P}{EI}}\,x\right) - \frac{V}{P}$$

$$\text{at } x = l, \quad \frac{dy}{dx} = 0$$

$$\Rightarrow \boxed{A\sqrt{\frac{P}{EI}}\cos\left(\sqrt{\frac{P}{EI}}\,l\right) = \frac{V}{P}} \quad \text{------ (iii)}$$

From (ii) and (iii)

$$\frac{1}{\sqrt{\frac{P}{EI}}}\tan\left(\sqrt{\frac{P}{EI}}\,l\right) = l$$

$$\tan\left(\sqrt{\frac{P}{EI}}\,l\right) = \sqrt{\frac{P}{EI}}\,l$$

The smallest root of this is

$$l\sqrt{\frac{P}{EI}} = 4.4934$$

$$\Rightarrow \frac{l^2P}{EI} = 20.1906 = 2.044\pi^2$$

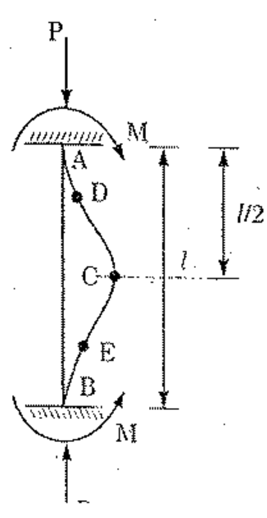
$$\Rightarrow \frac{l^2P}{EI} = 2\pi^2$$

$$P = \frac{2\pi^2 EI}{l^2}$$

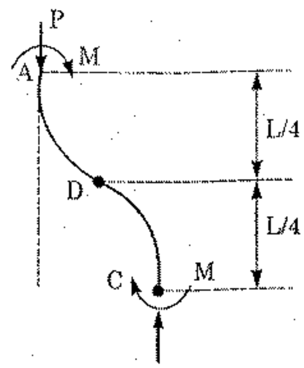
Thus in expression $\boxed{P_{cr} = \dfrac{\pi^2 E}{\left(\dfrac{l_{eff}}{r_{min}}\right)^2}}$, $\boxed{l_{eff} = \dfrac{l}{\sqrt{2}}}$

(c) Both Ends Fixed

For symmetry, shear V = 0 at supports



From symmetry, point 'C' should be $\frac{L}{2}$ from end A.

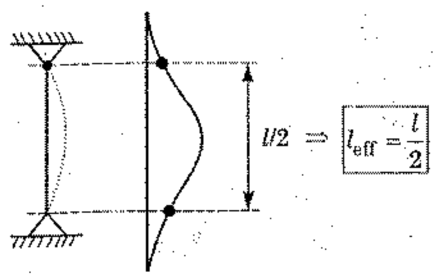


From the above situation it is clear that bending moment is zero at point D and there is symmetry at point D. Hence point D is the point of inflection.

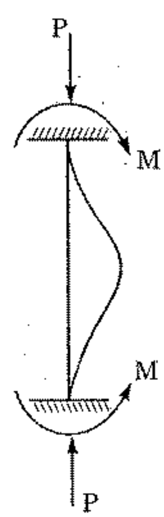
$$EI\frac{d^2y}{dx^2} = M - Py$$

$$\frac{d^2y}{dx^2} = \frac{M}{EI} - \frac{P}{EI}y$$

$$\frac{d^2y}{dx^2} + \frac{Py}{EI} = \frac{M}{EI}$$



Solution of this differential equation is



$$y = A\sin\left(\sqrt{\frac{P}{EI}}\,x\right) + B\cos\left(\sqrt{\frac{P}{EI}}\,x\right) + \frac{M}{P}x$$

at $x = 0,\ y = 0 \Rightarrow B = 0$

$$\frac{dy}{dx} = \sqrt{\frac{P}{EI}}\cos\left(\sqrt{\frac{P}{EI}}\,x\right) + \frac{M}{P}$$

at $x = 0,\ \frac{dy}{dx} = 0 \Rightarrow A\sqrt{\frac{P}{EI}} = -\frac{M}{P}$

at $x = l,\ \frac{dy}{dx} = 0$

$$\Rightarrow -\frac{M}{P}\cos\left(\sqrt{\frac{P}{EI}}\,l\right) + \frac{M}{P} = 0$$

$$\Rightarrow \cos\left(\sqrt{\frac{P}{EI}}\,l\right) = 1$$

$$\Rightarrow \sqrt{\frac{P}{EI}} \times l = 0, 2\pi, 4\pi \ldots$$

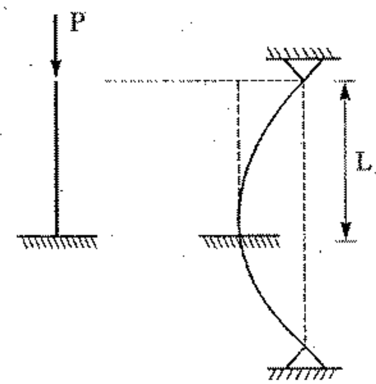
Min. Significant value is $\sqrt{\frac{P}{EI}}\,l = 2\pi$

$$\Rightarrow \quad \frac{Pl^2}{EI} = 4\pi^2$$

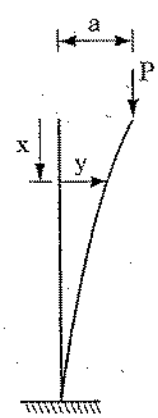
$$P = \frac{4\pi^2 EI}{l^2} = \frac{\pi^2 EI}{\left(\frac{l}{2}\right)^2}$$

$$\Rightarrow \quad \boxed{l_{eff} = \frac{l}{2}}$$

(d) One end fixed, other free



The column behaves as upper half of pin connected column.



$$\Rightarrow \quad l_{eff} = 2l$$

$$\frac{d^2y}{dx^2} = \frac{P(a-y)}{EI}$$

$$\frac{d^2y}{dx^2} + \frac{Py}{EI} = \frac{Pa}{EI}$$

Solution of this differential equation

$$y = A\sin\left(\sqrt{\frac{P}{EI}}\,x\right) + B\cos\left(\sqrt{\frac{P}{EI}}\,x\right) + a$$

at $x = 0$, $y = a \;\Rightarrow\; B = 0$

$$\frac{dy}{dx} = A\sqrt{\frac{P}{EI}}\cos\left(\sqrt{\frac{P}{EI}}\,x\right) + 0$$

at $x = l$, $\frac{dy}{dx} = 0$

$$\Rightarrow \quad \cos\left(\sqrt{\frac{P}{EI}}\,l\right) = 0$$

$$\Rightarrow \quad \sqrt{\frac{P}{EI}}\,l = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2} \ldots$$

⇒ Min loading corresponds to this is given by

$$\sqrt{\frac{P}{EI}}\, l = \frac{\pi}{2}$$

$$\frac{Pl^2}{EI} = \frac{\pi^2}{4} \Rightarrow P = \frac{\pi^2 EI}{4l^2}$$

$$\Rightarrow P = \frac{\pi^2 EI}{(2l)^2}$$

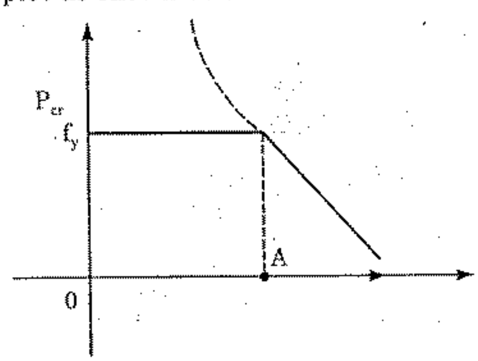
$$\Rightarrow \boxed{l_{eff} = 2l}$$

ASSUMPTION OF EULER'S THEORY

- It is valid for long columns.
- EI is uniform and material is isotropic.
- Load is purely axial.
- Axis of the shaft is perfectly straight when unloaded.

$$P_{cr} = \frac{\pi^2 E}{\lambda^2}$$

if we plot P_{cr} against λ we get a plot as shown below:



for $P_{cr} = f_y$, $\lambda = \sqrt{\frac{\pi^2 E}{f_y}}$

Hence Euler's formula cannot be applied if $\lambda < \sqrt{\frac{\pi^2 E}{f_y}}$ because for $\lambda < \sqrt{\frac{\pi^2 E}{f_y}}$, stress becomes more than f_y and crushing will occur. Crushing is a characteristic of short column whereas Euler's formula is only valid for long columns.

Note that for $f_y = 250$ and $E = 2 \times 10^5$ N/mm².

$\lambda_{max} = 88.89$ for Euler's law to be applicable.

RANKINE FORMULA

Rankine formula is valid both for short and long column. Hence failure by both crushing and buckling has been accounted for in this case

if P = Failure load

$$P_E = \text{Buckling load} = \frac{\pi^2 E}{l^2}$$

Then

$$\frac{1}{P} = \frac{1}{P_C} + \frac{1}{P_E} \quad \text{---} (\alpha)$$

$$P = \frac{P_C}{1 + \frac{P_C}{P_E}} = \frac{f_C A}{1 + \frac{f_C A}{\frac{\pi^2 EA}{\lambda^2}}}$$

$$P = \frac{f_C A}{1 + \left(\frac{f_C}{\pi^2 E}\right)\lambda^2} = \frac{f_C A}{1 + a\lambda^2}$$

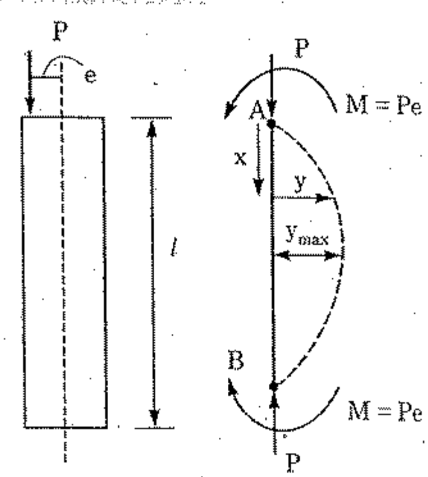
$$\boxed{P = \frac{f_C A}{1 + a\lambda^2}} \quad \text{Rankine formula}$$

a = Rankine constant
f_C = Cushing stress in compression
This is an emperical formula in which 'a' is determined experimentally.

***Note:** For solving problems formula 'α' is most frequently used.*

LONG COLUMN UNDER ECCENTRIC LOADING: SECANT FORMULA

Solution of Secant formula solution is obtained by trial and errors.



$$\frac{d^2y}{dx^2} = -\frac{Pe}{EI} - \frac{Py}{EI},$$

Let
$$\frac{P}{EI} = w^2$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} + w^2 y = -w^2 e$$

Solution of this equation is

$$y = A \sin wx + B \cos wx - e$$

at $x = 0$, $y = 0 \Rightarrow B = e$
at $x = l$, $y = 0 \Rightarrow A \sin wl = e(1 - \cos wl)$

$$A \sin wl = 2e \sin^2 \frac{wl}{2}$$

$$\Rightarrow \quad A = \frac{2e \sin^2 \frac{wl}{2}}{2 \sin \frac{wl}{2} \cos \frac{wl}{2}} = e \tan \frac{wl}{2}$$

$$\Rightarrow \quad y = e \tan \frac{wl}{2} \sin wx + e \cos wx - e$$

$$y = e\left(\tan\frac{wl}{2}\sin wx + \cos wx - 1\right)$$

y_{max} is obtained at $x = \frac{l}{2}$

$$\Rightarrow \boxed{y_{max} = e\left(\sec\frac{wl}{2} - 1\right)}$$

Note that y_{max} becomes ∞ if $\frac{wl}{2} = \frac{\pi}{2}$

i.e. $\sqrt{\frac{P}{EI}}\,\frac{l}{2} = \frac{\pi}{2}$

$\boxed{P = \frac{\pi^2 EI}{l^2}}$ Euler load

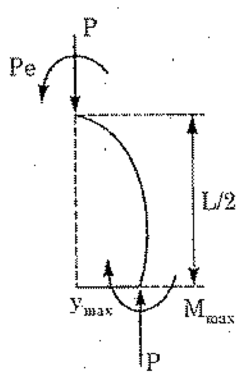
$$\Rightarrow P_{cr} = \frac{\pi^2 EI}{l^2}$$

$$\Rightarrow \boxed{y_{max} = e\left(\sec\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}} - 1\right)}$$

Max stress will occur in section where BM is max, This occurs at mid point

$$\Rightarrow \sigma_{max} = \frac{P}{A} + \frac{M_{max}C}{I}$$

Also



$Pe - M_{max} + Py_{max} = 0$

$M_{max} = P(y_{max} + e)$

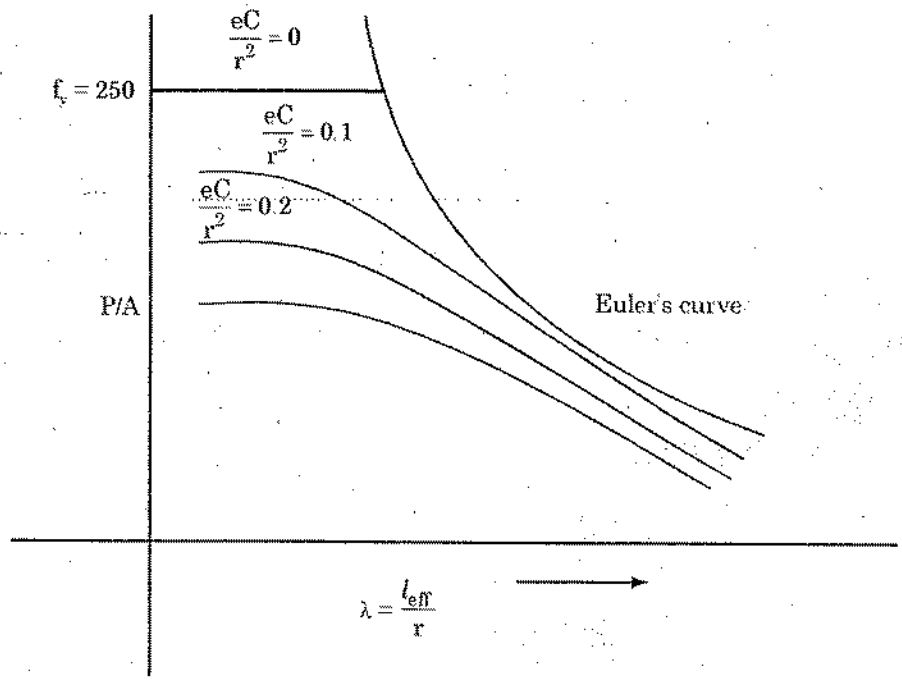
$$\sigma_{max} = \frac{P}{A}\left[1 + \frac{(y_{max} + e)\,C}{r^2}\right]$$

r = radius of gyration

$$\Rightarrow \sigma_{max} = \frac{P}{A}\left[1 + \frac{eC}{r^2}\sec\sqrt{\frac{P}{EI}} \times \frac{L}{2}\right]$$

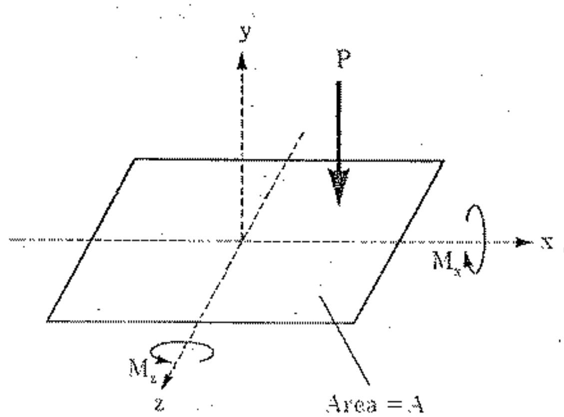
$$\sigma_{max} = \frac{P}{A}\left[1+\frac{eC}{r^2}\sec\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right]$$

$$\frac{P}{A} = \frac{\sigma_{max}}{1+\frac{eC}{r^2}\sec\left(\frac{1}{2}\sqrt{\frac{P}{EA}}\,\frac{l_{eff}}{r}\right)} \quad \text{Secant formula.}$$



For large value of λ, Secant formula tends to Eular's formula.
Hence Secant formula is basically used to calculate stress under intermediate column condition.
Design of Column: Column is designed using the following formula.

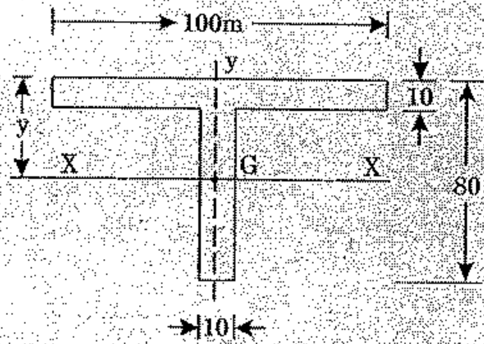
$\frac{P}{A} \le \sigma_{all}$ (σ_{all} = allowable stress) → For concentric loading



$$\frac{P/A}{(\sigma_{..})} + \frac{\left|\frac{M_x Z_{max}}{I_x}\right|}{(\sigma_{..})} + \frac{\left(\frac{M_z X_{max}}{I_z}\right)}{(\sigma_{..})} \le 1, \text{ for Eccentric loading}$$

Example 1

Determine the buckling load for a strut of tee section, the flange width being 100 mm, overall depth 80 mm and both flange and stem 10 mm thick. The strut is 3m long and is hinged at both ends. Take E = 200 GN/m².



Sol: Let centroid be at a depth $\bar{y}$ from top fibre. Then

$$\bar{y} = \frac{100\times10\times5+70\times10\times(35+10)}{100\times10+70\times10} = 21.47 \text{ mm}$$

$$I_x = \frac{1}{12}\times100\times10^3+100\times10(21.47-5)^2+\frac{1}{12}\times10\times70^3+10\times70(45-21.47)^2$$

$$= 952990.2 \text{ mm}^4$$

$$I_y = \frac{1}{12}\times10\times100^3+\frac{1}{12}\times70\times10^3$$

$$= 839166.67 \text{ mm}^4$$

$\therefore$ Least moment of inertia $I = I_y = 839166.67$ mm⁴

Since both ends are hinged

Effective length = Actual length

$L = l = 3\text{m} = 3000$ mm

$E = 200$ G N/m² = 200 × 1000 N/mm²

$$P_{cr} = \frac{\pi^2 EI}{L^2}$$

$$= \frac{\pi^2\times200\times1000\times839166.67}{(3000)^2}$$

$= 184049.85$ N

$= 184.05$ kN (Ans)

OBJECTIVE QUESTIONS

1. If the crushing stress in the material of a mild steel column is 3300 kg/cm^2. Euler's Formula for crippling load is applicable for slenderness ratio equal to/greater than
   (a) 40 (b) 50
   (c) 60 (d) 80

2. Which one of the following pairs is not correctly matched?

Boundary conditions of column	Euler's buckling load
(a) Pin-Pin	$\frac{\pi^2 EI}{L^2}$
(b) Fixed-Fixed	$\frac{4\pi^2 EI}{L^2}$
(c) Fixed-Free	$\frac{0.25\pi^2 EI}{L^2}$
(d) Fixed-Pin	$\frac{\sqrt{2}\pi^2 EI}{L^2}$

3. For a circular column having its ends hinged, the slenderness ratio is 160. The l/d ratio of the column is
   (a) 80 (b) 57
   (c) 40 (d) 20

4. In the statements "the buckling load for a slender column rigidly fixed at both ends is about X times that of a geometrically identical column but with hinged ends". X stands for
   (a) two (b) three
   (c) four (d) six

5. A circular column of length 2 m has Euler's crippling load of 1.5 kN. If the diameter of the column is reduced by 10% the reduction in the crippling load will be
   (a) 10% (b) 20%
   (c) 30% (d) more than 30%

6. **Assertion (A)**: Two identical slender columns, one of high strength alloy steel and the other of ordinary structural steel will have approximately the same buckling failure strength under axial load.
   Reason (R): The moduli of elasticity of different steels are approximately same.
   Of these statements
   (a) both A and R are true and R is the correct explanation of A
   (b) both A and R are true but R is not a correct explanation of A
   (c) A is true but R is false
   (d) A is false but R is true

7. **Assertion (A)**: The buckling load obtained by the use of Euler's formula may be much more than the actual buckling load.
   Reason (R): Euler's formula does not take into account the effect of direct compressive stress.
   Of these statements
   (a) both A and R are true and R is the correct explanation of A

(c) A is true but R is false
(d) A is false but R is true

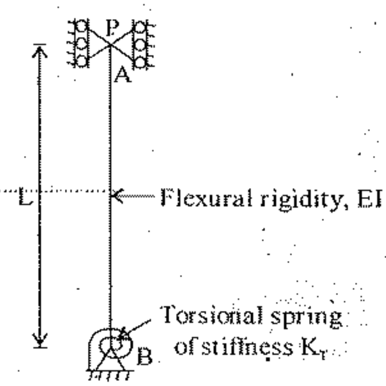
8. A long structural column (length = L) with both ends hinged is acted upon by an axial compressive load, P. The differential equation governing the bending of column is given by:

$$EI\frac{d^2y}{dx^2} = -Py$$

where y is the structural lateral deflection and EI is the flexural rigidity. The first critical load on column responsible for its buckling is given by

(a) $\frac{\pi^2 EI}{L^2}$ (b) $\frac{\sqrt{2}\pi^2 EI}{L^2}$
(c) $\frac{2\pi^2 EI}{L^2}$ (d) $\frac{4\pi^2 EI}{L^2}$

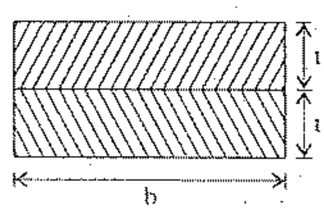
9. The buckling load $P = P_{cr}$ for the column AB in figure, as K_T approaches infinity, becomes $\alpha\frac{\pi^2 EI}{L^2}$



where α is equal to
(a) 0.25 (b) 1.00
(c) 2.05 (d) 4.00

10. A steel column, pinned at both ends, has a buckling load of 200 kN. If the column is restrained against lateral movement at its mid-height, its buckling load will be
(a) 200 kN (b) 283 kN
(c) 400 kN (d) 800 kN

11. Cross-section of a column consisting of two steel strips, each of thickness t and width b is shown in the figure below. The critical loads of the column with perfect bond and without bond between the strips are P and P_0 respectively. The ratio P/P_0 is



(a) 2 (b) 4
(c) 6 (d) 8

12. If the Euler load for a steel column is 1000 kN and crushing load is 1500 kN, the Rankine load is equal to

(a) 2500 kN (b) 1500 kN

(c) 1000 kN (d) 600 kN

13. **Assertion (A):** Rankine's theory is generally used for finding out the buckling load of intermediate columns.
Reason (R): Euler's theory gives higher values for buckling loads in intermediate columns.
Of these statements

(a) both A and R are true and R is the correct explanation of A
(b) both A and R are true but R is not a correct explanation of A
(c) A is true but R is false
(d) A is false but R is true

14. The slenderness ratio of a compression member in the context of Rankine's formula is defined as

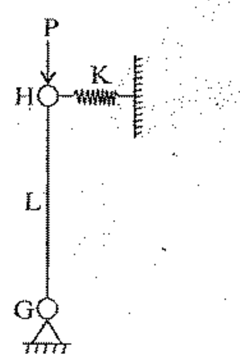
(a) $\dfrac{\text{Length}}{\text{Least lateral dimension}}$ (b) $\dfrac{\text{Effective length}}{\text{Least radius of gyration}}$

(c) $\dfrac{\text{Effective length}}{\text{Least lateral dimension}}$ (d) $\dfrac{\text{Length}}{\text{Least radius of gyration}}$

15. The axial load which just produces the condition of elastic instability in a column is

(a) Rankine load (b) Euler load
(c) Yield load (d) Crushing load

16. A rigid bar GH of length L is supported by a hinge and a spring of stiffness K as shown in the figure below. The buckling load, P_{cr}, for the bar will be



(a) 0.5 KL (b) 0.8 KL
(c) 1.0 KL (d) 1.2 KL

17. A column of height H and area at top A has the same strength throughout its length, under its own weight and applied stress p_0 at the top. Density of column material is ρ. To satisfy the above condition, the area of the column at the bottom should be

(a) $\dfrac{Ae^{Hp_0}}{\rho}$ (b) $\dfrac{Ae^{-\rho H}}{p_0}$

(c) $Ae^{(+\rho H/p_0)}$ (d) $\dfrac{A\rho^{H}}{e^{p_0}}$

18. The resultant cuts the base of a circular column of diameter 'd' with an eccentricity equal to one-fourth of 'd'. The ratio between the maximum compressive stress and the maximum tensile stress is

(a) 3 (b) 4
(c) 5 (d) infinity

ANSWERS

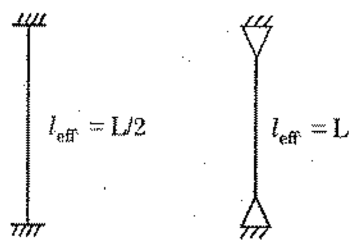
1. (d)	6. (a)	11. (b)	16. (c)
2. (d)	7. (a)	12. (d)	17. (c)
3. (c)	8. (a)	13. (a)	18. (a)
4. (c)	9. (a)	14. (b)	
5. (d)	10. (d)	15. (b)	

SOLUTION...

1. (d) Eulers formula is not valid.
When $\lambda < 80 \rightarrow$ Limitation
Another limitation : valid only in proportional limit.

3. (c)
$$\lambda = \frac{\text{left}}{r_{min}} = 160 = \frac{l}{d/4} = 160$$
$$r = \sqrt{\frac{I_{max}}{A}}$$
$$r = \sqrt{\frac{\pi d^4/64}{\pi d^2/4}} = \frac{d}{4} = \frac{160}{4} = 40$$
$$\frac{l}{d} = 40$$

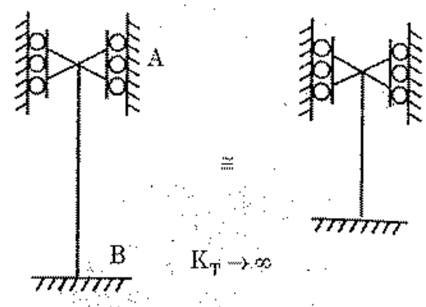
4. (c)



$$P = \frac{\pi^2 EI}{L^2} \text{ Eulers Formula}$$
$$\frac{P_{fixed}}{P_{hinged}} = \frac{(L)^2}{\left(\frac{L}{2}\right)^2} = 4$$

5. (d)
$$P = \frac{\pi^2 EI}{L^2}$$
$$P \propto d^4$$
$$I = \frac{\pi d^4}{64}$$
$$\frac{P}{P'} = \frac{d^4}{(0.9)^4 d^4} = 1.524$$
$$P' = 0.65P \quad \text{reduction } (> 30\%)$$
Reduction (> 30%)

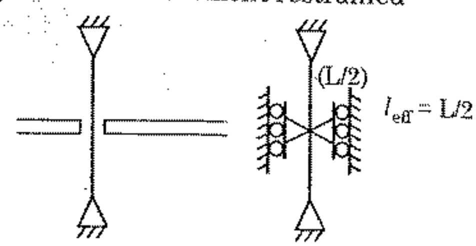
9. (c)



This means infinite stiffness against 'T'

$\Rightarrow$ no rotation / twist at B. So $l_{eff} = L/\sqrt{2}$

10. (d) Lateral movement restrained

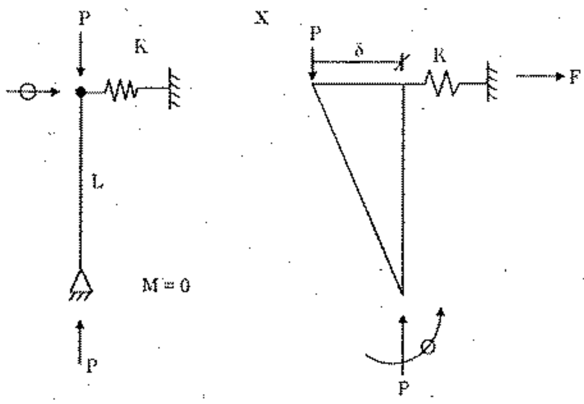


$$l_{eff} = l/2$$
$$P' = 4P$$

11. (b) $P \propto I$
$$I = \frac{bd^3}{12}$$
$$\frac{P}{P_0} = \frac{(2t)^3}{2 \times t^3} = 4$$

12. (d)
$$\frac{1}{P_R} = \frac{1}{P_C} + \frac{1}{P_R}$$
$$P_R = \frac{1000 \times 1500}{2500} = 600 \text{ kN}$$

16. (c)



No load } initial
⇒ No deflection }

Now give 'δ' deflection

$$P \times \delta = F \times L$$

$$P = \frac{K\delta L}{\delta} = KL$$

18. (a)

$$e = \frac{d}{4} \Rightarrow \frac{P}{A} \pm \frac{M}{Z}$$

$$= \frac{P}{A} \pm \frac{P \times \left(\frac{d}{4}\right)}{\frac{\pi d^3}{32} \frac{\pi d^2}{4} \times \left(\frac{d}{8}\right)}$$

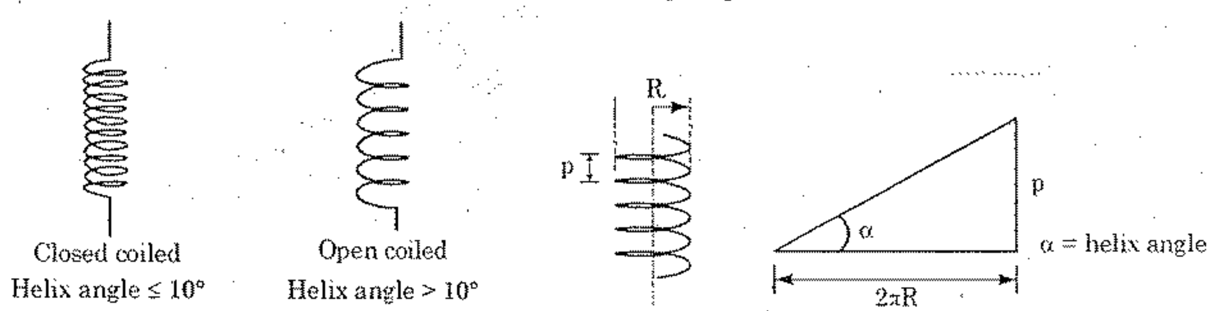
$$= \frac{P}{A}(1 \pm 2) = 3\frac{P}{A}, -\frac{P}{A}$$

10

Springs

INTRODUCTION

- The primary function of a spring is to deflect or distort under load and to recover its original shape when the load is released. During deflection or distortion, it absorbs energy and releases the same as and when required.
- Springs are used in many engineering applications such as automobiles and railway buffers in order to cushion, absorb or control energy due to shock and vibrations.
- Springs will suffer a sizeable change in form without being distorted permanently when the loads are applied.
- Springs are generally classified as leaf springs or helical springs.
- Leaf spring consist of a number of thin curved plates, each of same thickness and width but of different lengths all bent to the same curvature.
- Helical springs are formed by coiling thick spring wire into a helix.
- Helical springs are classified into two groups. When the helix angle is less than about 10°, it is named as close-coiled helical spring. In such springs the wire experiences too little bending or direct shear stress and their effect is neglected. Torsional stresses are predominant in such springs.
- If, however the helix angle is significant, then the wire experiences both torsional and bending stresses. Such type of spring is termed as open-coiled helical spring.



Closed coil helical spring is called <u>torsion spring</u> and open coiled helical spring is called <u>bending spring</u>.

Proof load

It is the greatest load that the spring can carry without getting permanently distorted.

Proof stress

Proof resilience

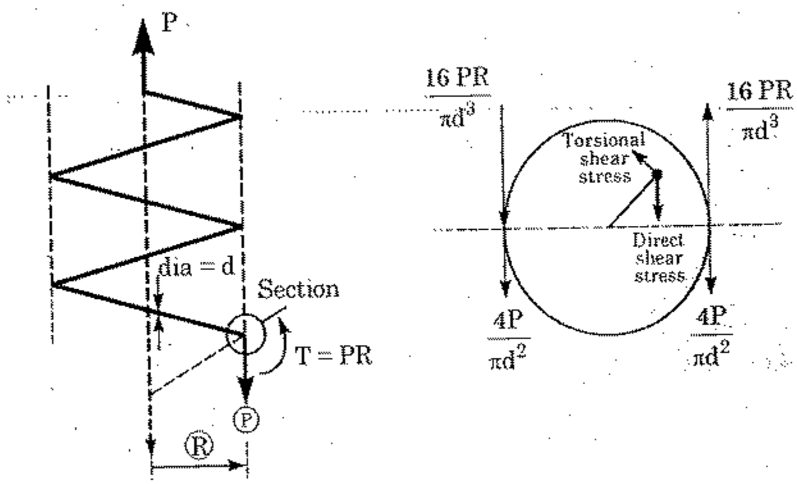
It is the strain energy stored in the spring when it has been subjected to the maximum load i.e. proof load.

Spring constant (stiffness of the spring)

It is the load per unit deflection. It is expressed in N/m or kN/m.

CLOSED COIL HELICAL SPRING

A helical spring is said to be close-coiled when the obliquity of the wire is small, i.e. the pitch of the coils is very small. Each turn can be regarded as practically lying in planes at right angles to the axis of the helix.



- Max shear stress occurs at the inner end of the spring

$$\tau_{inner} = \tau_{max} = \frac{16\,PR}{\pi d^3}\left[1 + \frac{d}{4R}\right]$$

$$\tau_{outer} = \frac{16\,PR}{\pi d^3}\left(1 - \frac{d}{4R}\right)$$

- Generally $\frac{d}{4R} << 1$, hence effect of direct shear is neglected.

Strain Energy Stored in Spring

$$U = \frac{T^2L}{2GJ}$$

$T = PR$

$L = 2\pi Rn$

n = no. of coils

$$J = \frac{\pi d^4}{32}$$

$$U = \frac{P^2R^2 \times 2\pi Rn}{2G \times \frac{\pi d^4}{32}}$$

$$U = \frac{32P^2R^3n}{\ldots}$$

Axial deflection of spring (δ)

$$\delta = \frac{\partial U}{\partial P} = \frac{64\,PR^3n}{Gd^4}$$

$$\boxed{\delta = \frac{64\,PR^3n}{Gd^4}}$$

Stiffness of Spring

Stiffness of spring is defined as the load required to produce unit deflection.

$$\boxed{P = K\cdot\delta}$$

K = Stifiness of spring

$$\Rightarrow \quad \boxed{K = \frac{Gd^4}{64\,R^3n}}$$

Proof load

Proof load is the maximum load carrying capacity of the spring, without getting permanently distored. We know that

$$f_s = \frac{16\,WR}{\pi d^3}$$

$$W_{max} = \frac{\pi d^3}{16R}\times(f_s)_{max}$$

$$\Rightarrow \quad \boxed{\text{Proof load} = \frac{\pi d^3}{16R}\times(f_s)_{max}}$$

where $(f_s)_{max}$ is the allowable shear stress of the material of the spring.

CHANGE OF STIFFNESS OF SPRING ON CUTTING

If stiffness of a spring of 'x_1' number of coils be $K_1 = \frac{Gd^4}{64\,R^3n_1}$ and spring is cut into two equal parts then stiffness of each part $[K_2]$

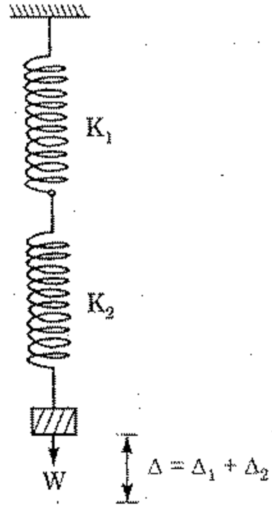
then,

$$K_2 = \frac{Gd^4}{64\,R^3\frac{n_1}{2}} = 2\times K_1$$

$$\boxed{K_2 = 2K_1}$$

SERIES AND PARALLEL COMBINATION OF SPRINGS

(i) Series Combination



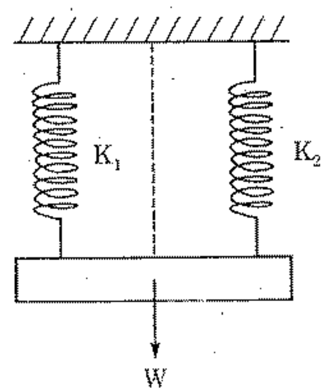
$$\Delta = \frac{W}{K_{eq}} = \frac{W}{K_1} + \frac{W}{K_2}$$

$$\Rightarrow \boxed{\frac{1}{K_{eq}} = \frac{1}{K_1} + \frac{1}{K_2}}$$ Series connection

$$\text{Flexibility} = f = \frac{1}{K}$$

$$\Rightarrow \boxed{f_{eq} = f_1 + f_2}$$ Series connection

(ii) Parallel Connection



$$\Delta = \Delta_1 = \Delta_2$$

$$\frac{W}{K_{eq}} = \frac{W_1}{K_1} = \frac{W_2}{K_2}$$

also, $W = W_1 + W_2$

$$\Rightarrow W = \frac{K_1 W}{K_{eq}} + \frac{K_2 W}{K_{eq}}$$

$$\Rightarrow \boxed{K_{eq} = K_1 + K_2}$$ Parallel connection

$$\boxed{\frac{1}{f_{eq}} = \frac{1}{f_1} + \frac{1}{f_2}}$$

f = flexibility

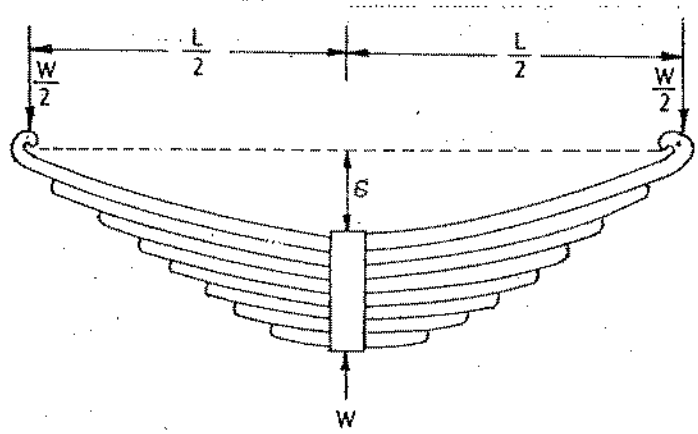
LEAF SPRINGS

- This type of springs are commonly used in carriages such as cars, railway wagons etc and they are also termed as laminated or carriage springs.

- It is made up of a number of leaves of equal width and thickness, but varying length placed in laminations and loaded as a beam.
- The lengths of the plates are so adjusted that the maximum bending stress remains same in every plate and thereby it behaves like a beam of uniform strength.
- It is assumed that each plate is free to slide relative to the adjacement plates as the spring deflects and the ends of each plate are tapered to provide a uniform change in effective breadth. Further it is assumed that the plates are bent to the same radius so that they contact only at their edges.

Stress in Springs

Figure shows a carriage spring carrying a central vertical load W, which is balanced by equal end reactions $\frac{W}{2}$



Let W = load on the spring
R = initial radius of curvature of plates
δ = initial central deflection
b = width of each plate
t = thickness of each plate
n = number of leaves (plates)
L = span of the spring
f = bending stress

- Section modulus for a single plate or laminate $= \frac{bt^2}{6}$
- Section modulus for the whole spring (having n laminates) $= n\left(\frac{bt^2}{6}\right)$

Maximum bending moment, $M = \frac{WL}{4}$

But $\frac{M}{I} = \frac{f}{y}$

or $M = f \cdot Z$

$$\frac{WL}{4} = f\left(\frac{n\,bt^2}{6}\right)$$

$$\boxed{f = \frac{3}{2} \times \frac{WL}{nbt^2}}$$

Strain Energy

$$\text{Resilience due to bending} = \frac{f^2}{6E}$$

$$\therefore \quad \text{Total strain energy, } U = \frac{f^2}{6E} \times (\text{Volume of the equivalent plate})$$

$$\text{Volume} = \left(\frac{nb}{2} Lt\right)$$

$$\therefore \quad \text{Strain Energy } U = \frac{f^2}{6E}\left(\frac{nb}{2} Lt\right)$$

Substituting for f,

$$\boxed{U = \frac{\left(\frac{3}{2} \times \frac{WL}{nbt^2}\right)^2}{6E} \times \left(\frac{nb}{2} Lt\right)}$$

$$\text{Work done by the load} = \frac{1}{2} \times W \times \delta$$

Equating these two

$$\frac{1}{2}(W \times \delta) = \frac{\left(\frac{3}{2} \times \frac{WL}{nbt^2}\right)^2}{12\,E} nbLt$$

$$\therefore \quad \boxed{\delta = \frac{3WL^3}{8Enbt^3}}$$

Stiffness of Springs

It is defined as the load required to produce unit deflection.

$$\therefore \quad \text{Spring constant, } S = \frac{W}{\delta}$$

$$= \frac{8\,Enbt^3}{3L^3}$$

Proof Load

If W_0 is the load required to make the spring flat, it is known as the proof load.
If δ_0 is the deflection corresponding to proof load W_0,

then, $\delta_0 = \dfrac{3W_0L^3}{8Enbt^3}$

$\therefore$ Proof load, $W_0 = \frac{8Enbt^3}{3L^3}\delta_0$

Practical Applications

Leaf springs are extensively used in railway carriages, railway wagons, trucks, trollies, buses and cars etc. The common purpose of all kinds of springs is to absorb energy and to release it as and when required. Carriage springs are used normally to absorb shock. In other words, they act as primarily shock absorbers.

Example 1

A helical spring in which the slope of the helix may be assumed small, is required to transmit a maximum pull of 1 kN and to extend 10 mm for 200 N load. If the mean diameter of the coil is to be the 80 mm, find the suitable diameter for the wire and number of coils required. Take G = 80 GPa and allowable shear stress as 100 MPa.

Sol: Shear stress, $f_s = \frac{8WD}{\pi d^3}$

$$\therefore d^3 = \frac{8WD}{\pi f_s} = \frac{8\times 1000\times 80}{\pi\times 100}$$

Here, we have W = 1000 N D = 80 mm

f_s = 100 MPa

$\therefore$ Diameter of spring wire 12.68 mm.

Now δ = 10 mm for W = 200 N.

$$\therefore \text{Spring constant, } k = \frac{W}{\delta} = \frac{200}{10} = 20 \text{ N/mm}$$

$$= \frac{20}{10^{-3}} = 2\times 10^4 \text{ N/m}$$

We have, G = 80 GPa

$$\delta = \frac{8WD^3n}{Gd^4}$$

$$\therefore n = \frac{Gd^4}{8\left(\frac{W}{\delta}\right)D^3} = \frac{(80\times 10^3)(12.68)^4}{8\times 20\times (80)^3}$$

$$= 25.28$$

Number of coils required = 25.28 say 26.

Example 2

A truck weighing 30 kN and moving at 6 km/hr has to be brought to rest by a buffer. Find how many springs each of 20 coils will be required to store the energy of motion during a compression of 200 mm. The spring is made out of 20 mm diameter steel rod coiled to a mean diameter of 200 mm. Take G = 100 GPa.

Sol: Weight of truck = 30 kN

$$\text{Mass of the truck} = \left(\frac{30\times10^3}{9.81}\right)\text{kg}$$

$$\text{Velocity of the truck} = 6 \text{ km/hr}$$

$$= \left(\frac{6\times10^3}{60\times60}\right)\text{m/s}$$

$$\text{Kinetic energy of the truck} = \frac{1}{2}mv^2$$

$$= \frac{1}{2}\left(\frac{30\times10^3}{9.81}\right)\left(\frac{6000}{60\times60}\right)^2$$

$$= 4248 \text{ Nm}$$

For the spring, let N = number of springs.

$n = 20$ $\delta = 200$ mm

$d = 20$ mm $G = 100$ GPa

$D = 200$ mm $R = 100$ mm

$$\text{Using the relationship, } \delta = \frac{64WR^3n}{Gd^4}$$

$$\therefore W = \frac{200\times(100\times10^3)(20)^4}{64\times(100)^3\times20} = 2500 \text{ Newton}$$

$$\text{Strain energy stored in one spring} = \frac{1}{2}W\delta$$

$$= \frac{1}{2}\times2500\times(200\times10^{-3})$$

$$\text{Strain energy stored in N springs} = \left(\frac{1}{2}\times2500\times200\times10^{-3}\right)N$$

$$= 250 \text{ N}$$

Equating the strain energy stored to kinetic energy of the truck,

$$250 \text{ N} = 4248$$

$$\therefore N = \frac{4248}{250} = 16.99$$

$\therefore$ Provide 17 springs.

Example 3

A close coiled helical spring of circular section has coils of 75 mm mean diameter. When loaded with an axial load of 250 N, it is found to extend 160 mm and when subjected to a twisting couple of 3 N m, there is an angular rotation of 60° degrees. Determine the Poisson's ratio for the material.

Sol: Here, we have $D = 75$ mm $W = 250$ N

$\delta = 160$ mm $T = 3$ N m

$$\phi = 60° = \left(60\times\frac{\pi}{180}\right) \text{radians.}$$

Let the Poisson's ratio be μ.

Using the relationship, $\delta = \dfrac{8WD^3n}{Gd^4}$ for an axial load,

$$\therefore G = \frac{8WD^3n}{\delta d^4}$$

$$= \frac{8\times250\times(75\times10^{-3})^3 n}{160\times10^{-3}\times d^4\times10^{-12}}$$

Therefore, $G = \dfrac{12.5\times75^3\times10^6 n}{d^4}$ N/mm² ——— (A)

For an axial torque, $\phi = \dfrac{64TDn}{Ed^4}$

Thus we get $E = \dfrac{64TDn}{\delta^4}$

$$= \frac{64\times3\times(75\times10^{-3})n}{\left(60\times\frac{\pi}{180}\right)(d^4\times10^{-12})}$$

Therefore, $E = \dfrac{576\times75\times10^9 n}{\pi d^4}$ ——— (B)

But we know the relationship E = 2G (1 + μ)

Substituting the values of E and G from (A) and (B),

$$\frac{576\times75\times10^9 n}{\pi d^4} = 2\times\frac{12.5\times75^3\times10^6 n}{d^4}(1+\mu)$$

$$\therefore (1+\mu) = \frac{576\times75\times10^9}{25\pi\times75^3\times10^6} = 1.304$$

$$\therefore \mu = 0.304$$

Example 4

Two close coiled helical springs are compressed between two parallel plates by a load of 1 kN. The springs have a wire diameter of 10 mm and the radii of coils are 50 and 75 mm. Each spring has 10 coils and is of the same initial length. If the smaller spring is placed inside the larger one such that both the springs are compressed by same amount, calculate:

(a) the total deflection, and

(b) the maximum stress in each spring.

Take G = 40 GPa for both the springs.

Sol: *Deflections*

Here, we denote spring 1 as larger and spring 2 as smaller.

$d_1 = 10$ mm $d_2 = 10$ mm

$R_1 = 75$ mm $R_2 = 50$ mm

$n_1 = 10$ $n_2 = 10$

Let W_1 and W_2 be the load carried by spring 1 and 2 respectively. Since deflection of both the springs is same, we get

$$\frac{64W_1R_1^3n_1}{Gd_1^4} = \frac{64W_2R_2^3n_2}{Gd_2^4}$$

$$\therefore \frac{W_1}{W_2} = \left(\frac{R_2}{R_1}\right)^3\left(\frac{d_1}{d_2}\right)^4\left(\frac{n_2}{n_1}\right)$$

$$= \left(\frac{50}{75}\right)^3\left(\frac{10}{10}\right)^4\left(\frac{10}{10}\right)$$

$$= 0.296$$

$$\therefore W_1 = 0.296\ W_2 \quad \text{.......... (A)}$$

Also, we have $W_1 + W_2 = 1000$ N .... (B)

From (A) and (B), we get, $W_1 = 228.5$ N, and $W_2 = 771.5$ N.

Thus, $$\delta_1 = \delta_2 = \frac{64\times 228.5\times 75^3\times 10}{(40\times 10^3)(10^4)} = \frac{64\times 771.5\times 50^3\times 10}{(40\times 10^3)(10^4)} = 154 \text{ mm}$$

$$(f_s)_{max} \text{ in spring 1} = \frac{16W_1R_1}{\pi d_1^3}$$

$$= \frac{16\times 228.5\times 75}{\pi(10)^3} = 87.3 \text{ N/mm}^2$$

$$(f_s)_{max} \text{ in spring 2} = \frac{16W_2R_2}{\pi(d_2)^3}$$

$$= \frac{16\times 771.5\times 50}{\pi(10)^3} = 196.46 \text{ N/mm}^2.$$

Example 5

A composite spring has two close coiled helical springs connected in series, each spring has 12 coils at a mean diameter of 25 mm. Find the diameter of the wire in one of the springs if the diameter of the wire in the other spring is 2.5 mm and stiffness of the composite spring is 700 N/m. Estimate the greatest load that can be carried by the composite spring for a maximum shearing stress of 180 MPa. Take G = 80 GPa.

Sol: Here, we have $n_1 = 12 \quad n_2 = 12$

$D_1 = 25$ mm $\quad D_2 = 25$ mm

Spring are connected in series

$d_1 = 2.5$ mm $\quad d_2 = ?$

For composite spring,

$$K = 700 \text{ N/m} = 0.7 \text{ N/mm}$$

$$G = 80 \text{ GPa} = 80\times 10^3 \text{ N/mm}^2$$

Using the relationship, $$K = \frac{W}{\delta}$$

$$\therefore \frac{1}{\delta} = \left(\frac{1}{0.7}\right)$$

$$\delta = \frac{64\,WR^3 n}{Gd^4}$$

$$\therefore \frac{1}{K} = \frac{\delta}{W} = \frac{64\,R^3 n}{Gd^4}$$

For spring 1,

$$\frac{\delta_1}{W} = \frac{1}{K_1} = \frac{64\,R_1^3 n_1}{Gd_1^4}$$

$$= \frac{64 \times (12.5)^3 \times 12}{80 \times 10^3 \times (2.5)^4}$$

For spring 2,

$$\frac{\delta_2}{W} = \frac{1}{K_2} = \frac{64 \times (12.5)^3 \times 12}{(80 \times 10^3)(d_2)^4}$$

For springs connected in series,

$$\frac{1}{K} = \frac{1}{K_1} + \frac{1}{K_2}$$

$$\frac{1}{0.7} = \frac{64 \times (12.5)^3 \times 12}{(80 \times 10^3)(2.5)^4} + \frac{64 \times (12.5)^3 \times 12}{(80 \times 10^3)(d_2)^4}$$

$$= \frac{64 \times 12.5^3 \times 12}{80 \times 10^3}\left(\frac{1}{(2.5)^4} + \frac{1}{(d_2)^4}\right)$$

$$\left(\frac{1}{(2.5)^4} + \frac{1}{(d_2)^4}\right) = 0.07619$$

$$\frac{1}{(d_2)^4} = 0.05058$$

$$\therefore \quad d_2 = 2.109 \text{ mm}$$

Given $(f_s)_{max} = 180$ MPa, then W = ?

$$f_s = \frac{8WD}{\pi d^3}$$

$$\therefore \quad W = \frac{f_s \times \pi d^3}{8D}$$

Since both the springs carry the same load W,

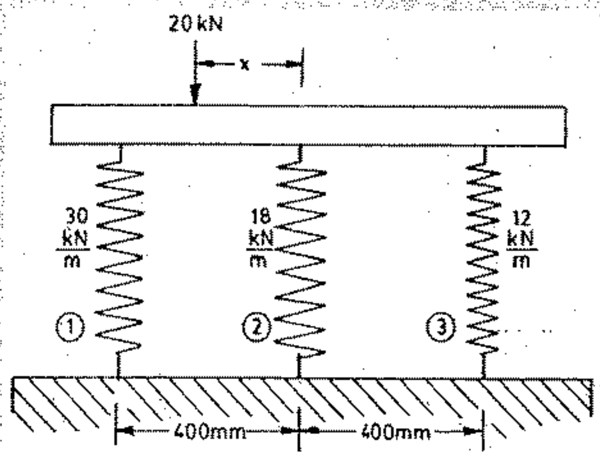
For spring 1, $W = \frac{180 \times \pi (2.5)^3}{8 \times 25} = 44.17$ N

For spring 2, $W = \frac{180 \times \pi (2.109)^3}{8 \times 25} = 26.51$ N

Thus, the greatest load the spring can carry based on the maximum shear stress criteria is lesser of these two, i.e. 26.51 N.

Example 6

A rigid bar weighing 5 kN and carrying a load of 20 kN is supported by 3 springs as shown in figure, having spring constants $S_1 = 30$ kN/m, $S_2 = 18$ kN/m and $S_3 = 12$ kN/m. If the unloaded springs are all of the same length, find the distance x such that the bar is horizontal.



Sol: Here the springs are connected in parallel.

We know, spring constant, $S = \left(\frac{W}{\delta}\right)$

Using the relationship, $\delta = \frac{8WD^3n}{Gd^4}$

We get, $S = \frac{W}{\delta} = \frac{Gd^4}{8D^3n}$

Here, deflection of all the 3 springs will be equal, i.e. $\delta = \frac{W}{S}$ = constant.

$$\therefore \quad \frac{W_1}{S_1} = \frac{W_2}{S_2} = \frac{W_3}{S_3}$$

where, W_1, W_2 and W_3 are the loads carried by these 3 springs and S_1, S_2 and S_3 are their respective stiffnesses.

$$S_1 = 30 \text{ kN/m}$$
$$S_2 = 18 \text{ kN/m}$$
$$S_3 = 12 \text{ kN/m}$$
$$W = W_1 + W_2 + W_3 = 20 + 5 = 25 \text{ kN} \quad \text{------ (i)}$$
$$\therefore \quad \frac{W_1}{30} = \frac{W_2}{18} = \frac{W_3}{12} \quad \text{------ (ii)}$$
$$W_2 = 0.6\ W_1$$
$$W_3 = 0.4\ W_1$$

Substituting in (i), $W_1 + W_2 + W_3 = 25$ kN

$$W_1 + 0.6\ W_1 + 0.4\ W_1 = 25$$
$$W_1 = 12.5 \text{ kN}$$
$$W_2 = 7.5 \text{ kN}$$
$$W_3 = 5.0 \text{ kN}$$

Taking moments about the spring 3,

$$(W_1 \times 0.8) + (W_2 \times 0.4) = (5 \times 0.4) + 20(0.4 + x)$$

$$(12.5 \times 0.8) + (7.5 \times 0.4) = 2 + 20\ (0.4 + x)$$

$$\therefore \quad x = 0.150 \text{ m}$$

The load of 20 kN must be placed 150 mm from the middle spring.

Example 7

A leaf spring 0.8 m long consists of 12 plates, each of them is 65 mm wide and 6 mm thick. It is simply supported at its ends. The greatest bending stress is not to exceed 180 MPa and the central deflection when the spring is fully loaded is not to exceed 20 mm. Estimate the magnitude of the greatest central load that can be applied to the spring. Take $E = 200 \times 10^3$ MPa.

Sol:

Here, we have, $L = 0.8$ m $\quad n = 12$

$b = 65$ mm $\quad t = 6$ mm

$f \leq 210$ N/mm^2 $\quad d \leq 20$ mm

Using the relationship, $f_{max} = \dfrac{3WL}{2nbt^2}$

$$180 = \frac{3W \times 800}{2 \times 12 \times 65 \times (6)^2}$$

$$\therefore \quad W = 4212 \text{ N}$$

Using the relationship, $\delta = \dfrac{3}{8} \times \dfrac{WL^3}{Enbt^3}$

$$20 = \frac{3}{8} \times \frac{W \times (800)^3}{(200 \times 10^3) \times 12 \times (65) \times (6)^3}$$

$$\therefore \quad W = 3510 \text{ N}$$

Thus, the greatest central load that can be applied is lesser of these two, i.e. 3.51 kN.

Example 8

A leaf spring is required to satisfy the following specification:

$L = 0.75$ m, $W = 5$ kN, $b = 75$ mm, maximum stress = 210 MPa,

Maximum deflection = 25 mm, E = 200 GPa.

Find the number of leaves, their thicknesses and initial radius of curvature.

Sol: Here, we have, $L = 0.75$ m $= 750$ mm $\quad W = 5$ kN $= 5000$ N

$\delta \leq 25$ mm $\quad f \leq 210$ N/mm^2

Maximum stress, $f = \dfrac{3}{2} \times \dfrac{WL}{nbt^2}$

$$210 = \frac{3}{2} \times \frac{5000 \times 750}{75 \times nt^2}$$

$$\therefore \quad nt^2 = 357.2$$

$$\text{Maximum deflection, } \delta = \frac{3}{8} \times \frac{WL^3}{Enbt^3}$$

$$25 = \frac{3}{8} \times \frac{5000 \times 750^3}{200 \times 10^3 \times (75) \times nt^3}$$

$$\therefore \quad nt^3 = 2109$$

From (i) and (ii), t = 5.905 mm

Thus, use 6 mm thick plates.

$$\therefore \quad n = \frac{357.2}{6^2} = 9.916$$

Adopt 10 leaves.

Using property of circle, $\frac{L}{2} \times \frac{L}{2} = \delta(2R - \delta)$

$$\frac{L^2}{4} = 2R\delta - \delta^2$$

On account of δ^2 being very small, can be neglected.

$$\therefore \quad \frac{L^2}{4} = 2R\delta$$

$$\text{or} \quad \delta = \frac{L^2}{8R}$$

$$\text{and} \quad R = \frac{L^2}{8\delta} = \frac{750^2}{8 \times 25} = 2815 \text{ mm}$$

$$\therefore \quad \text{Radius of curvature} = 2.815 \text{ m.}$$

OBJECTIVE QUESTIONS

1. A close coiled spring is cut into two identical halves. The stiffness of each of the resulting springs will
   (a) remain the same as the that of the original spring
   (b) reduce to half that of the original spring
   (c) become twice that of the original spring
   (d) become zero

2. Two springs of stiffness K_A and K_B are placed one inside the other so as to compress by the same amount under an applied axial load. The combined stiffness of the two springs will be
   (a) $\frac{1}{K_A}+\frac{1}{K_B}$ (b) $\frac{K_A \times K_B}{K_A + K_B}$
   (c) $\frac{K_A + K_B}{2}$ (d) $K_A + K_B$

3. If two springs of stiffness k_1 and k_2 are connected in series, the stiffness of the combined spring is
   (a) $\frac{k_1 k_2}{k_1 + k_2}$ (b) $\frac{k_1 + k_2}{k_1 k_2}$
   (c) $k_1 + k_2$ (d) $k_1 k_2$

4. Consider a close-coiled helical spring of radius R subject to a load P. The spring consists of 'n' turns of wire with wire radius 'r'. The stiffness of the spring is
   (a) $\frac{PR^3}{GI_Z}$ (b) $\frac{Gr^4}{4nR^3}$
   (c) $\frac{GI_Z n r^4}{2\pi R^3}$ (d) $\frac{PR^2 2\pi n}{GI_Z}$

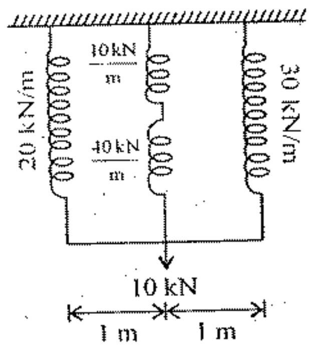
5. A close-coiled helical spring has 100 mm mean diameter and is made of 20 turns of 10 mm diameter steel wire. The spring carries an axial load of 100 N. Modulus of rigidity is 84 GPa. The shearing stress developed in the spring in N/mm^2 is
   (a) $120/\pi$ (b) $160/\pi$
   (c) $100/\pi$ (d) $80/\pi$

6. In an analysis of a closely coiled helical spring under axial load, which of the following is/are negligible?
   1. Torsion
   2. Bending
   3. Axial force in the wire

   Select the correct answer using the codes given below:
   (a) 1 alone (b) 2 and 3
   (c) 1 and 2 (d) 1 and 3

7. A close-coiled helical spring with n coils, mean radius R and diameter 'd' is subjected to an axial load W. What is the compression in spring ?
   (a) $\frac{64WR^3n}{Cd^3}$ (b) $\frac{64WR^3n}{Cd^4}$
   (c) $\frac{32WR^3n}{Cd^3}$ (d) $\frac{32WR^3n}{Cd^4}$

8. What is the equivalent spring stiffness for the system of springs shown in the figure given below?



(a) 43 kN/m (b) 50 kN/m
(c) 58 kN/m (d) 64 kN/m

ANSWERS

1.	(c)	3.	(a)	5.	(d)	7.	(b)
2.	(d)	4.	(b)	6.	(b)	8.	(c)

SOLUTION...

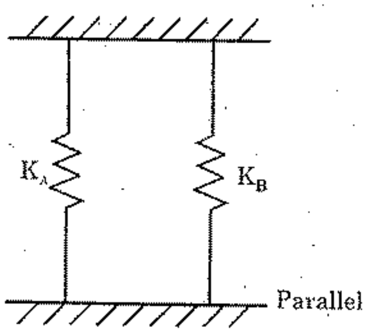
1. (c) Closely coiled spring

$$\delta = \frac{64WR^3n}{Nd^4} \quad [\text{Two halves} \Rightarrow n = \frac{n}{2}]$$

$$K = \frac{W}{\delta} \propto \frac{1}{n}$$

$$\frac{K_{new}}{K_{avg}} = 2$$

2. (d)



$$P = P_A + P_B$$

$$\delta_A = \delta_B = \delta$$

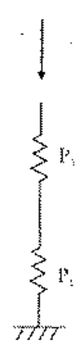
$$K = \frac{F}{\delta}$$

$$K.\delta = K_A\delta + K_B\delta$$

$$K = K_A + K_B$$

3. (a)

$$P_1 = P_2 = P \quad [K = \frac{P}{\delta}]$$



$$\delta = \delta_1 + \delta_2$$

$$\frac{P_1}{K_1} + \frac{P_2}{K_2} = \frac{P}{K}$$

$$K = \frac{K_1K_2}{K_1 + K_2}$$

4. (b) Stiffness $= \dfrac{W}{\delta} = \dfrac{Gd^4}{64R^3n} = \dfrac{Gr^4}{4R^3n}$

5. (d) R = 100 mm; G = 84 Gpa; n = 20; W = 100 N; d = 10 mm; T = WR; τ = ?

$$\tau = \frac{WR}{\left(\frac{\pi d^3}{16}\right)}$$

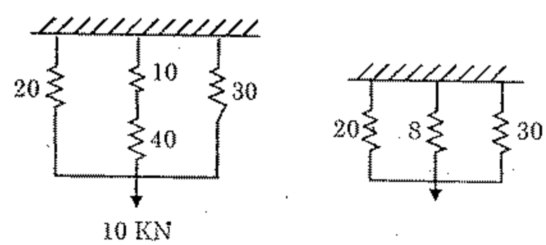
$$\tau = \frac{100 \times \left(\frac{100}{2}\right)}{\left(\frac{\pi \times 10^3}{16}\right)} = \frac{160}{2\pi}$$

$$= \frac{80}{\pi} \text{ N/mm}^2$$

6. (b) In closely coiled spring, torsion is predominant

∴ Others are neglected

8. (c)



$$K_{eq} = \frac{10 \times 40}{10 + 40} = \frac{400}{50} = \frac{40}{5} = 8 \text{ kN/m}$$

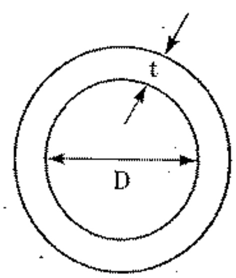
$$K_{eq} = 20 + 8 + 30 = 58 \text{ kN/m}$$

11

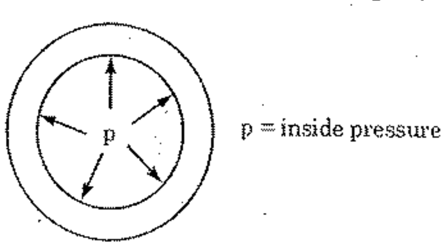
Thick and Thin Cylinder/Sphere

INTRODUCTION

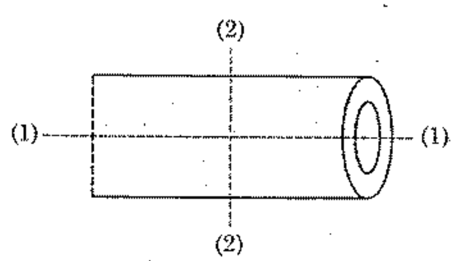
- In this chapter we study the effect of fluid pressure inside a thick or thin shells.
- A shell is called thin shell if $t < \frac{D}{10}$ to $\frac{D}{15}$.

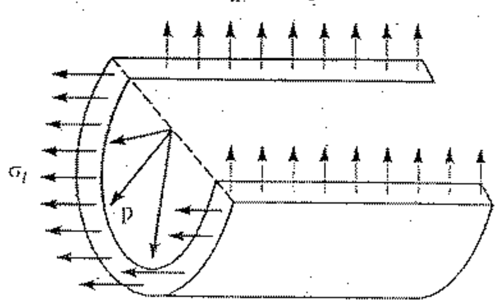


The fluid pressure inside the shell produces Hoop stress, longitudinal stress and radial stress in the shells. These stresses are shown as in the given below figure.

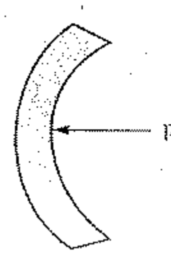


σ_h = Hoop stress





Section (1)-(1)



Radial pressure

σ_l = Longitudinal stress

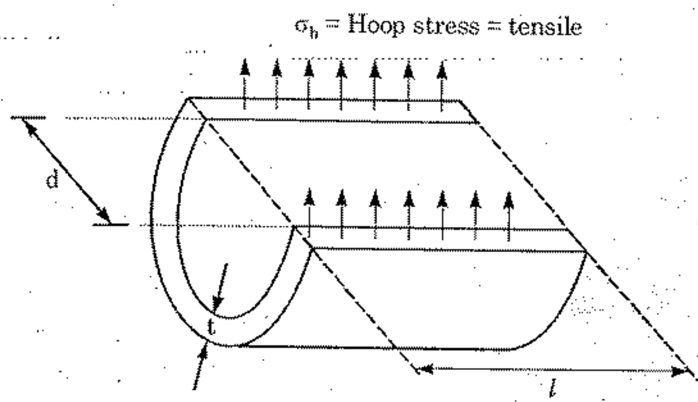
σ_h = Hoop stress = Meridional stress = Circumferential stress

p = radial pressure

Hoop stress (σ_h) → Constant throughout the thickness in thin shells.
However in thick shell, it is max at inner face and min at outer face.
Longitudinal Stress (σ_l) → constant across the thickness both for thin shells and thick shells.
Radial Pressure → max inside = p
→ min = 0 outside
Radial pressure is normally neglected being very small as compared to σ_h and σ_l.

THIN CELLS (CYLINDRICAL)

(a) Hoop Stress



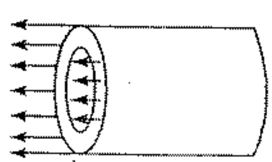
$p \times l \times d = \sigma_h \times 2l \times t$

$$\sigma_h = \frac{pd}{2t} \quad \Rightarrow \quad \boxed{\sigma_h = \frac{Pd}{2t}}$$

d = inner dia
t = thickness
p = fluid pressure

- No shear exists on the section on which Hoop stress acts.
- Hence $\sigma_h = \frac{pd}{2t}$ is a principal stress.
- σ_h is always tensile both in thick shells as well as in thin shells due to fluid pressure from inside. However for outside pressure σ_h will be compressive.

(b) Longitudinal Stress (σ_l)



$$\frac{\pi d^2}{4} \times p = \sigma_l \times \pi\left(d + 2 \times \frac{t}{2}\right) \times t$$

$$= \sigma_l \times \pi\,(d + t)\,t$$

Neglecting t^2 term, as it is very small as compared to $d \times t$

$$\frac{\pi d^2}{4} \times p = \sigma_l \times \pi \, dt$$

$$\boxed{\sigma_l = \frac{pd}{4t}}$$

As no shear, acts on section on which σ_l is acting $\Rightarrow$ σ_l = principal stress.

(c) Radial Pressure

Inside pressure = p (compressive)

Outside pressure = 0

(d) Hoop Strain

$$\frac{\Delta d}{d} = \varepsilon_h = \frac{\sigma_h - \mu\sigma_l}{E} \text{ (Neglecting radial pressure)}$$

$$= \frac{\sigma_h - \mu\sigma_l}{E} = \frac{pd}{2tE}\left(1 - \frac{\mu}{2}\right)$$

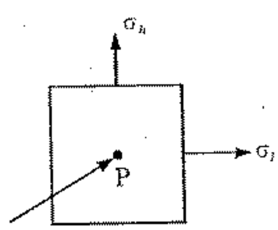
$$\boxed{\varepsilon_h = \frac{pd}{4tE}(2-\mu)}$$

(e) Longitudinal strain (ε_l)

$$\frac{\Delta l}{l} = \varepsilon_l = \frac{\sigma_l - \mu\sigma_h}{E} = \frac{pd}{4tE}(1-2\mu)$$

$$\boxed{\varepsilon_l = \frac{Pd}{4tE}(1-2\mu)}$$

Note:



If p radial pressure (compressive) is also considered $\frac{\mu p}{E}$ is added to ε_h and ε_l both.

$$\Rightarrow \varepsilon_h = \frac{\sigma_h - \mu\sigma_l - \mu(-p)}{E}$$

$$\varepsilon_h = \frac{\sigma_h - \mu\sigma_l}{E} + \frac{\mu p}{E}$$

$$\varepsilon_l = \frac{\sigma_l - \mu\sigma_h - \mu(-p)}{E}$$

$$\varepsilon_l = \frac{\sigma_l - \mu\sigma_h}{E} + \frac{\mu p}{E}$$

(f) Volumetric strain [Increase in the volume of cylinder)

$$V = \frac{\pi d^2}{4} \times l$$

$$dV = \frac{\pi}{4}\left[(d^2)\, dl + 2ld \cdot d(d)\right]$$

$$\frac{\delta V}{V} = \frac{\frac{\pi}{4}\left[d^2 \times dl + 2ld \times d(d)\right]}{\frac{\pi}{4} d^2 l}$$

$$\frac{\delta V}{V} = \frac{dl}{l} + \frac{2d(d)}{d}$$

$$\boxed{\varepsilon_v = \varepsilon_l + 2\varepsilon_h}$$

$$\varepsilon_v = \frac{pd}{4tE}[1 - 2\mu + 4 - 2\mu]$$

$$\boxed{\varepsilon_v = \frac{pd}{4tE}[5 - 4\mu]}$$

Note: If radial pressure is also considered additional term of $\frac{\mu p}{E} + \frac{2\mu p}{E}$ i.e. $\frac{3\mu p}{E}$ added to the ε_v

Shear Stress

Max shear stress in the plane of σ_h and σ_l:

$$(\tau_{max})_{in\,plane} = \frac{\sigma_h - \sigma_l}{2} = \frac{\frac{pd}{2t} - \frac{pd}{4t}}{2} = \frac{pd}{8t}$$

$$\boxed{(\tau_{max})_{in\,plane} = \frac{pd}{8t}}$$

- This is not the absolute max shear stress.
- Absolute max shear stress $= \max\left[\frac{|\sigma_{max}|}{2}, \frac{|\sigma_{max} - \sigma_{min}|}{2}\right]$

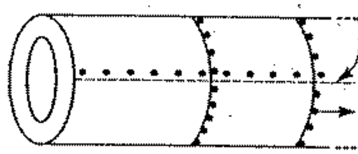
$$= \max\left(\frac{\frac{pd}{2t}}{2}, \frac{\frac{pd}{2t} - (-p)}{2}\right)$$

$$\boxed{\tau_{abs\,max} = \frac{pd}{4t} + \frac{p}{2}}$$

Neglecting radial pressure effect

$$\boxed{\tau_{abs\,max} = \frac{pd}{4t}}$$

RIVETTED CYLINDER



Due to reduction in the area of plate, on account of drilling of holes, the stresses will increase. Hence

$$\boxed{\sigma'_h = \frac{pd}{2t\eta_l}}$$

$$\boxed{\sigma'_l = \frac{pd}{4t\eta_h}}$$

where η_l and η_h are the efficiencies of longitudinal and circumferencial joints respectively.

Note: If S is the pitch of rivets then

$$(S - d')\, t\sigma'_h = St \times \sigma_h$$

$$\sigma'_h = \frac{\sigma_h}{\left(\frac{(S-d')\,t}{St}\right)} = \frac{\sigma_h}{\frac{(S-d')t\left(f_y/Fos\right)}{(St)\,(f_y/Fos)}} = \frac{\sigma_h}{\frac{(S-d')t)\times(f_y/F.o.s.)}{\left(St \times \frac{f_y}{F.o.s.}\right)}}$$

$$= \frac{\sigma_h}{\left(\frac{\text{Strength with deduction in hole}}{\text{Strength without deduction for note}}\right)}$$

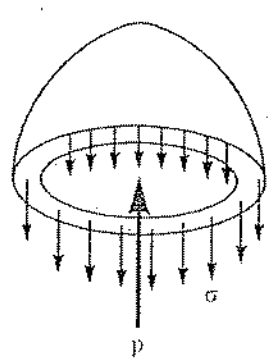
$$\sigma'_h = \frac{\sigma_h}{\text{Efficiency of longitudinal joint}}$$

$$\sigma'_h = \frac{\sigma_h}{\eta_l} = \frac{pd}{2t\eta_l}$$

$$\boxed{\sigma'_h = \frac{pd}{2t\eta_l}}$$

THIN CELLS (SPHERICAL)

From force equillibrium



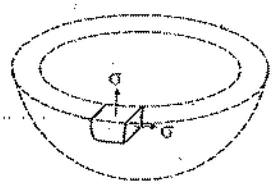
$$\sigma \times 2\pi\left(\frac{d+t}{2}\right) \times t = p \times \frac{\pi d^2}{4}$$

again neglecting 't^2' term as compared to d × t

$$\sigma \pi d t = p\,\frac{\pi d^2}{4}$$

$$\boxed{\sigma = \frac{pd}{4t}}$$

- In this case major and minor principal stresses are same.



- $$\boxed{\text{Max in plane shear stress} = \frac{\sigma_1 - \sigma_2}{2} = 0}$$
- Absolute max shear stress

$$\boxed{\tau_{\text{abs max}} = \frac{\sigma_1}{2} = \frac{pd}{8t}}$$

- As major and minor principal stresses are same, Mohr circle reduces to a point. Hence All planes are principal planes.

Hoop Strain $(\varepsilon_h) = \frac{\sigma_h}{E} - \mu\frac{\sigma_l}{E} = \frac{\sigma_h}{E} - \frac{\mu\sigma_h}{E}$ (since $\sigma_l = \sigma_h$)

$$\varepsilon_h = \frac{\sigma_h}{E}(1-\mu) = \frac{pd}{4tE}(1-\mu)$$

$$\text{Hoop strain } \boxed{\varepsilon_h = \frac{pd}{4tE}(1-\mu)} = \frac{\Delta d}{d} = \frac{\text{Change in diameter}}{\text{Original diameter}}$$

Volumetric Strain (increase in the storage capacity of spherical shell)

$$V = \frac{4\pi}{3}\left(\frac{d}{2}\right)^3 = \frac{\pi d^3}{6}$$

$$dV = \frac{\pi \times 3d^2}{6}\,\partial(d)$$

$$\frac{dV}{V} = \frac{3\pi d^2 \times \partial(d)}{6 \times \frac{\pi d^3}{6}} = 3\,\frac{\partial d}{d}$$

$$\frac{dV}{V} = 3\varepsilon_h = \frac{3pd}{4tE}(1-\mu)$$

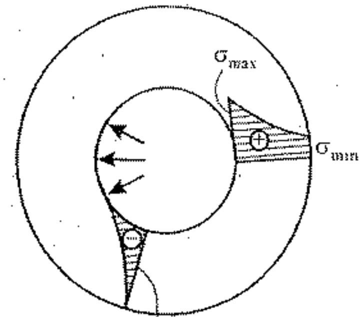
$$\boxed{\varepsilon_v = \frac{3pd}{4tE}(1-\mu)}$$

THICK CYLINDER

- **Longitudinal Stress due to Internal Pressure Constant throughout the Thickness (Tensile)**
- **Hoop Stress (Due to internal pressure)**
  (a) Max at inside
  (b) Min at outside
  (c) Tensile throughout

Radial Pressure

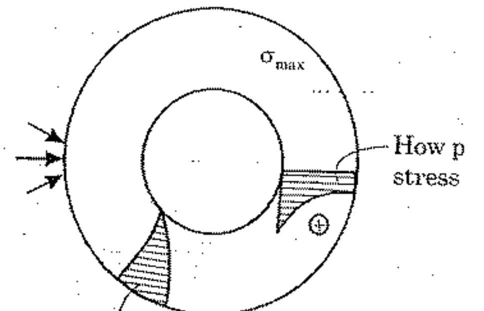
(a) Max inside
(b) Zero outside
(c) Compressive throughout



Radial pressure (due to internal pressure)

Hoop stress (due to external pressure)

(a) Max inside
(b) Min outside
(c) Compressive throughout

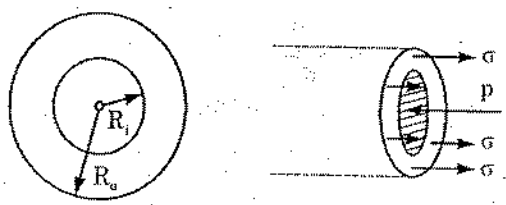


Radial pressure (due to external pressure)

⇒ Analysis of thick shell is done using Lame's Theorem

⇒ In this theorem material is assumed homogeneous and isotropic and longitudinal strain is assumed constant at every point in the thickness.

Longitudinal Stress



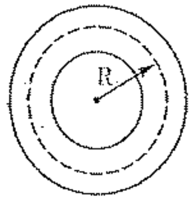
$$p \times \pi R_i^2 = \sigma_l \times \pi\left(R_0^2 - R_i^2\right)$$

$$\Rightarrow \boxed{\sigma_l = \frac{p.R_i^2}{R_0^2 - R_i^2}}$$

Hoop Stress

Hoop stress at any radius 'R' from centre is given by, $\boxed{\sigma_h = \frac{B}{R^2} + A}$ (Tensile) (i)

Lame's equation



A and B are Lame's constants.

Radial Pressure (P_R)

$$\boxed{P_R = \frac{B}{R^2} - A} \quad \text{(Compression)} \quad \text{(iii)}$$

A and B are Lame's constant.

From (i) and (ii)

$$\sigma_h - P_R = 2A$$

at $R = R_0,\ P_R = 0$

$$\Rightarrow \quad A = \frac{B}{R_0^2}$$

at $R = R_i,\ P_R = P$

$$\Rightarrow \quad P = \frac{B}{R_i^2} - \frac{B}{R_0^2}$$

$$\Rightarrow \quad B = \frac{P}{\left(\frac{1}{R_i^2} - \frac{1}{R_0^2}\right)}$$

$$\Rightarrow \quad \boxed{\sigma_h = \frac{P}{R^2\left(\frac{1}{R_i^2} - \frac{1}{R_0^2}\right)} + \frac{P}{R_0^2\left(\frac{1}{R_i^2} - \frac{1}{R_0^2}\right)}}$$

THICK SPHERE

Hoop and longitudinal stresses

Hoop and longitudinal stresses are equal and varies from max at the inner face to min at outer face.

$$\boxed{\sigma_h = \frac{B}{R^3} + A} = \sigma_l \quad \text{(Tensile)}$$

Radial Pressure

$$\boxed{P_R = \frac{2B}{R^3} - A} \quad \text{(Compressive)}$$

A and B are Lame's constant.

at $R = R_i$, $P_R = P$

at $R = R_0$, $P_R = 0$

Note: *Lame's constant 'A' and 'B' are both positive for internal pressure and both negative for external pressure.*

Example 1

A cylindrical compressed air drum is 2 m in diameter with plates 12.5 mm thick. The efficiencies of the longitudinal (η_l) and circumferential (η_c) joints are 85% and 45% respectively. If the tensile stress in the plating is to be limited to 100 MN/m^2, find the maximum safe air pressure.

Sol: The efficiency of the joint influences the stresses induced. For a seamless shell (with no joints), efficiency is 100%. When the efficiency of joint is less than 100%, the stressess are increased accordingly.

Hence, if η is the efficiency of a joint in the longitudinal direction, influencing the hoop stress, then the stress will be given as,

$$\sigma_n = \frac{pd}{4t \times \eta_l}$$

Here, the diameter $d = 2$ m = 2000 mm.
Thickness, $t = 12.5$ mm.
Limiting tensile stress = 1000 mN/m² = 100 N/mm².
Considering the circumferential joint which influences the longitudinal stress

$$\frac{pd}{4t \times \eta_l} = 100$$

$$\frac{p \times 2000}{4 \times 12.5 \times 0.45} = 100$$

$$p = 1.125 \text{ N/mm}^2$$

Similarly, considering the longitudinal joint which influences the hoop stress,

$$\frac{pd}{2t \times \eta_c} = 100$$

$$\frac{p \times 2000}{2 \times 12.5 \times 0.85} = 100$$

$$p = 1.063 \text{ N/mm}^2$$

Evendently, safe pressure is governed by hoop stress.
Hence, maximum safe air pressure = 1.063 N/mm².

Example 2

A cylindrical shell, 0.8 m in diameter and 3 m long is having 10 mm wall thickness. If the shell is subjected to an internal pressure of 2.5 N/mm², determine (a) change in diameter (b) change in length and (c) change in volume. Take E = 200 GPa and Poisson's ratio = 0.25.

Sol: Diameter of the shell d = 0.8 m = 800 mm
Thickness of the shell, t = 10 mm
Internal pressure p = 2.5 N/mm²

Hoop stress, $\sigma_n = \frac{pd}{2t}$

$$= \frac{2.5 \times 800}{2 \times 10} = 100 \text{ N/mm}^2$$

Longitudinal stress, $\sigma_l = \frac{pd}{4t}$

$$= \frac{2.5 \times 800}{4 \times 10} = 50 \text{ N/mm}^2$$

Hoop strain, $\varepsilon_h = \frac{1}{E}(\sigma_h - \varepsilon\sigma_l) = \frac{1}{2\times10^5}(100 - 0.25\times50)$

$= 4.375 \times 10^{-4}$

Longitudinal strain, $\varepsilon_l = \frac{1}{E}(\sigma_l - \varepsilon\sigma_h) = \frac{1}{2\times10^5}(50 - 0.25\times100)$

$= 1.25 \times 10^{-4}$

Volumetric strain $= 2\varepsilon_h + \varepsilon_l = 2 \times 4.375 \times 10^{-4} + 1.25 \times 10^{-4}$

$= 10 \times 10^{-4} = 10^{-3}$

Increase in diameter = Hoop strain × original diameter

$= 4.375 \times 10^{-4} \times 800 = 0.35$ mm

Increase in length = Longitudinal strain × original length

$= 1.25 \times 10^{-4} \times 3000 = 0.375$ mm

Increase in volume = Volumetric strain × original volume

Original volume $= \frac{\pi d^2}{4} \times l = \frac{\pi}{4} \times 800^2 \times 3000 = 1507 \times 10^6$ mm^3

Increase in volume $= 10^{-3} \times 1507 \times 10^6 = 1507 \times 10^3$ mm^3.

Example 3

A copper tube of 50 mm diameter and 1200 mm length has a thickness of 1.2 mm with closed ends. It is filled with water at atmospheric pressure. Find the increase in pressure when an additional volume of 32 cc of water is pumped into the tube.

Take E for copper = 100 GPa, Poisson's ratio = 0.3 and K for water = 2000 N/mm^2.

Sol: The additional quantity of water pumped accounts for the change in volume of the shell as well as the compression of the water in it.

Hence, if p is the increase in pressure in water, then,

Hoop stress $= \frac{pd}{2t} = \frac{p\times50}{2\times1.2} = 20.83\,p$

Longitudinal stress $= \frac{pd}{4t} = \frac{p\times50}{4\times1.2} = 10.42\,p$

Hoop strain $= \frac{1}{E}[\sigma_h - \mu\sigma_l] = \frac{1}{E}[20.83p - 0.3\times10.42] = \frac{17.7p}{E}$

Longitudinal strain $= \frac{1}{E}[\sigma_l - \mu\sigma_h] = \frac{1}{E}[10.42p - 0.3\times20.83] = \frac{4.17p}{E}$

Volumetric strain $= 2\varepsilon_h + \varepsilon_l = \frac{2\times17.7p}{E} + \frac{4.17p}{E} = \frac{39.57p}{E}$ (increase)

Due to compression in water, its volumetric strain $\frac{p}{K}$ (decrease)

$\therefore$ Additional volume pumped = increase in volume of cylinder + decrease in volume of water i.e. $32\times10^3 = \frac{39.57p}{E}\times V\times\frac{p}{K}V$

$$= \left(\frac{39.57}{1\times10^5}+\frac{1}{2000}\right)\times p\times\frac{\pi d^2}{4}\times l$$

$$= 87.57\times10^{-5}\times p\times\frac{\pi}{4}\times50^2\times1200$$

$\therefore$ p = 15.51 N/mm²

Example 4

The internal and external diameters of a thick hollow cylinder are 80 mm and 120 mm respectively. It is subjected to an external pressure of 40 N/mm² and an internal pressure of 120 N/mm². Calculate the circumferential stress at the external and internal surfaces and determine the radial and circumferential stresses at the mean radius.

Sol: We know that $p_x = \frac{b}{x^2} - a$

Thus, at x = 40, p_x = 120 N/mm², and

at x = 60, p_x = 40 N/mm²

Substituting these values, we get,

$$120 = \frac{b}{40^2} - a \text{ and}$$

$$40 = \frac{b}{60^2} - a$$

On solving, we get, a = 24 and b = 230400.

Circumferential stress is given by, $f_x = \frac{b}{x^2} + a$

Thus, at x = 40, $f_x = \frac{230400}{40^2} + 24 = 168$ N/mm²

at x = 60, $f_x = \frac{230400}{60^2} + 24 = 88$ N/mm²

At the mean radius, i.e. $\frac{(40+60)}{2} = 50$ mm

Radial stress = $\frac{230400}{50^2} - 24 = 68.16$ N/mm², and

Circumferential stress = $\frac{230400}{50^2} + 24 = 116.16$ N/mm².

Example 5

A thick cylinder of 0.5 m external diameter and 0.4 m internal diameter is subjected simultaneously to internal and external pressures. If the internal pressure is 25 MN/m² and the hoop stress at the inside of the cylinder is 45 MN/m² (tensile), determine the intensity of the external pressure.

Sol: Using Lame's expression for hoop stress and radial stress, which are

$$f_x = \frac{b}{x^2} + a \text{ and}$$

$$p_x = \frac{b}{x^2} - a$$

For the internal surface, i.e. x = 0.2 m,

$$25 = \frac{b}{0.2^2} - a$$

$$45 = \frac{b}{0.2^2} + a$$

On solving, we get, b = 1.4 and a = 10.

Thus, the intensity of external pressure at x = 0.25 will be

$$p_x = \frac{1.4}{0.25^2} - 10$$

$$= 12.4 \text{ MN/m}^2$$

Example 6

The cylinder of a hydraulic press has an internal diameter of 0.3 m and is to be designed to withstand a pressure of 10 MN/m^2 without the material being stressed over 20 MN/m^2. Determine the thickness of the metal and the hoop stress on the outer side of the cylinder.

Sol: Internal radius $= \frac{0.3}{2} = 0.15$ m.

Let the external radius be R.

Hoop stress will be maxinum at the inner side.

Thus, $$20 = \frac{b}{0.15^2} + a$$

Radial pressure at the inner side = 10 MN/m^2.

Therefore $$10 = \frac{b}{0.15^2} - a$$

On solving, we get, b = 0.3375 and a = 5.

For external pressure = 0, we get,

$$0 = \frac{0.3375}{R^2} - 5$$

$$R = 0.26 \text{ m.}$$

Thus, metal thickness = 0.26 – 0.15 = 0.11 m = 110 mm

Finally, hoop stress at the outside of the cylinder will be,

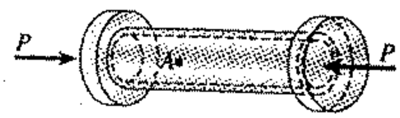
$$= \frac{0.3375}{0.26^2} + 5$$

$$= 11.00 \text{ MN/m}^2$$

Example 7

A thin-walled cylindrical pressure vessel with a circular cross section is subjected to internal gas pressure p and simultaneously compressed by an axial load P = 55 kN(Fig. a). The cylinder has inner radius r = 50 mm and wall thickness t = 4mm.

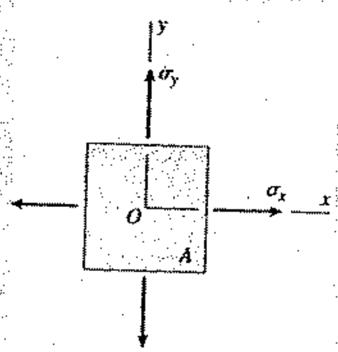
Determine the maximum allowable internal pressure p_{allow} based upon an allowable shear stress of 45 MPa in the wall of the vessel.



(a)

Sol: The stresses in the wall of the pressure vessel are caused by the combined action of the internal pressure and the axial force. Since both actions produce uniform normal stresses throughout the wall, we can select any point on the surface for investigation. At a typical point, such as point A (Fig a), we isolate a stress element. The x axis is parallel to the longitudinal axis of the pressure vessel and the y axis is circumferential. Note that there are no shear stresses acting on the element.

Principal Stresses: The longitudinal stress σ_x is equal to the tensile stress σ_2 produced by the internal pressure



$$\sigma_x = \frac{pr}{2t} - \frac{P}{A} = \frac{pr}{2t} - \frac{P}{2\pi rt} \qquad \text{......(f)}$$

in which A = $2\pi rt$ is the cross-sectional area of the cylinder. (Note that for convenience we are using the inner radius r in all calculations.)

The circumferential stress σ_y is equal to the tensile stress σ_1 produced by the internal pressure

$$\sigma_y = \frac{pr}{t}$$

Note that σ_y is algebraically larger than σ_x

Since no shear stresses act on the element (Fig.), the normal stresses σ_x and σ_y are also the principal stresses:

$$\sigma_1 = \sigma_y = \frac{pr}{t} \qquad \sigma_2 = \sigma_x = \frac{pr}{2t} - \frac{P}{2\pi rt}$$

Now substituting numerical values, we obtain

$$\sigma_1 = \frac{pr}{t} = \frac{p(50\text{ mm})}{4\text{ mm}} = 12.5p$$

$$\sigma_2 = \frac{pr}{2t} - \frac{P}{2\pi rt} = \frac{p(50\text{ mm})}{2(4\text{ mm})} - \frac{55\text{ kN}}{2\pi(50\text{ mm})(4\text{ mm})}$$

= 6.2p − 43.77 MPa

in which σ_1, σ_2 and p have units of megapascals (MPa).

In-plane shear stresses. The maximum in-plane shear stress is

$$\tau_{max} = \frac{\sigma_1 - \sigma_2}{2} = \frac{1}{2}(12.5p - 6.25p + 43.77\text{ MPa}) = 3.125p + 21.88\text{ MPa}$$

Since τ_{max} is limited to 45 MPa, the preceding equation becomes

45 MPa = 3.125 p + 21.88 MPa

from which we get

$$p = \frac{23.12\text{ MPa}}{3.125} = 7.39\text{ MPa} \quad \text{or } (p_{allow})_1 = 7.3\text{ MPa}$$

Out-of-plane shear stresses. The maximum out-of-plane shear stress is either

$$\tau_{max} = \frac{\sigma_2}{2} \quad \text{or } \tau_{max} = \frac{\sigma_1}{2}$$

From the second equation we get

45 MPa = 3.125 p − 21.88 MPa or $(p_{allow})_2$ = 21.4 MPa

From the second equation we get

45 MPa = 6.25p   or $(p_{allow})_3$ = 7.2 MPa

Allowable internal pressure. Comparing the three calculated values for the allowable pressure, we see that $(p_{allow})_3$ governs, and therefore the allowable internal pressure is

p_{allow} = 7.2 MPa

At this pressure the principal stresses are $\sigma_1 = 90$ MPa and $\sigma_2 = 1.23$ MPa. These stresses have the same signs, thus confirming that one of the out-of-plane shear stresses must be the largest shear stress.

Example 8

A 120 mm diameter and 6 mm thick pipe is welded to a 14 mm plate by fillet weld. The pipe is subjected to a vertical load of 3 kN at 1.00 m from the welded end and a twisting moment of 1.2 kNm. Design the joint.

Sol: Direct load = 3×10^3 N

Bending moment = $3 \times 10^3 \times 1000 = 3 \times 10^6$ Nmm

Twisting moment = 1200×10^3 Nmm

Let t = Effective throat thickness of the weld.

Polar moment of inertia, $I_{zz} = 2\pi r^3 t$

$$= 2\pi \times (60)^3 \times t$$

$$= 1357168 \text{ t mm}^4$$

$$I_{xx} = \frac{I_{zz}}{2}$$

$$= \frac{1357168t}{2}$$

$$= 678584.01\, t \text{ mm}^4$$

$$\text{Shear stress due to direct load} = \frac{3\times10^3}{2\pi\times60\times t}$$

$$= \frac{7.957}{t} \text{N/mm}^2$$

$$\text{Bending stress due to bending moment} = \frac{3\times10^6\times60}{67.85\times10^4\times t}$$

$$= \frac{265.29}{t} \text{N/mm}^2$$

This bending stress is treated as shear since, the actual failure will be along the throat of the weld.

$$\text{Shear stress due to twisting moment} = \frac{1200\times10^3\times60}{135.7168\times10^4\times t}$$

$$= \frac{53.05}{t} \text{N/mm}^2$$

$$\left\{\left(\frac{7.957}{t}\right)^2 + \left(\frac{265.29}{t}\right)^2 + \left(\frac{53.05}{t}\right)^2\right\}^{1/2}$$

$$\frac{270.65}{t} = 108$$

$$t = \frac{270.65}{108}$$

$$= 2.50 \text{ mm}$$

$$\approx 3\text{mm}$$

Hence, provide a 3 mm fillet weld.

OBJECTIVE QUESTIONS

1. A thin cylindrical steel pressure vessel of diameter 6 cm and wall thickness 3 mm is subjected to an internal fluid pressure of intensity 'p'. If the ultimate strength of steel 3600 kg/cm^2, the bursting pressure will be
   (a) 18 kg/cm^2 (b) 36 kg/cm^2
   (c) 180 kg/cm^2 (d) 360 kg/cm^2

2. Two closed thin vessels, one cylindrical and the other spherical with equal internal diameter and wall thickness are subjected to equal internal fluid pressure. The ratio of hoop stresses in the cylindrical to that of spherical vessel is
   (a) 4.0 (b) 2.0
   (c) 1.0 (d) 0.5

3. A thin cylindrical tube with closed ends is subjected to:
   1. Longitudinal stress $\sigma_1 = 14$ N/mm^2
   2. Hoop stress $\sigma_2 = 2$ N/mm^2
   3. Shearing stress $\tau = 8$ N/mm^2

   Then the maximum shearing stress is
   (a) 14 N/mm^2 (b) 12 N/mm^2
   (c) 10 N/mm^2 (d) 8 N/mm^2

4. A thin cylindrical tube closed at ends is subjected to internal pressure. A torque is also applied to the tube. The principal stresses p_1 and p_2 developed are 80.0 units and 20.0 units respectively. If the yield stress is 240 units, then what is the factor of safety according to maximum shear stress theory?
   (a) 3.00 (b) 4.00
   (c) 5.00 (d) 6.00

5. Which of the following statements regarding thin cylindrical shells are correct?
   1. If the thickness of the wall of the shell is less than 1/10 and 1/15 of the diameter of the shell, then it is treated as a thin shell.
   2. It is assumed that the normal stress (tensile or compressive) is uniformly distributed through the thickness of the wall.
   3. The intensity of longitudinal stress is one-half of the intensity of the hoop stress.

   Select the correct answer using the codes given below:
   (a) 1, 2 and 3 (b) 1 and 2
   (c) 2 and 3 (d) 1 and 3

6. A seamless pipe with 80 cm diameter carries a fluid under a pressure of 2 N/mm^2. If the permissible tensile stress is 100 N/mm^2, the minimum required thickness of the pipe is
   (a) 2mm (b) 4mm
   (c) 8mm (d) 16mm

7. A thin cylindrical shell of internal diameter D and thickness 't' is subjected to internal pressure 'p'. The change in diameter is given by
   (a) $\frac{pD^2}{4tE}(2-\mu)$ (b) $\frac{pD^2}{4tE}(1-2\mu)$
   (c) $\frac{pD^2}{2tE}(1-2\mu)$ (d) $\frac{pD^2}{2tE}(2-\mu)$

8. What is the change in diameter D of a thin spherical shell of wall thickness t when subjected to an internal fluid pressure p?
(E = Young's Modulus and μ = Poisson's ratio)

(a) $\frac{pD^2}{3\,tE}(1-\mu)$ (b) $\frac{pD}{4\,tE}(1-\mu)$

(c) $\frac{pD^2}{4\,tE}(1-\mu)$ (d) $\frac{pD^2}{4\,tE}(1-2\mu)$

9. A thin-walled long cylindrical tank of inside radius r is subjected simultaneously to internal gas pressure p and axial compressive force F at its ends. In order to produce 'pure shear' state of stress in the wall of the cylinder, F should be equal to

(a) $p\pi r^2$ (b) $2p\pi r^2$
(c) $3p\pi r^2$ (d) $4p\pi r^2$

10. A thin cylindrical shell of diameter 'd', length 'l' and thickness 't' is subjected to an internal pressure 'p'. What is the ratio of longitudinal strain to hoop strain in terms of poissons ratio 'μ'.

(a) $\frac{\mu-2}{2\mu-1}$ (b) $\frac{1-2\mu}{2-\mu}$

(c) $\frac{2\mu-1}{2+\mu}$ (d) $\frac{\mu-2}{2\mu+1}$

11. Which one of the following pairs is not correctly matched?

(a) Lame's constants : Thick cylinder
(b) Macaulay's method : Deflection of beam
(c) Euler's method : Theory of column
(d) Eddy's theorem : Torsion of shafts

12. Consider the following statements in respect of a thick cylinder subjected to internal pressure :

1. The stress on an element on the outer wall is unidirectional.
2. The stresses on an element on the inner wall are principal stresses.
3. The constants of the Lame's equation are positive.

Which of these statements are correct?

(a) 1 and 2 (b) 1 and 3
(c) 2 and 3 (d) 1, 2 and 3

13. The variation of the hoop stress across the thickness of a thick cylinder is

(a) linear (b) uniform
(c) parabolic (d) hyperbolic

14. A thick cylinder is subjected to external pressure. The magnitude of hoop stress at internal radius will be

(a) equal to the magnitude of hoop stress at external radius
(b) less than the magnitude of hoop stress at external radius
(c) greater than the magnitude of hoop stress at external radius
(d) equal to the magnitude of radial stress at internal stress

15. Match the following :

List 1

(A) Longitudinal stress in thick cylindrical shell
(B) Radial pressure in thick spherical shell

(C) Hoop stress in thick spherical shell

(D) Hoop stress in thick cylindrical shell

List 2

(a) $\dfrac{pR_i^2}{R_0^2 - R_i^2}$

(b) $\dfrac{B}{R^2} - A$

(c) $\dfrac{B}{R^3} + A$

(d) $\dfrac{B}{R^2} + A$

Codes:

	A	B	C	D
(a)	1	3	4	2
(b)	1	2	3	4
(c)	1	2	4	3
(d)	1	3	2	4

16. In a hollow thick cylinder the radial stress σ_r under an internal pressure 'p'

(a) increases from a minimum at the innermost surface to a maximum value at the outermost surface

(b) decreases from a maximum at the innermost surface to a minimum value at the outermost surface

(c) increases from zero at the innermost surface to a value $\sigma_r = p$ at the outermost surface

(d) decreases from a value $\sigma_r = p$ at the innermost surface to zero at the outermost surface

ANSWERS

1. (d)	5. (a)	9. (c)	13. (d)
2. (b)	6. (c)	10. (b)	14. (c)
3. (c)	7. (a)	11. (d)	15. (b)
4. (a)	8. (c)	12. (c)	16. (d)

SOLUTION...

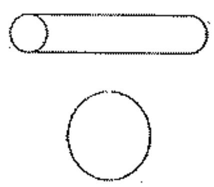
1. (d) $f = \frac{Pd}{2t}$

Critical Stress = Hoop's stress

$$3600 = \frac{P \times 6}{2 \times 0.3}$$

$$P = 360 \text{ kg/cm}^2$$

2. (b) $f_h = \frac{Pd}{2t}$



$$f_h = \frac{Pd}{4t}$$

$$\frac{f_{cylinder}}{f_{sphere}} = \frac{1}{(1/2)} = 2$$

3. (c) $\tau_{max} = \frac{\sigma_1 - \sigma_2}{2}$

$$\sigma_{1,2} = \frac{14+2}{2} \pm \sqrt{\left(\frac{14-2}{2}\right)^2 + 8^2}$$

$$8 \pm 10 \Rightarrow 18, -2$$

$$\tau_{max} = \frac{\sigma_1 - \sigma_2}{2} = \frac{18-(-2)}{2}$$

$$\Rightarrow \tau_{max} = \frac{20}{2} = 10 \text{ N/mm}^2$$

4. (b) $P_1 = 80$; $P_2 = 20$; $f_y = 240$

Max. shear stress theory

$$\tau_{max} = \left[\left(\frac{\sigma_1}{2}, \frac{\sigma_2}{2}\right)\left(\frac{\sigma_1 - \sigma_2}{2}\right)\right]_{max}$$

$$\Rightarrow 40 = \frac{240/2}{F.O.S}$$

$$F.O.S = \frac{240}{2 \times 40} = 3.0$$

6. (c) $f_h = \frac{Pd}{2t}$

$$100 = \frac{2 \times 800}{2 \times t}$$

$$t = 8 \text{ mm}$$

7. (a) Cylinder $\sigma_h = \frac{Pd}{2t}$

$$\sigma_l = \frac{Pd}{4t}$$

$$\frac{\Delta d}{d} = \varepsilon_h = \frac{\sigma_h - \mu\sigma_l}{E}$$

$$\Delta d = \frac{Pd^2}{4tE}(2-\mu)$$

8. (c) Sphere $\sigma_h = \sigma_l = \frac{Pd}{4t}$

$$\frac{\Delta d}{d} = \frac{\sigma_h - \mu\sigma_l}{2E}$$

$$\Delta d = \frac{Pd}{4tE}(1-\mu)$$

9. (c) For pure shear case

$$\sigma_h = F/A - \sigma_l$$

$$\frac{Pr}{t} = \frac{F}{2\pi rt} - \frac{Pr}{2t}$$

$$\Rightarrow F = 3\pi r^2 P$$

12

Moment of Inertia

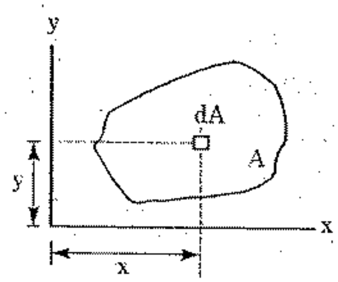
INTRODUCTION

In the design of any structural or mechanical system, we require the knowledge of the geometrical properties of a section. The geometrical properties are area, centroid, moment of inertia, product of inertia etc.

CENTROID

Coplaner area

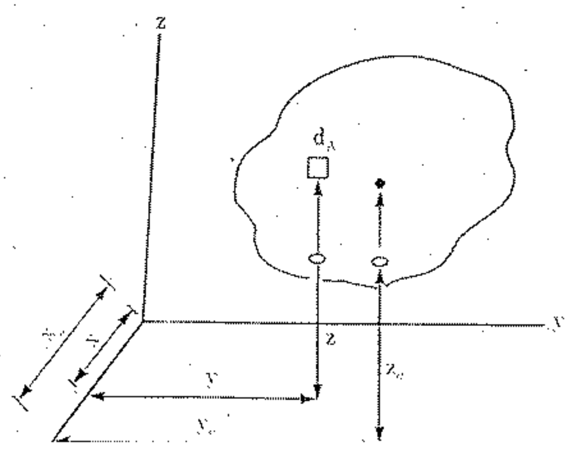
For a coplaner area A as shown in figure below.



Centroidal co-ordinate is given by (x_c, y_c) where

$$x_c = \int_A \frac{x dA}{A} \qquad y_c = \int_A \frac{y dA}{A}$$

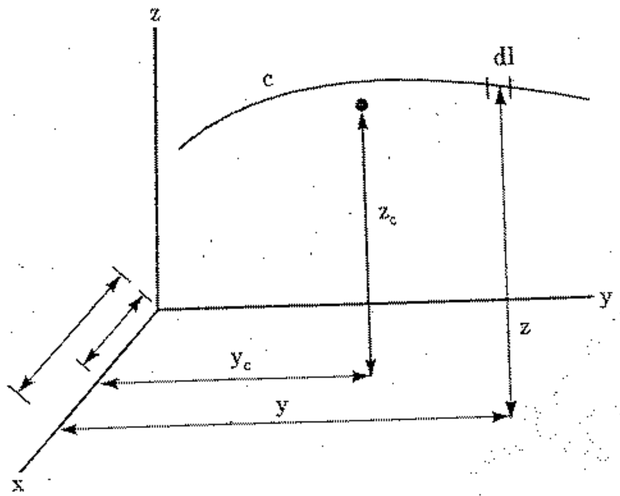
Curved surface



Centroidal co-ordinate is given by (x_c, y_c and z_c)

where $$x_c = \frac{\int_A x\,dA}{\int dA},\ y_c = \frac{\int_A y\,dA}{\int dA},\ z_c = \int_A z.dA$$

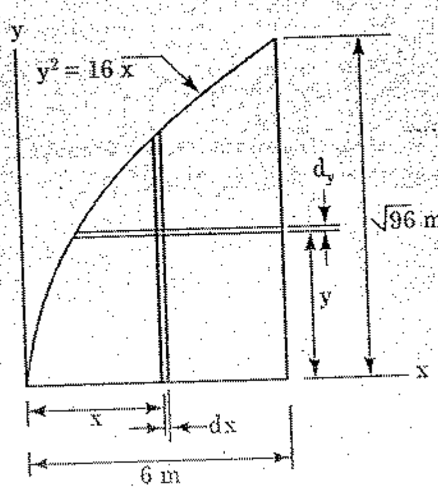
Centroid of line



$$z_c = \frac{\int z\,dA}{\int dA},\ y_c = \frac{\int y\,dA}{\int dA},\ x_c = \frac{\int_A x\,dA}{\int dA}$$

Example 1

Shown in Figure is a plane surface bounded by the x axis, the curve $y^2 = 16\,x$ and a line parallel to the y axis. Determine the centroidal co-ordinates.



Sol: We shall first compute $\int_A y\,dA$ and $\int_A x\,dA$ for this area. Using vertical infinitesimal area elements of thickness d_x and height y, we have,

$$\int x\,dA = \int_0^4 x(y.dx) = \int_0^4 x(4\sqrt{x})\,dx$$

$$\therefore \int x\,dA = \left[\frac{4.x^{5/2}}{5/2}\right]_0^4 = \frac{4\times 4^{5/2}}{5/2} = 51.2\ m^3$$

To compute $\int x\,dA$ we may use horizontal area elements of thickness d_y a shown in figure. Thus, we have,

$$\int y\,dA = \int_0^{\sqrt{96}} y\,(6-x)\,dy = \int_0^{\sqrt{96}}\left(6y-\frac{y^3}{16}\right)dy$$

$$\int y\,dA = \left[6.\frac{y^2}{2}-\frac{y^4}{64}\right]_0^{\sqrt{96}} = 144\ m^3$$

We could also have used vertical strips for computing $\int y\,dA$ as follows:

$$\int y\,dA = \int_0^6 \frac{y}{2}(y\,dx) = \int_0^6 \frac{16x}{2}\,dx$$

$$= 8\left(\frac{x^2}{2}\right)_0^6 = 144\ m^3$$

To compute the position of the centroid (x_c, y_c) we will need the area A of the surface,

Thus, $$A = \int_0^6 y\,dx = \int_0^6 4\sqrt{x}\,dx = 4\left(\frac{x^{3/2}}{3/2}\right)_0^6$$

$$= 39.192\ m^2$$

The centroidal co-ordinates are accordingly,

$$x_c = \frac{51.200}{39.192} = 1.307\ m$$

$$y_c = \frac{144.000}{39.192} = 3.674\ m$$

SECOND MOMENTS AND THE PRODUCT OF INERTIA OF A PLANE AREA

The second moments of the area A about the x and y axes denoted as I_{xx} and I_{yy} respectively are defined as

$$I_{xx} = \int_A y^2 dA, \text{ and}$$

$$I_{yy} = \int_A x^2 dA$$

The second moment of area cannot be negative in contrast to the first moment.

In an analogy to the centroid, a single point may be located in the area where the entire area may be concentrated to give the same second moment of area for a given reference

Thus,
$$A\,k_x^2 = \int_A y^2 dA, \quad \therefore k_x^2 = \frac{\int_A y^2 dA}{A}$$

$$A\,k_y^2 = \int_A x^2 dA, \quad \therefore k_y^2 = \frac{\int_A x^2 dA}{A}$$

The distances k_x and k_y are called the radii of gyration.
The product of inertia relates an area directly to a set of axes by the following formulation.

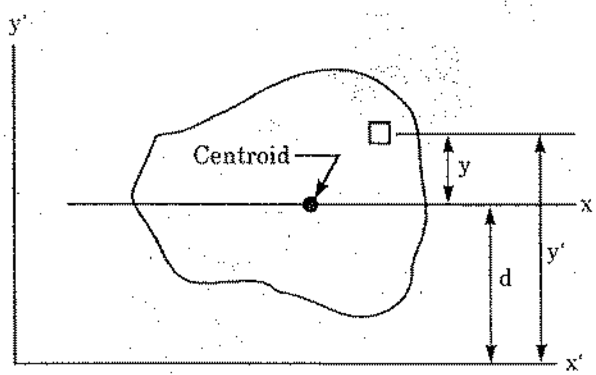
$$I_{xy} = \int_A xy\,dA$$

This quantity may be negative.

TRANSFER THEOREMS

In the figure shown below x′ axis is parallel to and is at a distance 'd' from an axis x going through the centroid of the area. The latter axis is called a centroidal axis. The second moment of area about the x′ axis is

$$I_{x'x'} = \int_A (y')^2 dA = \int_A (y+d)^2 dA$$



where the distance y′ has been replaced by (y + d). Integration leads to the result.

$$I_{x'x'} = \int_A y^2 dA + 2d\int_A y\,dA + A.d^2$$

The first term on the right hand-side is clearly I_{xx}. The second term involves the first moment of area about the x axis. But since the x axis here is a centroidal axis, the distance y_c from this axis is zero and so the second term is zero. We can now state the transfer theorem (Some times called the parallel axis theorem) as follows:

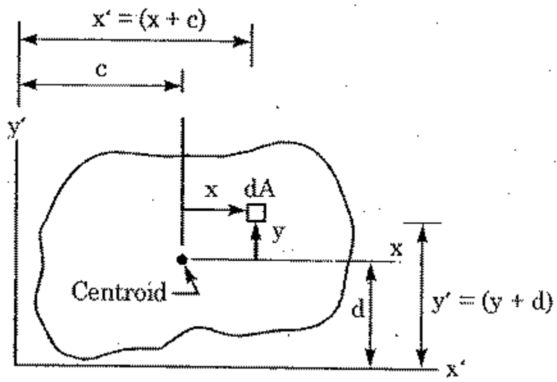
$$(I)_{\text{about any axis}} = (I)_{\text{about a parallel axis at centroid}} + Ad^2$$

PRODUCT OF INERTIA FOR PARALLEL-AXIS THEOREM

- Two parallel references are shown in figure, one at the centroid and the other positioned, so that the x and x′ are separated by distance d while the y and y′ axes are separated by distance c.
- The product of inertia about the non centroidal axis x′y′ can then be given as

$$I_{x'y'} = \int_A x'y'\,dA = \int_A (x+c)(y+d)\,dA$$

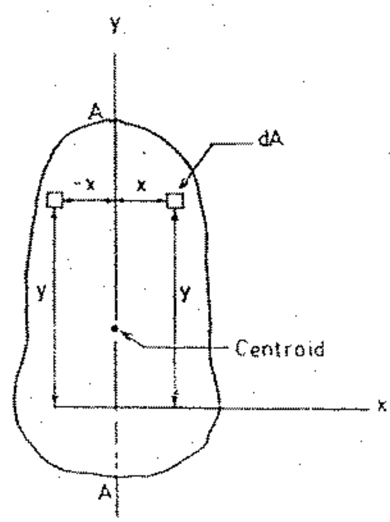
$$I_{x'y'} = \int_A xy\,dA + c\int_A y\,dA + d\int_A x\,dA + dc\,A$$



- The first term on the right side is I_{xy}, while the next two terms are zero, as explained earlier. Thus, we arrive at a parallel axis theorem for products of inertia of the form as

$$I_{x'y'\text{ for any set of axis}} = I_{xy\text{ a parallel set of axis at centroid}} + dc.A$$

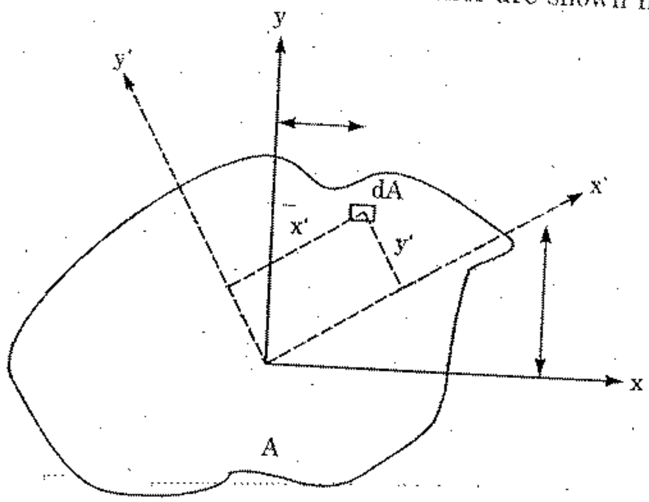
- If the area under consideration has an axis of symmetry, the product of inertia for this axis and any axis orthogonal to this axis must be zero in value.
- Consider the area in figure below which is symmetrical about the axis YY. The centroid is somewhere along this axis. The axis of symmetry has been indicated as the y axis, and an arbitrary x axis has been shown. Also indicated are two elemental areas that are positioned as mirror images about the y axis. The contribution to the product of inertia of each element is xy dA, but with opposite signs and so the net result is then zero. Since the entire area can be considered to be composed of such pairs, it becomes evident that the product of inertia of such cases is zero.
- This should not be taken to mean that a non-symmetric area cannot have a zero product of inertia about a set of axis.



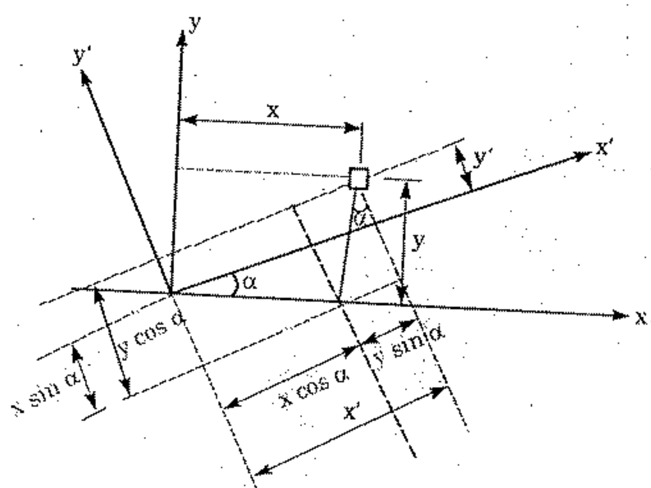
RELATION BETWEEN MOMENTS AND PRODUCTS OF INERTIA

- We can ascertain moments and products of inertia for a reference if for some other reference that has the same origin, these quantities are known.

- Two such references, rotated at an angle α from each other are shown in figure.



The relation between the co-ordinates of the area elements dA between the two references is given as follows:



$$x' = x\cos\alpha + y\sin\alpha$$
$$y' = -x\sin\alpha + y\cos\alpha$$

We can express $I_{x'x'}$ in the following manner:

$$I_{x'x'} = \int_A (y')^2 dA = \int_A (-x\sin\alpha + y\cos\alpha)^2 dA$$

$$I_{x'x'} = \sin^2\alpha \int_A x^2 dA - 2\sin\alpha\cos\alpha \int_A xy\,dA + \cos^2\alpha \int_A y^2 dA$$

$$\boxed{I_{x'x'} = I_{yy}\sin^2\alpha + I_{xx}\cos^2\alpha - 2I_{xy}\sin\alpha\cos\alpha}$$

A more common form of the desired relation can be formed by using the following trigonometric relations:

$$\cos^2\alpha = \frac{1}{2}(1+\cos 2\alpha)$$

$$\sin^2\alpha = \frac{1}{2}(1-\cos 2\alpha)$$

$$2\sin\alpha\cos\alpha = \sin 2\alpha$$

Then, we have,

$$\boxed{I_{x'x'} = \frac{I_{xx}+I_{yy}}{2} + \frac{I_{xx}-I_{yy}}{2}\cos 2\alpha - I_{xy}\sin 2\alpha}$$

To determine $I_{y'y'}$, we need only to replace the α in the preceding result by $\left(\alpha + \frac{\pi}{2}\right)$.

$$I_{y'y'} = \frac{I_{xx}+I_{yy}}{2} + \frac{I_{xx}-I_{yy}}{2}\cos(2\alpha+\pi) - I_{xy}\sin(2\alpha+\pi)$$

Note that $\cos(2\alpha + \pi) = -\cos 2\alpha$ and

$\sin(2\alpha + \pi) = -\sin 2\alpha$. Hence, the above equation becomes:

$$\boxed{I_{y'y'} = \frac{I_{xx}+I_{yy}}{2} - \frac{I_{yy}-I_{xx}}{2}\cos 2\alpha + I_{xy}\sin 2\alpha}$$

Note that $\boxed{I_{xx} + I_{yy} = I_{x'x'} + I_{y'y'}}$

Next, the product of inertia $I_{x'y'}$ can be computed in a similar manner.

$$I_{x'y'} = \int_A x'y'\,dA = \int_A (x\cos\alpha + y\cos\alpha)(-x\sin\alpha + y\cos\alpha)\,dA$$

$$I_{x'y'} = \sin\alpha\cos\alpha\,(I_{xx} - I_{yy}) + (\cos^2\alpha - \sin^2\alpha)\,I_{xy}$$

using trigonometric identities,

$$\boxed{I_{x'y'} = \frac{I_{xx}-I_{yy}}{2}\sin 2\alpha + I_{xy}\cos 2\alpha}$$

Thus, we see that if we know the quantities I_{xx}, I_{yy} and I_{xy}, the moments and products of inertia for every set of axis at the corresponding point can be computed.

For $I_{x'x'}$ to be maximum or minimum, $I_{x'y'} = 0$. The inclination of the principal axis can be calculated using the following formula:

$$\boxed{\tan 2\alpha = \frac{2I_{xy}}{I_{yy} - I_{xx}}} \quad \text{------ (iv)}$$

Using trigonometric relations we can write,

$$\cos 2\alpha = \frac{I_{yy} - I_{xx}}{2\sqrt{\left(\frac{I_{xx}-I_{yy}}{2}\right)^2 + I_{xy}^2}}$$

$$\sin 2\alpha = \frac{I_{xy}}{2\sqrt{\left(\frac{I_{xx}-I_{yy}}{2}\right)^2 + I_{xy}^2}}$$

$$\Rightarrow \quad \boxed{(I_{x'x'})_{max,min} = \frac{I_{xx}+I_{yy}}{2} \pm \sqrt{\left(\frac{I_{xx}-I_{yy}}{2}\right)^2 + I_{xx}^2}} \quad \text{------ (v)}$$

$I_{x'x', max}$ is usually denoted by I_{uu} and $I_{x'x' min}$ is denoted by $I_{vv,}$ thus

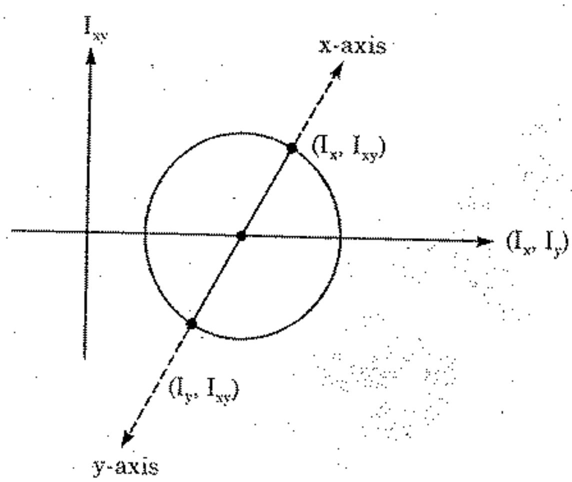
$$I_{uu} = \frac{I_{xx}+I_{yy}}{2} + \sqrt{\left(\frac{I_{xx}-I_{yy}}{2}\right)^2 + I_{xy}^2} \qquad \text{------(vi)}$$

$$I_{vv} = \frac{I_{xx}+I_{yy}}{2} - \sqrt{\left(\frac{I_{xx}-I_{yy}}{2}\right)^2 + I_{xy}^2} \qquad \text{------(vii)}$$

- The axis about which moment of inertia is max is called major principal axis and the axis about which moment of inertia is min is called minor principal axis.
- Note that product of inertia for principal axis = 0.

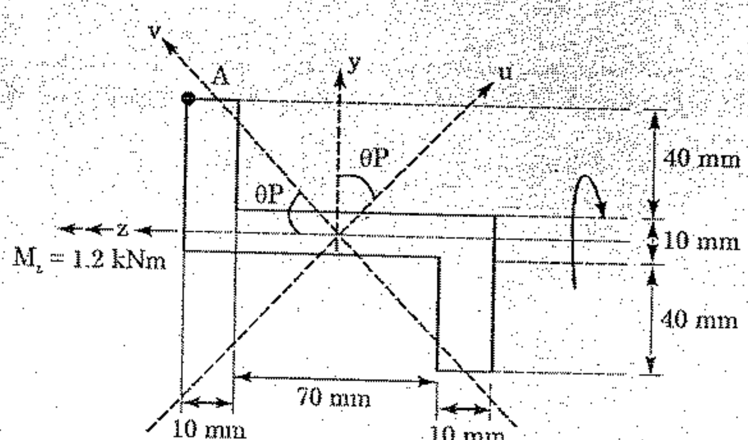
USE OF MOHR CIRCLE

Plot (I_x, I_{xy}) and (I_y, $-I_{xy}$) and use already discused approach as in Mohr Circle for stresses to find out moment of inertia and product of inertia about any axis.



Example 2

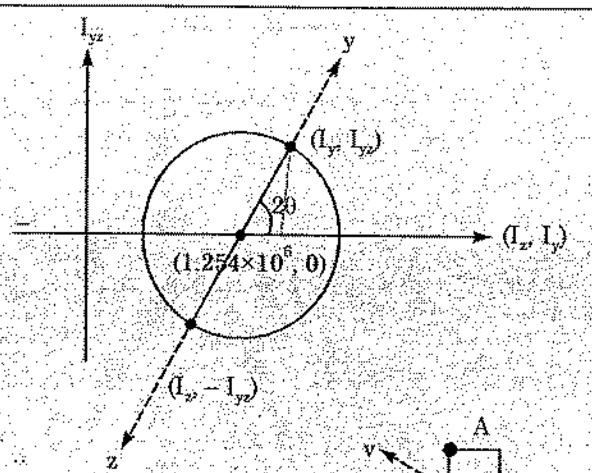
The couple M_z acts in a vertical plane and is applied to a beam of the x-section shown. Determine the stress at point 'A'.

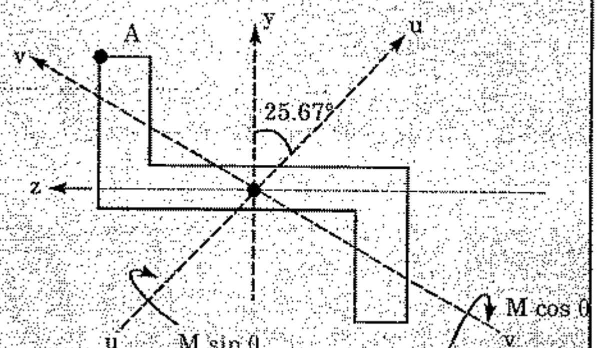


$I_y = 1.894 \times 10^6$ mm^4

$I_z = 0.614 \times 10^6$ mm^4

$I_{yz} = 0.800 \times 10^6$ mm^4





$I_u = (1.245 + 1.0245) \times 10^{-6} = 2.2785 \times 10^6 \text{ mm}^4$

$I_v = 0.2295 \times 10^6 \text{ mm}^4$

$$\sin 2\theta = \frac{0.8}{1.0245} = 0.7809$$

$2\theta = 51.34° \Rightarrow \theta = 25.67°$

$u_A = y\cos(-\theta) + z\sin(-\theta)$

$v_A = -y\sin(-\theta) + z\cos(-\theta)$

$u_A = y\cos\theta - z\sin\theta = 45\cos(25.67°) = 45\sin(25.67°) = 21.06$

$v_A = y\sin\theta + z\cos\theta = 45\sin(25.67°) + z\cos(25.67°) = 60.05$

$M_v = M\cos\theta = 1.08 \times 10 \times 10^3 \text{ Nm}$

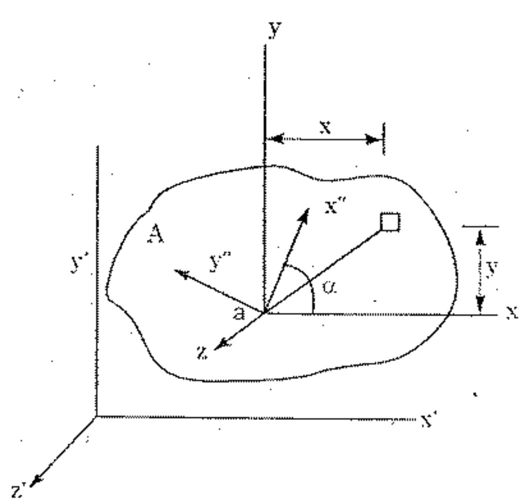
$M_u = M\sin\theta = -0.5148 \times 10^3 \text{ Nm}$

$$\sigma = \frac{-M_v u}{I_{vv}} + \frac{M_u v}{I_{uu}}$$

$$\Rightarrow \sigma_A = \frac{-1.0816\times10^6\times21.06}{0.2295\times10^6} + \frac{(-0.5198\times10^3)\times60.05}{2.2785\times10^6} = -112.95 \text{ N/mm}^2$$

POLAR MOMENT OF INERTIA

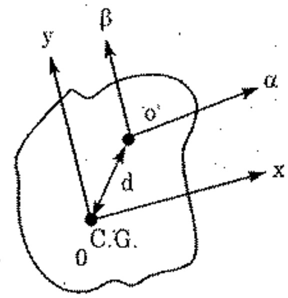
Moment of inertia of the given section about z-axis is called polar moment of inertia.



$$I_{xx} + I_{yy} = \int_A y^2 dA + \int_A x^2 dA$$

$$= \int_A (x^2 + y^2)\, dA = \int_A r^2 dA$$

Since, r^2 is independent of the inclination of the co-ordinate system; it becomes apparent from the above equation, that the sum $I_{xx} + I_{yy}$ is independent of the inclination of the reference. Therefore, the sum of moments of inertia about orthogonal axis is a function, only of the position of the origin for the axis. This sum is termed the polar moment of inertia, I_p.



By parallel axis theorem

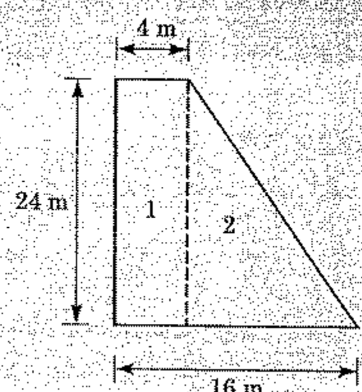
$$I_{p \text{ about } O'} = I_{p \text{ about } O} + Ad^2$$

Thus,

$$I_p = I_{xx} + I_{yy} = I_{x''x''} + I_{y''y''}$$

Example 3

Determine the centroid of the elementary profile of a dam whose cross section is shown in figure.



Sol: We will divide the cross section into two segments, i.e. one rectangle and one triangle.

$$A_1 = 24 \times 4 = 96 \text{ m}^2$$

$$A_2 = \frac{1}{2} \times 12 \times 24 = 144 \text{ m}^2$$

$$x_1 = \frac{4}{2} = 2 \text{ m}$$

$$x_2 = 4 + \frac{12}{3} = 8 \text{ m}$$

$$y_1 = \frac{24}{2} = 12 \text{ m}$$

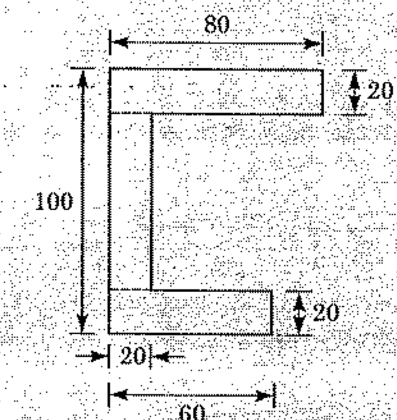
$$y_2 = \frac{24}{3} = 8 \text{ m}$$

$$\bar{x} = \frac{(96\times2)+(144\times8)}{96+144} = 5.6 \text{ m}$$

$$\bar{y} = \frac{(96\times12)+(144\times8)}{96+144} = 9.6 \text{ m}$$

Example 4

Find the centroid of the section shown in the figure below.



Sol: Firstly, we will devide the section into three rectangular segments.

$$A_1 = 80 \times 20 = 1600 \text{ mm}^2$$
$$A_2 = 60 \times 20 = 1200 \text{ mm}^2$$
$$A_3 = 60 \times 20 = 1200 \text{ mm}^2$$

$$x_1 = \frac{80}{2} = 40 \text{ mm}$$

$$x_2 = \frac{20}{2} = 10 \text{ mm}$$

$$x_3 = \frac{60}{2} = 30 \text{ mm}$$

$$y_1 = 100 - \frac{20}{2} = 90 \text{ mm}$$

$$y_2 = \frac{100}{2} = 50 \text{ mm}$$

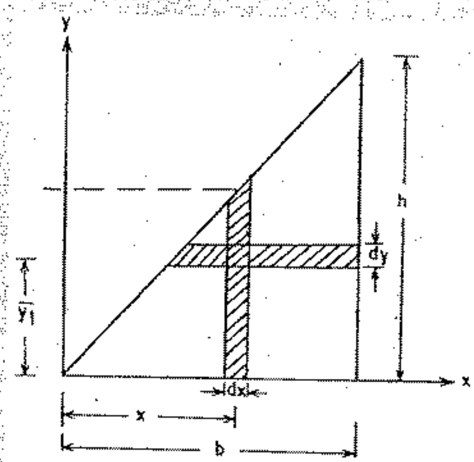
$$y_3 = \frac{20}{2} = 10 \text{ mm}$$

$$\bar{x} = \frac{1600\times40+1200\times10+1200\times30}{1600+1200+1200}$$
$$= 28 \text{ mm}$$

$$\bar{y} = \frac{1600\times90+1200\times50+1200\times10}{1600+1200+1200}$$
$$= 54 \text{ mm}$$

Example 5

Calculate I_{xx}, I_{yy} and I_{xy} for the triangle shown in figure.



Sol: Centroidal co-ordinates for this triangle is $G\left(\frac{2}{3}b, \frac{h}{3}\right)$

$$I_{xx} = \int y^2 dA$$

$$\bar{y} = \frac{h}{b}x$$

$$I_{xx} = \int_0^b dx \int_0^{\bar{y}} y^2 dy$$

$$I_{xx} = \int_0^b dx \int_0^{h/b.x} y^2 dy$$

$$= \int_0^b dx \left[\frac{y^3}{3}\right]_0^{y/b.x}$$

$$= \int_0^b \frac{1}{3}\frac{h^3}{b^3}x^3 dx$$

$$= \frac{1}{3} \times \frac{h^3}{b^3}\int_0^b x^3 dx$$

$$= \frac{h^3}{3b^3}\left(\frac{x^4}{4}\right)_0^b = \frac{bh^3}{12}$$

Similarly, $$I_{yy} = \frac{b^3h}{12}$$

Now, we will consider an axis passing through centroid of the triangle and the moment of inertia about the centroidal axis.

$$x = X - \frac{2}{3}b$$

$$y = Y - \frac{h}{3}$$

$$I_{xx} = \int_0^b dX \int_0^{h/b.X} \left(Y - \frac{h}{3}\right)^2 dY$$

$$= \int_0^b dX \int_0^{h/b.X} \left(Y - \frac{h}{3}\right)^2 dY$$

$$= \int_0^b dX \left[\frac{1}{3}\left(\frac{h}{b}X\right)^3 - \frac{2}{3}\frac{h}{2}\left(\frac{h}{b}X\right)^2 + \frac{h^2}{9} \times \frac{h}{b} \times X\right]$$

$$= \int_0^b dX \left[\frac{1}{3}\frac{h^3}{b^3}X^3 - \frac{h^3X^2}{3b^2} + \frac{h^3}{9b}X\right]$$

$$= \left[\frac{1}{3}\frac{h^3}{b^2} \times \frac{X^4}{4} - \frac{h^3X^3}{3 \times 3b^2} + \frac{h^3}{2 \times 9b}X^2\right]_0^b$$

$$= \frac{bh^3}{12} - \frac{bh^3}{9} + \frac{bh^3}{18} = \frac{bh^3}{36}$$

Similarly, $I_{yy} = \frac{b^3h}{36}$

$$I_{xy} = \int_0^b \left(X - \frac{2}{3}b\right) dX \int_0^{h/b.X} \left(Y - \frac{h}{3}\right) dY$$

$$= \int_0^b \left(X - \frac{2}{3}b\right) dX \left[\frac{Y^2}{2} - \frac{h}{3}.y\right]_0^{\frac{h}{b}X}$$

$$= \int_0^b \left(X - \frac{2}{3}b\right)\left[\frac{h^2}{2b^2}X^2 - \frac{h}{3}.\frac{h}{b}X\right] dX$$

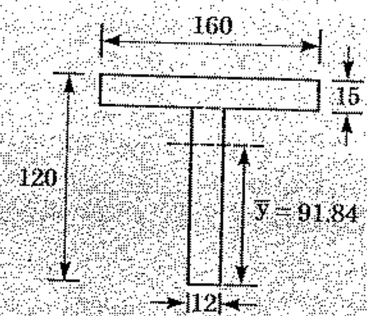
$$= \int_0^b \left[\frac{h^2}{2b^2}X^3 - \frac{h^2}{3b}X^2 - \frac{2}{3}b \times \frac{h^2}{2b^2}X^2 + \frac{2}{3} \times b \times \frac{h^2}{3b}X\right] dX$$

$$= \left[\frac{h^2}{2b^2}\frac{X^4}{4} - \frac{2}{3}\frac{h^2}{b}\frac{X^3}{3} + \frac{2}{9}h^2\frac{X^2}{2}\right]_0^b$$

$$= \frac{b^2h^2}{72}$$

Example 6

Calculate I_{xx}, I_{yy} and I_{xy} for section shown in figure.



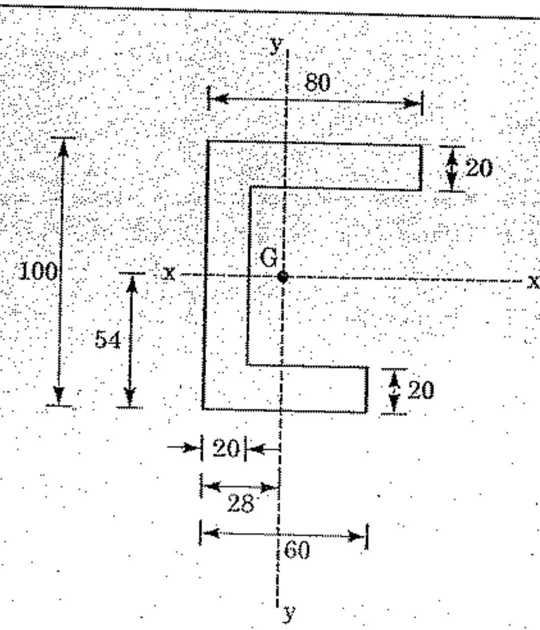
Sol:

$$\bar{y} = 91.84 \text{ mm}$$
$$A_1 = 160 \times 15 = 2400 \text{ mm}^2$$
$$A_2 = 105 \times 12 = 1260 \text{ mm}^2$$
$$\bar{y}_1 = 112.5 - 91.84 = 20.66 \text{ mm}$$
$$\bar{y}_2 = 52.5 - 91.84 = -39.34 \text{ mm}$$
$$I_{xx} = \frac{1}{12}\times160\times15^3 + 2400\times20.66^2 + \frac{1}{12}\times12\times105^3 + 1260\times(-39.34)^2$$
$$= 4.17705 \times 10^6 \text{ mm}^4$$
$$I_{yy} = \frac{1}{12}\times15\times160^3 + \frac{1}{12}\times105\times12^3 = 5.13512\times10^6 \text{mm}^4$$
$$I_{xy} = 0$$

Example 7

Find the principle moments of inertia and the inclination of the principal axes for the cross section shown in the figure.

$$I_{xx} = 4.86933 \times 10^6 \text{ mm}^4$$
$$I_{yy} = 1.87733 \times 10^6 \text{ mm}^4$$
$$I_{xy} = 6.72 \times 10^5 \text{ mm}^4$$



Sol:

$I_{xx} = 4.86933 \times 10^6 \text{ mm}^4$
$I_{yy} = 1.87733 \times 10^6 \text{ mm}^4$
$I_{xy} = 6.72 \times 10^5 \text{ mm}^4$

$$I_{uu} = \left[\frac{4.86933+1.87733}{2} + \sqrt{\left(\frac{4.86933-1.87733}{2}\right)^2 + 0.672^2}\right] \times 10^6$$

$= 5.0133 \times 10^6 \text{ mm}^4$

$$I_{vv} = \left[\frac{4.86933+1.87733}{2} - \sqrt{\left(\frac{4.86933-1.87733}{2}\right)^2 + 0.672^2}\right] \times 10^6$$

$= 1.7333 \times 10^6 \text{ mm}^4$

$$\tan 2\alpha = \frac{2I_{xy}}{I_{yy} - I_{xx}} = \frac{2 \times 0.672 \times 10^6}{(1.87733 - 4.86933) \times 10^6} = -0.4492$$

$= -12° \, 5' \, 41''$ or $77° \, 54' \, 19''$

The principal axes are shown in figure. Since α is negative, it is measured in the clockwise direction from x-axis. If it is positive, it is measured above x axis in the anticlockwise direction.